International Journal of Pure and Applied Mathematics
Volume 70 No. 4 2011, 503-520

CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

R.A. Rashwan[1 §], S.M. Altwqi[2]
[1,2]Department of Mathematics
Faculty of Science
Assiut University
Assiut, EGYPT

Abstract: In this paper, we prove some strong and weak convergence theorems for generalized three step iterative scheme to approximate common fixed points of three asymptotically nonexpansive nonself mappings in a uniformly convex Banach space. Our results extend and improve the recent results in, [20] and [21], and many others.

AMS Subject Classification: 47H10, 54H25
Key Words: asymptotically nonexpansive nonself-mapping, weak and strong convergence, completely continuous, opial's condition, common fixed points

1. Introduction

Let C be a nonempty subset of a real Banach space E and let $T : C \to C$ be a nonlinear mapping. The mapping T is said to be asymptotically nonexpansive if for each $n \geq 1$, there exists a positive constant $\{k_n\}$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \text{ for all } x, y \in C. \tag{1.1}$$

T is nonexpansive if $k_n = 1$ for $n = 1, 2, ...$

Received: February 2, 2011

[§]Correspondence author

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \text{ for all } n \geq 1 \textit{ and } x, y \in C. \tag{1.2}$$

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [3] in 1972, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point. Iterative techniques for approximating fixed points of nonexpansive self-mappings have been studied by various authors (see, e.g., [1, 4, 6, 12, 13, 14]), using the Mann iteration process or the Ishikawa iteration process. For nonself nonexpansive mappings, some authors (see, e.g., [5, 9, 16, 18, 22]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces.

In 1991, Schu [15] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. More precisely, he proved the following theorem.

Theorem 1.1. (see [14]) *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of E. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [0, \infty)$ for all $n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1, n \geq 1$, for some constant a, b. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \; n \geq 1 \tag{1.3}$$

converges strongly to some fixed point of T.

Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces (see, e.g. [8, 10, 11, 13, 14]).

The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume [2] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.1. (see [2]) Let C a nonempty subset of real normed linear space E. Let $P : E \to C$ be the nonexpansive retraction of E onto C. A nonself mapping $T : C \to E$ is called asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|. \tag{1.5}$$

By studying the following iteration process:

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad x \in C \tag{1.6}$$

Remark 1.1. If T is a self-mapping, then P becomes the identity mapping so that (1.4) and (1.5) reduce to (1.1), (1.2) and (1.6) reduces to (1.3) respectively.

Chidume, Ofoedu and Zegeye [2] got the following strong and weak convergence theorems for nonself asymptotically nonexpansive mapping.

Theorem 1.2. *Let E be a real uniformly convex Banach space, C nonempty closed convex subset of E. Let $T : C \to E$ be completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by (1.6). Then $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem 1.3. *Let E be a real uniformly convex Banach space which has a Fréchet differentiable norm, C nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive map with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.6). Then $\{x_n\}$ converges weakly to some fixed point of T.*

Recently, Wang [21] generalized the iteration process (1.6) as follows:

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak convergence of the iterative scheme (1.7) under proper conditions. Meanwhile, the results of [21] generalized some results of [2].

Remark 1.2. As $T_1 = T_2 = T$ and $\beta_n = 0$, for all $n \geq 1$. The iterative scheme (1.7) reduce to (1.6).

More recently Sornsak Thianwan [20] introduce and study a new class of iterative schemes. The scheme is defined as follows,

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1 \end{aligned} \tag{1.8}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak convergence of the iterative scheme (1.8) under proper conditions

Remark 1.3. As $T_1 = T_2 = T$ and $\beta_n = 0$, for all $n \geq 1$. The iterative scheme (1.8) reduce to (1.6).

The main purpose of this paper is to construct an iteration scheme (2.1) below for approximating common fixed points of three nonself asymptotically nonexpansive mappings and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces. Our results extend and improve the corresponding ones due to [20] and [21] and others.

2. Preliminaries

Let E be a real uniformly convex Banach space, C a nonempty closed convex subset of E, which is also a nonexpansive retract of E with retraction P. Let $T_1, T_2, T_3 : C \to E$ be nonself asymptotically nonexpansive mappings. For approximating the common fixed points of three nonself asymptotically nonexpansive mappings, we introduce a new three-step iteration scheme further our scheme generalize the iteration scheme (1.8) as follows

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n), \\ z_n &= P((1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1)$.

Remark 2.1. (i) As $T_2 = T_3$ and $\gamma_n = 0$, for all $n \geq 1$. The iterative scheme (2.1)reduce to (1.8)

(ii) As $T_1 = T_2 = T_3 = T$ and $\gamma_n = \beta_n = 0$, for all $n \geq 1$. The iterative scheme (2.1)reduce to (1.6)

Now we list the following definitions and results which are useful in the sequel.

Let E be Banach space with dim $E \geq 2$, the modulus of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon\}.$$

The Banach space E is uniformly convex if and only if with $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

The Banach space E is said to satisfy the Opial's condition [11] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

A map $T : C \to E$ is called demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \to x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A mapping $T : C \to C$ is completely continuous if and only if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in C.

A subset C of E is said to be a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$.

Note that: Every closed convex subset of uniformly convex Banach space is retract.

A map $P : E \to E$ is a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

A mapping $T : C \to E$ is said to be semicompact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q \in C$.

Three mappings $T_1, T_2, T_3 : C \to C$, where C is a subset of normed space E, are said to satisfy condition A'[7] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) \geq 0$ for all $r \in (0, \infty)$ such that either $\|x - T_1x\| \geq f(d(x, F))$ or $\|x - T_2x\| \geq f(d(x, F))$ or $\|x - T_3x\| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - q\| : q \in F = F(T_1) \bigcap F(T_2) \bigcap F(T_3)\}$.

The following lemmas are needed to prove our main results.

Lemma 2.1. (see [19]) *Let $\{s_n\}$ and $\{t_n\}$ be two nonnegative real sequences satisfying*

$$s_{n+1} \leq s_n + t_n$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.
Moreover if there exists a subsequence, $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} \to 0$ as $j \to \infty$ then $s_n \to 0$ as $n \to \infty$

Lemma 2.2. (see [15]) *Let E be a uniformly convex Banach space, and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ holed for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. (see [2]) *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : C \to E$ be nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to 1$ Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $\|x_n - Tx_n\| \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.4. (see [17]) *Let E be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequence of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3. Main Results

In this section, we will prove the strong and weak convergence of the iteration scheme (2.1) to a common fixed point for three asymptotically of nonexpansive nonself mappings in a uniformly convex Banach space. We first prove the following lemmas.

Lemma 3.1. *Let E be a uniformly convex Banach space and C be a nonempty closed convex nonexpansive retract of E with P as anonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, $\sum_{n=1}^{\infty} (r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ as in (2.1). If $q \in \bigcap_{i=1}^{3} F(T_i)$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in \bigcap_{i=1}^{3} F(T_i)$. Setting $k_n = 1+u_n$, $l_n = 1+v_n$, $r_n = 1+w_n$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, $\sum_{n=1}^{\infty} (r_n - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n <$

∞, $\sum\limits_{n=1}^{\infty} w_n < \infty$, using (2.1) we have

$$\begin{aligned}
\|z_n - q\| &= \|P((1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n) - q\| \\
&\leq \|(1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n - q\| \\
&= \|(1-\gamma_n)(x_n - q) + \gamma_n(T_3(PT_3)^{n-1}x_n - q)\| \\
&\leq (1-\gamma_n)\|x_n - q\| + \gamma_n\|T_3(PT_3)^{n-1}x_n - q\| \\
&\leq (1-\gamma_n)\|x_n - q\| + \gamma_n(1+w_n)\|x_n - q\| \\
&\leq (1+w_n)\|x_n - q\|,
\end{aligned}$$

and using (2.1) again

$$\begin{aligned}
\|y_n - q\| &= \|P((1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n) - q\| \\
&\leq \|(1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n) - q\| \\
&= \|(1-\beta_n)(z_n - q) + \beta_n(T_2(PT_2)^{n-1}z_n - q)\| \\
&\leq (1-\beta_n)\|z_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\
&\leq (1-\beta_n)\|z_n - q\| + \beta_n(1+v_n)\|z_n - q\| \\
&\leq (1+\beta_n v_n)\|z_n - q\| \\
&\leq (1+v_n)\|z_n - q\| \\
&\leq (1+v_n)(1+w_n)\|x_n - q\| \\
&\leq (1+v_n+w_n+v_n w_n)\|x_n - q\|,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - q\| \\
&\leq \|(1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n - q\| \\
&= \|(1-\alpha_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - q\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n(1+u_n)\|y_n - q\| \\
&\leq (1+u_n)\|y_n - q\| \\
&\leq (1+u_n)(1+v_n+w_n+v_n w_n)\|x_n - q\| \\
&= (1+u_n+v_n+w_n+u_n v_n+u_n w_n+v_n w_n+u_n v_n w_n)\|x_n - q\| \\
&\leq e^{\sum\limits_{n=1}^{\infty}(u_n+v_n+w_n+u_n v_n+u_n w_n+v_n w_n+u_n v_n w_n)}\|x_n - q\|,
\end{aligned}$$

since $\sum\limits_{n=1}^{\infty}(u_n + v_n + w_n + u_nv_n + u_nw_n + v_nw_n + u_nv_nw_n) < \infty$ then $\{x_n\}$ is bounded, this implies that there exists a constant $M > 0$ such that $\|x_n - q\| \leq M$ for all $n \geq 1$, so

$$\|x_{n+1} - q\| \leq \|x_n - q\| + (u_n + v_n + w_n + u_nv_n + u_nw_n + v_nw_n + u_nv_nw_n)M.$$

Hence by using Lemma (2.1), $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in \bigcap\limits_{i=1}^{3} F(T_i)$. This completes the proof.

Lemma 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum\limits_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum\limits_{n=1}^{\infty}(r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $\bigcap\limits_{i=1}^{3} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (2.1). Then $\lim\limits_{n\to\infty} \|x_n - T_1x_n\| = \lim\limits_{n\to\infty} \|x_n - T_2x_n\| = \lim\limits_{n\to\infty} \|x_n - T_3x_n\| = 0$.*

Proof. Let $q \in \bigcap\limits_{i=1}^{3} F(T_i)$. Set $k_n = 1 + u_n$, $l_n = 1 + v_n$, $r_n = 1 + w_n$. By Lemma (3.1), we see that $\lim_{n\to\infty} \|x_n - q\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - q\| = c$. Using (2.1), we have

$$\|z_n - q\| \leq (1 + w_n)\|x_n - q\|. \tag{3.1}$$

Taking the lim sup on both sides in the inequality (3.1), we have

$$\limsup_{n\to\infty} \|z_n - q\| \leq c. \tag{3.2}$$

In addition $\|T_2(PT_2)^{n-1}z_n - q\| \leq I_n\|z_n - q\|$, taking the lim sup on both sides in the inequality, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - q\| \leq c. \tag{3.3}$$

From (2.1), we have

$$\|y_n - q\| \leq \|(1 - \beta_n)z_n + \beta_nT_2(PT_2)^{n-1}z_n - q\|$$

$$\leq \quad (1+v_n+w_n+v_nw_n)\|x_n-q\|. \tag{3.4}$$

Taking the lim sup on both sides in the inequality (3.4), we have (where $\sum\limits_{n=1}^{\infty}(u_n+v_n+u_nv_n)<\infty$)

$$\limsup_{n\to\infty}\|y_n-q\|\leq c. \tag{3.5}$$

In addition $\|T_1(PT_1)^{n-1}y_n-q\|\leq k_n\|y_n-q\|$, taking the lim sup on both sides in the inequality, we have

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}y_n-q\|\leq c. \tag{3.6}$$

From (2.1), we have

$$\begin{aligned}\|x_{n+1}-q\| &\leq \|(1-\alpha_n)y_n+\alpha_nT_1(PT_1)^{n-1}y_n-q\|\\ &\leq (1+u_n+v_n+w_n+u_nv_n+u_nw_n+v_nw_n+u_nv_nw_n)\\ &\quad\times\|x_n-q\|. \end{aligned}\tag{3.7}$$

Since $\sum\limits_{n=1}^{\infty}(u_n+v_n+w_n+u_nv_n+u_nw_n+v_nw_n+u_nv_nw_n)<\infty$ and $\lim\limits_{n\to\infty}\|x_{n+1}-q\|=c$, letting $n\to\infty$ in inequality (3.7) we have

$$\lim_{n\to\infty}\|(1-\alpha_n)(y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\|=c. \tag{3.8}$$

By using (3.5), (3.6) and Lemma (2.2), we have

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}y_n-y_n\|=0. \tag{3.9}$$

In addition $\|T_3(PT_3)^{n-1}x_n-q\|\leq r_n\|x_n-q\|$, taking the lim sup on both sides in the inequality, we have

$$\limsup_{n\to\infty}\|T_3(PT_3)^{n-1}x_n-q\|\leq c. \tag{3.10}$$

Using (2.1) again

$$\begin{aligned}\|x_{n+1}-q\| &\leq \|(1-\alpha_n)(y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\|\\ &\leq (1-\alpha_n)\|y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-y_n+y_n-q\|\\ &\leq \|y_n-q\|+\|T_1(PT_1)^{n-1}y_n-y_n\| \end{aligned}\tag{3.11}$$

Taking the $\liminf$ on both sides in the inequality (3.11), by (3.9) and $\lim_{n\to\infty} \|x_{n+1} - q\| = c$, we have

$$\liminf_{n\to\infty} \|y_n - q\| \geq c. \tag{3.12}$$

It follows from (3.5) and (3.12) that $\lim_{n\to\infty} \|y_n - q\| = c$. This implies that

$$c = \lim_{n\to\infty} \|y_n - q\| \leq \lim_{n\to\infty} \|(1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n) - q\| \\ \leq \lim_{n\to\infty} \|y_n - q\| = c$$

and so

$$\lim_{n\to\infty} \|(1-\beta_n)(z_n - q) + \beta_n(T_2(PT_2)^{n-1}z_n - q)\| = c$$

Using (3.3) and lemma (2.2) we have

$$\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - z_n\| = 0 \tag{3.13}$$

Using (2.1), we get

$$\begin{aligned} \|y_n - q\| &\leq (1-\beta_n)\|z_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - z_n + z_n - q\| \\ &\leq (1-\beta_n)\|z_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - z_n\| + \beta_n\|z_n - q\| \\ &\leq \|z_n - q\| + \|T_2(PT_2)^{n-1}z_n - z_n\| \end{aligned} \tag{3.14}$$

Taking the $\liminf$ on both sides in the inequality (3.14), by (3.13) and $\lim_{n\to\infty} \|y_n - q\| = c$, we obtain

$$\lim_{n\to\infty} \|z_n - q\| \geq c. \tag{3.15}$$

It follows from (3.2) and (3.15) that $\lim_{n\to\infty} \|z_n - q\| = c$. This implies that

$$c = \lim_{n\to\infty} \|z_n - q\| \leq \lim_{n\to\infty} \|(1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n) - q\| \\ \leq \lim_{n\to\infty} \|z_n - q\| = c.$$

And also

$$\lim_{n\to\infty} \|(1-\gamma_n)(x_n - q) + \gamma_n(T_3(PT_3)^{n-1}x_n - q)\| = c$$

Then by (3.10) and lemma (2.2) we have

$$\lim_{n\to\infty} \|T_3(PT_3)^{n-1}x_n - x_n\| = 0. \tag{3.16}$$

From $z_n = P((1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n)$ and

$$\begin{aligned} \|z_n - x_n\| &= \|P((1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n) - x_n\| \\ &\le (1-\gamma_n)\|x_n - x_n\| + \gamma_n\|T_3(PT_3)^{n-1}x_n - x_n\| \\ &\le \|T_3(PT_3)^{n-1}x_n) - x_n\| \\ &\to 0 \qquad (\text{as } n\to\infty). \end{aligned} \tag{3.17}$$

From $y_n = P((1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n)$ and

$$\begin{aligned} \|y_n - z_n\| &= \|P((1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n) - z_n\| \\ &\le (1-\beta_n)\|x_n - z_n\| + \beta_n\|T_2(PT_2)^{n-1}z_n - z_n\| \\ &\le \|T_2(PT_2)^{n-1}z_n - z_n\| \\ &\to 0 \qquad (\text{as } n\to\infty). \end{aligned} \tag{3.18}$$

From (3.17), (3.18) we have

$$\begin{aligned} \|y_n - x_n\| &\le \|y_n - z_n\| + \|z_n - x_n\| \\ &\to 0 \qquad (\text{as } n\to\infty). \end{aligned} \tag{3.19}$$

In addition

$$\begin{aligned} \|T_1(PT_1)^{n-1}x_n - x_n\| &= \|T_1(PT_1)^{n-1}x_n - y_n + y_n - x_n\| \\ &\le \|T_1(PT_1)^{n-1}x_n - y_n\| + \|y_n - x_n\| \\ &= \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n + T_1(PT_1)^{n-1}y_n \\ &\quad -y_n\| + \|y_n - x_n\| \\ &\le \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n \\ &\quad -y_n\| + \|y_n - x_n\| \\ &\le k_n\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\|. \end{aligned}$$

Thus, it follows from (3.9) and (3.19) that

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \tag{3.20}$$

By using (2.1), we have

$$\|x_{n+1} - x_n\| \le (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - x_n\|$$

$$\begin{aligned}
&\leq (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n + y_n - x_n\| \\
&\leq (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n\| + \alpha_n\|y_n - x_n\| \\
&\leq \|y_n - x_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\|.
\end{aligned}$$

It follows from (3.9) and (3.19)that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.21}$$

Using (3.21) and (3.20), we have

$$\begin{aligned}
\|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-1}x_n \\
&\quad + T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n+1}x_{n+1} \\
&\quad - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| \\
&\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\to 0 \qquad (\text{as } n \to \infty).
\end{aligned} \tag{3.22}$$

In addition

$$\begin{aligned}
\|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-2}x_n \\
&\quad + T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2}x_{n+1} \\
&\quad - T_1(PT_1)^{n-2}x_n\| + \|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + L\|x_{n+1} - x_n\| \\
&\quad + \|T_1(PT_1)^{n-2}x_n - x_n\|.
\end{aligned}$$

where $L = sup\,\{k_n : n \geq 1\}$. It follows from (3.21)and (3.22 that

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| = 0. \tag{3.23}$$

We denote by $(PT_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (3.22) and (3.23), we obtain

$$\begin{aligned}
\|x_{n+1} - T_1x_{n+1}\| &= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1} + T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad + \|T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\|
\end{aligned}$$

$$\begin{aligned}
&\quad +\|T_1(PT_1)^{1-1}(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{1-1}x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad +L\|(PT_1)^{n-1}x_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad +L\|(PT_1)(PT_1)^{n-2}x_{n+1} - P(x_{n+1})\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad +L\|T_1(PT_1)^{n-2}x_{n+1} - x_{n+1}\| \\
&\to 0 \qquad (\text{as } n \to \infty),
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$. Similarly , we may show that $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_3x_n\| = 0$. The proof is completed

Theorem 3.1. *Let E be a uniformly convex Banach space and C be a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$.Let $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ be the sequences defined by (2.1). If one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.1 $\{x_n\}$ is bounded. In addition, by lemma 2.2, $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_3x_n\| = 0$, and then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{nj}\}$ of $\{T_1x_n\}$ such that $T_1x_{nj} \to q$ as $j \to \infty$. It follows from Lemma 3.2 that $\lim_{j\to\infty} \|x_{n_j} - T_1x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - T_2x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - T_3x_{n_j}\| = 0$. So by the continuity of T_1 and Lemma 2.3, we have $\lim_{j\to\infty} \|x_{n_j} - q\| = 0$ and $q \in \bigcap_{i=1}^{3} F(T_i$. Furthermore, by Lemma 3.1, we get that $\lim_{n\to\infty} \|x_n - q\|$ exists. Thus $\lim_{n\to\infty} \|x_n - q\|= 0$. From (3.19), we have $\lim_{n\to\infty} \|y_n - x_n\| = 0$, and it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$. and From (3.17), we have $\lim_{n\to\infty} \|z_n - x_n\| = 0$, and it follows that $\lim_{n\to\infty} \|z_n - q\| = 0$. The proof is completed.

Theorem 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by (2.1). If one of T_1, T_2 and T_3 is demicompact, then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Since one of T_1, T_2 and T_2 is demicompact, $\{x_n\}$ is bounded. and $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_3x_n\| = 0$, and then there exists subsequence $\{T_1x_{nj}\}$ of $\{T_1x_n\}$ such that x_{n_j} converge strongly to q.It follows from Lemma 2.3 that $q \in F(T_1) \cap F(T_2) \cap F(T_3)$, Thus $\lim_{n\to\infty} \|x_n - q\|$ exists by Lemma 3.1. Since the subsequence $\{x_{n_j}\}$ of x_n such that x_{n_j} converges strongly to q, then $\{x_n\}$ converges strongly to the common fixed point $q \in F(T_1) \cap F(T_2) \cap F(T_3)$. From (3.17)and (3.19)we have $\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|z_n - x_n\| = 0$, and we have $\lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \|z_n - q\| = 0$. The proof is completed.

In the next result, we prove the strong convergence of the scheme (2.1) under condition A' which is weaker than the compactness of the domain of the mappings.

Theorem 3.3. *Let E be a uniformly convex Banach space and C be a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by (2.1). converge strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_3x_n\| = 0$. It follows from condition A' that $\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - T_1x_n\| = 0$ or $\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ or

$\lim\limits_{n\to\infty} f(d(x_n, F)) \le \lim\limits_{n\to\infty} \|x_n - T_3x_n\| = 0$. In the both case, $\lim\limits_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ we obtain that $\lim\limits_{n\to\infty} f(d(x_n, F)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim\limits_{n\to\infty} f(d(x_n, F)) = 0$. and $\sum\limits_{n=1}^{\infty}(u_n + v_n + w_n + u_nv_n + u_nw_n + v_nw_n + u_nv_nw_n) < \infty$, given $\epsilon > 0$ there exists a natural number n_0 such that $d(x_n, F) < \frac{\epsilon}{4}$ and $\sum\limits_{n=1}^{\infty}(u_n+v_n+w_n+u_nv_n+u_nw_n + v_nw_n + u_nv_nw_n) < \frac{\epsilon}{2}$ for all $n \ge n_0$. So we can find $y^* \in F$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{4}$. For $n \ge n_0$ and $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \|x_{n+m} - y^*\| + \|x_n - y^*\| \\
&\le \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| \\
&\quad + \sum_{n=1}^{\infty}(u_n + v_n + w_n + u_nv_n + u_nw_n + v_nw_n + u_nv_nw_n)M \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2}
\end{aligned}$$

Thus shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim\limits_{n\to\infty} x_n = u$. Now $\lim\limits_{n\to\infty} f(d(x_n, F)) = 0$ gives that $d(u, F) = 0$. F is closed, therefore $u \in F$. From (3.17) and (3.19) we have $\lim\limits_{n\to\infty} \|z_n - x_n\| = 0$ and $\lim\limits_{n\to\infty} \|y_n - x_n\| = 0$ and it follows that $\lim\limits_{n\to\infty} \|z_n - u\| = 0$ and $\lim\limits_{n\to\infty} \|y_n - u\| = 0$. This completes the proof.

Finally, we prove the weak convergence of the iterative scheme (2.1) for three asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.4. *Let E be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to E$ be three asymptotically nonexpansive nonself mappings of C with sequences $\{k_n\}, \{l_n\}, \{r_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum\limits_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum\limits_{n=1}^{\infty}(r_n - 1) < \infty$, $k_n, l_n, r_n \to 1$ as $n \to \infty$, and $\bigcap\limits_{i=1}^{3} F(T_i) \ne \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by (2.1). converge weakly to a common fixed point of T_1, T_2 and T_3.*

Proof. It follows from Lemma 3.2, that $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_3x_n\| = 0$. Since E is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3, we have $u \in \bigcap_{i=1}^{3} F(T_i)$. Suppose that subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ and$\{x_{o_k}\}$ of $\{x_n\}$ converge weakly to u, v and w, respectively. From lemma 2.3, $u, v, w \in \bigcap_{i=1}^{3} F(T_i)$. By lemma 3.1, $\lim_{n\to\infty}\|x_n - u\|$, $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - w\|$ exist. It follows from lemma 2.4, that $u = v = w$. Therefor $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. Moreover $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and $\lim_{n\to\infty}\|y_n - x_n\| = 0$ as proved in lemma 3.2 and $x_n \to u$ weakly as $n \to \infty$, $y_n \to u$ weakly as $n \to \infty$ and therefore $z_n \to u$ weakly as $n \to \infty$. This completes the proof of the theorem.

References

[1] S.S. Chang, Y.J. Cho, H. Zhou, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, *J. Korean Math. Soc.*, **38** (2001), 1245-1260.

[2] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **280** (2003), 364-374.

[3] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **35** (1972), 171-174.

[4] S. Ishikawa, Fixed pointsand iteration of nonexpansive mappings of in a Banach spaces, *Proc. Amer. Math. Soc.*, **73** (1976), 61-71.

[5] J.S. Jung, S.S. Kim, Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces, *Nonlinear Anal.*, **33** (1998), 321-329.

[6] S.H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings,*Nonlinear Anal.*, **61** (2005), 1295-1301.

[7] H. Fukhar-ud-din, S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, *J. Math. Anal. Appl.*, **328** (2007), 821-829.

[8] M. Maiti, M.K. Ghosh, Approximating fixed points by Ishikawa iterates, *Bull. Austral. Math. Soc.*, **40** (1989)113-117.

[9] S.Y. Matsushita, D. Kuroiwa, Strong convergence of averaging iteration of nonexpansive nonself-mappings, *J. Math. Anal. Appl.*, **294** (2004), 206-214.

[10] M. Aslam Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251** (2000), 217-229.

[11] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.

[12] M.O. Osilike, A. Udomene, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, *Math. Comput.* Modelling **32** (2000), 1181-1191.

[13] M.O. Osilike, A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, *J. Math. Anal. Appl.*, **256** (2001), 431-445.

[14] J. Schu, Iterative construction of a fixed points of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **158** (1991), 407-413.

[15] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. *Austral. Math. Soc.*, **43** (1991), 153-159.

[16] N. Shahzad, Approximating fixed points of non-self nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, **61** (2005), 1031-1039.

[17] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **311** (2005), 506-517.

[18] W. Takahashi, G.E. Kim, Strong convergence of approximants to fixed points of nonexpansive nonself-mappings, *Nonlinear Anal.*, **32** (1998), 447-454.

[19] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301-308.

[20] S. Thianwan, Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, *J. Comput. Appl. Math.*, **224** (2009), 688-695.

[21] L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings,*J. Math. Anal. Appl.*, **323** (2006), 550-557.

[22] H.K. Xu, X.M. Yin, Strong convergence theorems for nonexpansive nonself-mappings, *Nonlinear Anal.*, **242** (1995), 223-228.