International Journal of Pure and Applied Mathematics
Volume 70 No. 4 2011, 549-557

A NOTE ON THE INTEGRAL FUNCTION $Ni(x)$ AND ITS ELEMENTARY PROPERTIES

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Abstract: In this note, we discuss the integral function *Ni(x)* which has recently been introduced. Some features of this function make it attractive in the solution to non-homogeneous Airy's differential equation. We will examine some of its elementary properties, its derivatives, and the differential equations it satisfies.

AMS Subject Classification: 33E99
Key Words: Airy's non-homogeneous equation, *Ni(x)* function

1. Introduction

In their analysis of flow over porous layers, Nield and Kuznetsov, [5], formulated the physical problem by introducing a transition layer of variable thickness between a porous layer and a free-space channel. This resulted in a mathematical problem involving the non-homogeneous Airy's differential equation (see [1], [8]), namely

$$y'' - xy = f(x). \tag{1}$$

In their elegant analysis, Nield and Kuznetsov, [5], introduced an integral function, *Ni(x)*, to facilitate obtaining the solution to their resulting governing equation. Some interesting properties of *Ni(x)* were discussed by Nield and Kuznetsov, [5], and point out to the need to look further into this function and its promising features. The aim of the current work is to investigate some elementary properties of *Ni(x)*, its connection to the Airy and Scorer functions, its *nth* derivative, and the ordinary differential equations this function satisfies.

Received: February 7, 2011

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This work is organized as follows. In Section 2, we provide an overview of solutions to equation (1). In Section 3 we discuss *Ni(x)* and some of its properties. We devote Section 4 to establishing the relationships between *Ni(x)*, the Airy and Scorer functions. Sections 5 and 6 are devoted, respectively, to establishing derivatives of *Ni(x)* and the values of the derivatives at $x = 0$. In Section 7 we provide a few of the differential equations that *Ni(x)* satisfies. Throughout this work, prime notation is used to denote ordinary differentiation.

2. Solutions to Airy's Equation

As is well-known, (cf. [1], [2], [6], [7], [8]), general solution to equation (1) takes the following forms. When $f(x)$= -1/π, solution is given by

$$y = c_1 Ai\,(x) + c_2 Bi\,(x) + Gi(x) \tag{2}$$

and when $f(x)$= 1/π, solution is given by

$$y = c_1 Ai\,(x) + c_2 Bi\,(x) + Hi(x). \tag{3}$$

The functions *Ai(x)* and *Bi(x)* are the linearly independent homogeneous Airy's functions defined by:

$$Ai\,(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right) dt, \tag{4}$$

$$Bi\,(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt + \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt, \tag{5}$$

with their Wronskian given by, [1]:

$$W\left(Ai\,(x)\,, Bi\,(x)\right) = Ai\,(x)\,Bi'(x) - Bi\,(x)\,Ai'(x) = \frac{1}{\pi}. \tag{6}$$

The functions *Gi(x)* and *Hi(x)* are the Scorer functions, see [1], [2], [6], (or the non-homogeneous Airy's function, as they represent particular solutions to equation (1)), which are defined by:

$$Gi\,(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt, \tag{7}$$

$$Hi\,(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt. \tag{8}$$

It can be seen from (7), (8) and (5) that

$$Gi\,(x) + Hi\,(x) = Bi(x). \tag{9}$$

Particular values of Airy's and Scorer's functions at $x = 0$, expressed in terms of the Gamma function are as follows, [1], [2], [8]:

$$Gi\,(0) = \frac{1}{2}Hi\,(0) = \frac{1}{3}Bi\,(0) = \frac{1}{\sqrt{3}}Ai\,(0) = \frac{1}{3^{7/6}\Gamma(\frac{2}{3})}, \tag{10}$$

$$Gi'\,(0) = \frac{1}{2}Hi'\,(0) = \frac{1}{3}Bi'\,(0) = -\frac{1}{\sqrt{3}}Ai'\,(0) = \frac{1}{3^{5/6}\Gamma\left(\frac{1}{3}\right)}. \tag{11}$$

3. The Integral Function *Ni(x)*

Nield and Kuznetsov [5], defined $Ni\,(x)$ by

$$Ni\,(x) = Ai\,(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt, \tag{12}$$

with its first derivative given by

$$Ni'\,(x) = Ai'\,(x)\int_0^x Bi\,(t)dt - Bi'\,(x)\int_0^x Ai\,(t)dt \tag{13}$$

and obtained the following interesting zeroes of $Ni\,(x)$ and $Ni'\,(x)$:

$$Ni\,(0) = Ni'\,(0) = 0. \tag{14}$$

We recognize here that another important derivative of $Ni\,(x)$ is its second, which we express in the following form that involves the Wronskian, $W(Ai\,(x)$, $Bi\,(x))$:

$$N''i(x) = Ai''(x)\int_0^x Bi\,(t)dt - Bi''(x)\int_0^x Ai(t)dt - W\left(Ai\,(x)\,, Bi\,(x)\right). \tag{15}$$

where, as given in equation (6), $W\left(Ai\,(x)\,, Bi\,(x)\right) = \frac{1}{\pi}$.

The value of the second derivative at $x = 0$ is given in terms of the said Wronskian, namely

$$Ni''\,(0) = -W\left(Ai\,(0)\,, Bi\,(0)\right) = -\frac{1}{\pi}. \tag{16}$$

Now, using the method of variation of parameters, and with the help of (16), the following expression for a particular solution to (1) when $f(x)$ = constant = R, is obtained with great ease:

$$y_p = \frac{1}{Ni''(0)}\left\{Ai(x)\int_0^x f(t)\,Bi(t)\,dt - Bi(x)\int_0^x f(t)\,Ai(t)\,dt\right\}$$
$$= \frac{R}{Ni''(0)}Ni(x) \quad (17)$$

For the special cases of $f(x)$= -1/π, and $f(x)$= 1/π, general solutions to (1) take the following forms, respectively:

$$y = c_1 Ai(x) + c_2 Bi(x) + Ni(x), \quad (18)$$

$$y = c_1 Ai(x) + c_2 Bi(x) - Ni(x). \quad (19)$$

Clearly, solutions (18) and (19) offer some advantages over solutions (2) and (3), which include the use of one function, *Ni(x)*, in the particular solutions of equation (1), as opposed to the two functions *Gi(x)* and *Hi(x)*

4. Relationships between *Ni(x)* and the Scorer Functions

There exist relationships between *Ni(x)* and the Scorer functions *Gi(x)* and *Hi(x)* that we can establish with the knowledge of the following integrals of the Airy's homogeneous function, reported in [8]:

$$\int_0^x Ai(t)dt = \frac{1}{3} + \pi\left\{Ai'(x)\,Gi(x) - Gi'(x)\,Ai(x)\right\}, \quad (20)$$
$$\int_0^x Ai(t)dt = -\frac{2}{3} - \pi\left\{Ai'(x)\,Hi(x) - Hi'(x)\,Ai(x)\right\}, \quad (21)$$
$$\int_0^x Bi(t)dt = \pi\left\{Bi'(x)\,Gi(x) - Gi'(x)\,Bi(x)\right\}, \quad (22)$$
$$\int_0^x Bi(t)dt = -\pi\left\{Bi'(x)\,Hi(x) - Hi'(x)\,Bi(x)\right\}. \quad (23)$$

Multiplying (22) by $Ai(x)$ and (20) by $Bi(x)$, then subtracting the latter product from the former, and making use of the Wronskian of $Ai(x)$ and $Bi(x)$, we obtain:

$$Ni(x) = Gi(x) - \frac{1}{3}Bi(x). \quad (24)$$

In a similar fashion, multiplying (23) by $Ai(x)$ and (21) by $Bi(x)$, then subtracting the latter product from the former, and making use of the Wronskian of $Ai\,(x)$ and $Bi\,(x)$, we obtain:

$$Ni\,(x) = \frac{2}{3}Bi\,(x) - Hi(x). \tag{25}$$

Now, from (24) and (25) we obtain:

$$Ni\,(x) = \frac{2}{3}Gi(x) - \frac{1}{3}Hi(x). \tag{26}$$

The following integral representation for *Ni(x)* can then be obtained by using any one of equations (24), (25) or (26) together with the appropriate integral representations of equations (4), (5), (7) and (8):

$$Ni\,(x) = \frac{2}{3\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt - \frac{1}{3\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt. \tag{27}$$

5. Derivatives of *Ni(x)*

First derivative of $Ni\,(x)$ was obtained by Nield and Kuznetsov, [5], and is given by equation (13). We have obtained the second derivative in equation (15). In what follows we will develop an iterative formula for higher derivatives of $Ni\,(x)$ that are of utility when dealing with *nth* order differential equations that $Ni\,(x)$ satisfies.

We can write the second derivative of $Ni\,(x)$ in the following equivalent forms:

$$Ni''\,(x) = x\left[Ai\,(x)\int_0^x Bi\,(t)dt - Bi(x)\int_0^x Ai\,(t)dt\right] - W\left(Ai\,(x)\,, Bi\,(x)\right), \tag{28}$$

$$Ni''\,(x) = xNi\,(x) - W(Ai\,(x)\,Bi\,(x)). \tag{29}$$

Similarly, third-order derivative of $Ni\,(x)$ can be written in the following equivalent forms:

$$Ni'''\,(x) = \left[Ai\,(x) + xAi'\,(x)\right]\int_0^x Bi\,(t)dt - [Bi\,(x)$$

$$+xBi'(x)\Big]\int_0^x Ai(t)dt, \quad (30)$$

$$Ni'''(x) = Ni(x) + xNi'(x). \quad (31)$$

We can obtain higher derivatives by repeated differentiation of (30) or (31), as follows. Repeated differentiation of (31) yields the following iterative formula:

$Ni^{(n)}(x) = (n-2)Ni^{(n-3)}(x) + xNi^{(n-2)}(x);$ n¿2. ...(32)

Alternatively, we can express second and higher derivatives in terms of *Ni(x)* $Ni'(x)$ and $W(Ai(x), Bi(x)) = \frac{1}{\pi}$ by assuming that the *nth* derivative is of the form:

$$N^{(n)}(x) = g(x)\,Ni(x) + h(x)\,N'i(x) - p(x)\,W(Ai(x)\,Bi(x)), \quad (32)$$

where $g(x)$, $h(x)$, and $p(x)$ are the coefficients of *Ni(x),* $Ni'(x)$ and *W(Ai(x),Bi(x)),* respectively, that appear in the *nth* derivative Then the $n+1^{st}$ derivative is given by:

$$N^{(n+1)}(x) = \left[g'(x) + xh(x)\right]Ni(x) + \left[g(x) + h'(x)\right]Ni'(x) - [h(x) + p'(x)]W(Ai(x)\,Bi(x)). \quad (33)$$

Equation (33) takes the following form in terms of $Ai(x)$ and $Bi(x)$:

$$N^{(n+1)}(x) = \left\{\left[g'(x) + xh(x)\right]Ai(x) + \left[g(x) + h'(x)\right]Ai'(x)\right\}\int_0^x Bi(t)dt - \left\{\left[g'(x) + xh(x)\right]Bi(x) + \left[g(x) + h'(x)\right]B'i(x)\right\}\int_0^x Ai(t)dt - \left[h(x) + P'(x)\right]W(Ai(x), Bi(x)) \quad (34)$$

6. Values of *Ni(x)* and its Derivatives at $x = 0$

In this section, we derive iterative formulae for evaluating the derivatives of *Ni(x)* at $x = 0$ They may be of utility when dealing with initial value problems of *nth* order differential equations that *Ni(x)* satisfies.

Nield and Kuznetsov, [5] provided the following values:

$$Ni(0) = Ni'(0) = 0. \quad (35)$$

In this work we have shown that:

$$Ni''(0) = -W(Ai(0), Bi(0)) = -\frac{1}{\pi} \quad (36)$$

Now, taking $x = 0$ in the n+1st derivative (equation (33) or (34)), we obtain:
$N^{(n+1)}(0) = -\frac{1}{\pi}[h(0) + p'(0)]$; n = 2, 3, 4, ...(38)

In using (38) we need the values of $h(0)$and $p^{'}(0)$, which must be determined from the *nth* derivative. Alternatively, we develop the following iterative formula for more convenient computations:

$Ni^{(n+1)}(0) = (n-1)\, Ni^{(n-2)}(0)$ for n = 2,3, 4, ...(39)

As an illustration of the resulting pattern, using (39) gives the following first eleven derivatives of $Ni(x)$ at $x = 0$:

$$Ni(0) = Ni^{'}(0) = Ni^{'''}(0) = N^{iv}(0) = N^{vi}(0) = N^{vii}(0) = N^{ix}(0) = N^{x}(0) = 0. \quad (37)$$

$$Ni^{''}(0) = -\frac{1}{\pi}; \quad Ni^{v}(0) = -\frac{3}{\pi}; Ni^{viii}(0) = -\frac{18}{\pi}; \quad Ni^{xi}(0) = -\frac{162}{\pi}. \quad (38)$$

7. Differential Equations that *Ni(x)* Satisfies

The function *Ni(x)* satisfies the following *nth* order ordinary differential equation, obtained from (33):

$$y^{(n)} - \left[g(x) + h^{'}(x)\right] y^{'} - \left[g^{'}(x) + xh(x)\right] y = -\frac{1}{\pi}[h(x) + P^{'}(x)]. \quad (39)$$

For example, *Ni(x)* satisfies the following equations of orders two to seven, computed iteratively using (39):

$$y^{''} - xy = -\frac{1}{\pi}, \quad (40)$$

$$y''' - xy' - y = 0, \quad (41)$$

$$y^{iv} - 2y' - x^2 y = -\frac{1}{\pi}x, \quad (42)$$

$$y^{v} - x^2 y^{'} - 4xy = -\frac{3}{\pi}, \quad (43)$$

$$y^{vi} - 6xy' - \left[4 + x^3\right] y = -\frac{1}{\pi}x^2, \quad (44)$$

$$y^{vii} - \left(10 + x^3\right) y^{'} - 9x^2 y = -\frac{8}{\pi}x. \quad (45)$$

It is worth noting that equation (41) is a special case of the so-called comparison differential equation that received interest in the literature (see [8], page 108, and Langer, [3,4]), namely:

$$y''' - xy' - \mu y = 0. \tag{46}$$

When $\mu = 1$ (46) reduces to (41). Solutions to (46) have been discussed in detail in [8]. However, we see here that $Ni\,(x)$ is a solution to (46) for the case of $\mu = 1$. Furthermore, as discussed in [8], if y is a solution of (46) then $z = y'$ is a solution of the following differential equation:

$$z''' - xz' - (1 + \mu)z = 0. \tag{47}$$

Accordingly, since the function $y = Ni\,(x)$ is a solution to (41), then the function $z = Ni'\,(x)$ is a solution of

$$z''' - xz' - 2z = 0. \tag{48}$$

In fact, this is easily verified by computing the fourth derivative of $Ni\,(x)$ and substituting in equation (48).

Conclusion

The main theme of this work has been a discussion and development of elementary properties the function *Ni(x)* possesses. This integral function, introduced by Nield and Kuznetsov, [5], can offer a viable alternative to the solution of the non-homogeneous Airy's differential equation. We discussed the various ordinary differential equations that *Ni(x)* satisfies. In particular, *Ni(x)* satisfies a third-order equation that is a special case of Langer's comparison equation, [3,4]

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