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REFLECTION AND REFRACTION AT AN INTERFACE BETWEEN TWO DISSIMILAR THERMALLY CONDUCTING VISCOUS LIQUID HALF-SPACES

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Abstract: The present investigation is concerned with the problem of reflection and refraction of plane waves at an interface between two dissimilar thermally conducting viscous liquid half-spaces. The required boundary conditions at the interface are satisfied by particular solutions in both the half-spaces to obtain the relations between the amplitude ratios of different reflected and refracted waves for both Lord-Shulman (L-S) and Green-Lindsay (G-L) theories of generalized thermoelasticity. The complex absolute values of the amplitude ratios are computed numerically for a particular example of the model. In the presence as well as absence of thermal and viscous parameters, these values of amplitude ratios are plotted against the angle of incidence to observe the effects of viscosity and thermal fields.

AMS Subject Classification: 74Jxx
Key Words: thermally conducting, viscous liquid, plane waves, reflection, refraction, amplitude ratios, thermal relaxation times

1. Introduction

The classical dynamical coupled theory of thermoelasticity was extended to generalized thermoelasticity theories by Lord and Shulman [1] and Green and

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Lindsay [2]. These theories consider heat propagation as a wave phenomenon rather than a diffusion phenomenon and predict a finite speed of heat propagation. Wave propagation in thermoelasticity has many applications in various engineering fields. Problems on reflection and refraction in coupled or generalized thermoelasticity have been a topic of research for various authors [3]-[12].

The nature of the layers beneath the Earth's surface has not been explored completely yet. Therefore, it is appropriate to consider various models in Earth's crust for the purpose of theoretical as well as numerical investigation of wave phenomenon. Waves and vibration problems become more significant in the field seismology, when we study the problems with additional parameters, e.g. viscosity, thermal disturbance, porosity, microrotation, anisotropy etc. Singh [13] studied the reflection of plane waves from a free surface of a thermally conducting viscous liquid half-space with thermal relaxation times.

In the present paper a problem on reflection and refraction of plane waves at an interface between two different thermally conducting viscous liquid is considered. The required boundary conditions are satisfied by appropriate potentials to obtain a system of six non-homogeneous equations in terms of amplitude ratios. These amplitude ratios are computed for a particular numerical example of the model with and without thermal and viscous parameters.

2. Formulation of the Problem and Solution

We introduce rectangular Cartesian co-ordinates (x, y, z) and place the origin at the interface separating the two thermally conducting viscous liquid half space as shown in the Figure 1. We consider these two half spaces in welded contact along a plane interface z = 0. The positive z-axes is taken in medium M_1 (lower half space). Following Lord and Shulman [1] and Green and Lindsay [2], the constitutive equation and the equations of motion for a thermally conducting viscous liquid in absence of body forces, body couples and heat sources are written as

$$t_{kl} = (\mathrm{K} - \frac{2}{3}\eta\frac{\partial}{\partial t})\, u_{i,i}\, \delta_{kl} + 2\eta\frac{\partial}{\partial t}\, e_{kl} - \beta\,(\mathrm{T} + \tau_1\dot{\mathrm{T}})\, \delta_{kl}, \tag{1}$$

$$(\mathrm{K} + \frac{4}{3}\eta\frac{\partial}{\partial t})\, \nabla\,(\nabla.\mathbf{u}) - \eta\frac{\partial}{\partial t}\nabla \times (\nabla \times \mathbf{u}) - \beta\,\nabla\,(\mathrm{T} + \tau_1\dot{\mathrm{T}}) = \rho\,\ddot{\mathbf{u}}, \tag{2}$$

$$\rho C^*\left(\dot{T} + \tau_0\ddot{T}\right) + \beta T_0\,[\dot{u}_{i,i} + \Delta\tau_0\ddot{u}_{i,i}] = \mathrm{K}^*\nabla^2 T, \tag{3}$$

where K is bulk modulus, ρ is density of fluid, η is fluid viscosity, T is temperature variable, T_0 is the reference temperature, u is the displacement vector,

t_{kl} are components of the force stress tensor, e_{kl} are the components of strain tensor, δ_{kl} is Kronecker delta, K^* is the coefficient of thermal conductivity, C^* is the specific heat at constant strain, τ_0, τ_1 are the relaxation times, α_t is the coefficient of linear expansion, k wave number, c is phase velocity, ω is the angular frequency and $\beta = 3\mathrm{K}\alpha_t$. The symbol Δ in the equation (3) makes these fundamental equations possible for the two different theories of the generalized thermoelectricity. For L-S theory $\tau_1 = 0$, $\Delta = 1$ and for G-L theory $\tau_1 > 0$ and $\Delta = 0$. The thermal relaxation times $\tau_0$andτ_1 satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$ for G-L theory only. The dots over variables represent the time derivatives and the subscripts followed by comma denote spatial derivatives.

To solve the basic equations, we decompose the displacement vector **u** into scalar and vector potentials ϕ andψ as

$$\mathbf{u} = \nabla\phi + \nabla \times \psi \,, \qquad \nabla.\psi = 0 \,, \tag{4}$$

Using equation (4), equation (2) reduces to

$$v_1^2 \nabla^2 \phi = \ddot{\phi} + \bar{\beta}(T + \tau_1 \dot{T}), \tag{5}$$

$$v_2^2 \nabla^2 \psi = \ddot{\psi} \,, \tag{6}$$

where,

$$v_1^2 = \left(\mathrm{K} + \frac{4}{3}\eta\frac{\partial}{\partial t}\right)/\rho, \quad v_2^2 = \left(\eta\frac{\partial}{\partial t}\right)/\rho, \quad \bar{\beta} = \frac{\beta}{\rho}, \psi = -\vec{\psi}.$$

From equation (5), we have

$$T = \left(v_1^2 \nabla^2 \phi - \ddot{\phi}\right)/\bar{\gamma}, \quad \bar{\gamma} = \bar{\beta}(1 + \tau_1 \frac{\partial}{\partial t}). \tag{7}$$

Eliminating T from equation (3) and (7), we get

$$\nabla^4\phi - [\frac{C^*}{\bar{\mathrm{K}}^*}\{(1 + \tau_0\frac{\partial}{\partial t}) + \varepsilon(1 + \tau_1\frac{\partial}{\partial t})(1 + \Delta\tau_0\frac{\partial}{\partial t})\} + \frac{1}{v_1^2}\frac{\partial}{\partial t}]\frac{\partial}{\partial t}\nabla^2\phi + \frac{C^*}{\bar{\mathrm{K}}^*}\frac{1}{v_1^2}(1 + \tau_0\frac{\partial}{\partial t})\frac{\partial^3\phi}{\partial t^3} = 0 \,, \tag{8}$$

where, $\varepsilon = \bar{\beta}^2 T_0 / v_1^2 C^*$, $\bar{\mathrm{K}}^* = \mathrm{K}^*/\rho$.

We seek the solution of the equation (8) in the form

$$\phi = f(z)\exp\{ik(ct - x)\} \qquad (c > \quad v_1).$$

With the help of equation (9), equation (8) reduces to

$$\frac{d^4 f(z)}{dz^4} + A\frac{d^2 f(z)}{dz^2} + Bf(z) = 0, \tag{10}$$

where,

$$A = k^2(\frac{c^2}{v_1^2} - 2) - kc\,(\frac{C^*}{\bar{K}^*})\,[(i - \tau_0\,kc) + \varepsilon\,(i - \tau_1 kc\,)\,(1 + ikc\tau_0\,\Delta)],$$

$$B = k^4(1 - \frac{c^2}{v_1^2}) + ck^3(\frac{C^*}{\bar{K}^*})[(\,i - \tau_0\,kc) + \varepsilon\,(\,i - \tau_1 kc\,)\,(1 + ikc\tau_0\,\Delta) - \frac{c^2}{v_1^2}(\,i - \tau_0 kc\,)].$$

and $c = v/SinI$, where, v is the velocity of the incident wave and Idenotes the angle of incidence.

The general solution of equation (10) is written as

$$f(z) = A_1 \exp(ikm_1 z) + A_2 \exp(-ikm_1 z) + A_3 \exp(ikm_2 z) + A_4 \exp(-ikm_2 z), \tag{11}$$

where, $m_1^2 = [\{(A^2 - 4B)^{\frac{1}{2}} + A\}/2k^2]$, $m_2^2 = [\{-(A^2 - 4B)^{\frac{1}{2}} + A\}/2k^2]$ correspond to the coupled longitudinal wave and thermal wave and A_1, A_2, A_3, A_4 are arbitrary constants.

We seek the solution of equation (6) in the form

$$\psi = g(z) \exp\{ik\,(ct - x)\} \qquad (c > v_2).$$

With the help of equation (12) in equation (6), we have

$$g(z) = [A_5 \exp(ikm_3 z) + A_6 \exp(-ikm_3 z)] \exp\{ik(ct - x)], \tag{13}$$

where,$m_3^2 = (\frac{c^2}{v_2^2} - 1)$ corresponds to transverse wave and A_5, A_6 are arbitrary constants.

The displacement components of vector **u** in x-z plane are

$$u_1 = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial z}, \quad u_3 = \frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial x}. \tag{14}$$

The field variables and constants with primes in the following sections correspond to the upper half space.

3. Reflection and Refraction

We consider the propagation of plane wave in $x - z$ plane which makes an angle I with the normal to the boundary. For an incident longitudinal wave $c = m_1/\sin I$ and for an incident transverse wave, $c = m_3/\sin I$. For incidence of longitudinal wave and transverse wave, we get three reflected waves in lower half-space and three refracted waves in upper half-space. The complete geometry showing these reflected and refracted waves is shown in Figure 1.

The appropriate potentials for lower and upper media are as follows:

For M_1:

$$\phi = B_0 \exp\{ik(ct - x + m_1 z)\} + B_1 \exp\{ik(ct - x - m_1 z)\} + B_2 \exp\{ik(ct - x - m_2 z)\}, \quad (15)$$

$$T = (\frac{1}{\bar{\gamma}_0})[a_1 B_0 \exp\{ik(ct - x + m_1 z)\} + a_1 B_1 \exp\{ik(ct - x - m_1 z)\} + a_2 B_2 \exp\{ik(ct - x - m_2 z)\}], \quad (16)$$

$$\psi = B_0 \exp\{ik(ct - x + m_3 z)\} + B_3 \exp\{ik(ct - x - m_3 z)\}. \quad (17)$$

For M_2:

$$\phi' = B_1' \exp\{ik(ct - x + m_1' z)\} + B_2' \exp\{ik(ct - x + m_2' z)\}, \quad (18)$$

$$T' = (\frac{1}{\bar{\gamma}_0'})[a_1' B_1' \exp\{ik\,(ct - x + m_1' z\,)\} + a_2' B_2' \exp\{\,ik\,(ct - x + m_2' z)\}], \quad (19)$$

$$\psi' = B_3' \exp\{ik(ct - x + m_3' z)\}, \quad (20)$$

where $B_i (i = 0, 1, 2, 3)$are arbitrary constants, $a_{1,2} = k^2\{(c^2 - v_1^2) - m_{1,2}^2 v_1^2$, $\bar{\gamma}_0 = \bar{\beta}(1 + i\omega\tau_1)$, and similar expressions with primes correspond to medium M_2

4. Boundary Conditions

For two dimensional motion in x-z plane the appropriate boundary conditions at the interface z =0 of two half spaces are the continuity of the normal force stress, tangential force stress, tangential displacement component, normal displacement component, normal heat flux component and temperature, i.e.

$$t_{zz} = t'_{zz},\ t_{zx} = t'_{zx},\ u_1 = u'_1,\ u_3 = u'_3,\ \mathrm{K}^* \frac{\partial T}{\partial z} = \mathrm{K}^{*\prime} \frac{\partial T'}{\partial Z},\ T = T'. \qquad (21)$$

Making use of the potentials given by the equations (15) to (20) in boundary conditions (21), after using the equations (1) and (14), we get a system of six non homogeneous equations as

$$\sum_{j=1}^{6} a_{ij} z_j = d_i \qquad (i = 1, 2, ..., 6), \qquad (22)$$

where

$$a_{11} = -[(\mathrm{K} + \tfrac{4}{3}\eta i\omega) m_1^2 + (\mathrm{K} - \tfrac{2}{3}\eta i\omega) + \rho \tfrac{a_1}{k^2}]$$
$$a_{12} = -[(\mathrm{K} + \tfrac{4}{3}\eta i\omega) m_2^2 + (\mathrm{K} - \tfrac{2}{3}\eta i\omega) + \rho \tfrac{a_2}{k^2}], \quad a_{13} = 2\eta,\ i\omega m_3$$
$$a_{14} = [(\mathrm{K}' + \tfrac{4}{3}\eta' i\omega) m'^2_1 + (\mathrm{K}' - \tfrac{2}{3}\eta' i\omega) + \rho' \tfrac{a'_1}{k^2}]$$
$$a_{15} = [(\mathrm{K}' + \tfrac{4}{3}\eta' i\omega) m'^2_2 + (\mathrm{K}' - \tfrac{2}{3}\eta' i\omega) + \rho' \tfrac{a'_2}{k^2}], \quad a_{16} = 2\eta' i\omega m'_3,$$

$a_{21} = 2m_1\eta, a_{22} = 2m_2\eta, a_{23} = -(1 - m_3^2)\,\eta, a_{24} = 2m'_1\eta', a_{25} = 2m'_2\eta',$
$a_{26} = (1 - m'^2_3)\eta',$
$a_{31} = \bar{\gamma}'_0 a_1 m_1, a_{32} = \bar{\gamma}'_0 a_2 m_2,\ a_{33} = 0,\ a_{34} = \bar{\gamma}_0 a'_1 m'_1 \frac{\mathrm{K}^{*\prime}}{\mathrm{K}^*},\ a_{35} = \bar{\gamma}_0 a'_2 m'_2 \frac{\mathrm{K}^{*\prime}}{\mathrm{K}^*},$
$a_{36} = 0,$
$a_{41} = 1,\ a_{42} = 1,\ a_{43} = m_3,\ a_{44} = -1,\ a_{45} = -1,\ a_{46} = m'_3,$
$a_{51} = m_1,\ a_{52} = m_2,\ a_{53} = -1,\ a_{54} = m'_1,\ a_{55} = m'_2,\ a_{56} = 1,$
$a_{61} = \bar{\gamma}'_0 a_1,\ a_{62} = \bar{\gamma}'_0 a_2,\ a_{63} = 0,\ a_{64} = -\bar{\gamma}_0 a'_1,\ a_{65} = -\bar{\gamma}_0 a'_2,\ a_{66} = 0.$

(a) For incident longitudinal wave:

$$d_1 = -a_{11},\ d_2 = a_{21},\ d_3 = a_{31},\ d_4 = -a_{41},\ d_5 = a_{51},\ d_6 = -a_{61}. \qquad (23)$$

(b) For incident transverse wave:

$$d_1 = a_{13},\ d_2 = -a_{23},\ d_3 = a_{33},\ d_4 = a_{43},\ d_5 = -a_{53},\ d_6 = a_{63}, \qquad (24)$$

$$z_1 = \frac{B_1}{B_0},\ z_2 = \frac{B_2}{B_0},\quad z_3 = \frac{B_3}{B_0}$$

are the amplitude ratios for various reflected waves, and

$$z_4 = \frac{B'_1}{B_0},\quad z_5 = \frac{B'_2}{B_0}, z_6 = \frac{B'_3}{B_0}$$

are the amplitude ratios for various refracted waves.

It may be noted here that if we remove the upper liquid half space, the equation (22) reduces to that as obtained by Singh in [13].

5. Numerical Results and Discussion

We now consider a numerical example to explain the analytical procedure presented earlier for L-S theory only. The complex absolute values of amplitude ratios are computed for the angle of incidence varying from 0^o to 90^o. Numerical computations of these amplitude ratios are also made for the particular cases. Following Fehler [14], the physical constants used for medium M_1 are

$$\rho = 1.01\, gm/cm^3, \quad \mathrm{K} = 0.0119 \times 10^{11} dyne/cm^2, \quad \varepsilon = 0.053,$$

$$\eta = 0.0014\, gm/cm\, s \quad , \quad K^* = 0.0048 \quad Cal/cms^\circ C, \quad \omega = 5\, s^{-1},$$

$$C^* = 0.206\, cal/gm\, C^\circ, \quad \tau_1 = 0.05s.$$

Following physical constants are considered for medium M_2

$$\rho' = 1.64\, gm/cm^3, \quad \mathrm{K}' = 0.0112 \times 10^{11} dyne/cm^2, \quad \varepsilon' = 0.049,$$

$$\eta' = 0.0012\, gm/cm\, s, \quad K^{*\prime} = 0.0045, \quad Cal/cms^\circ C,$$

$$C^{*\prime} = 0.195\, cal/gm\, C^\circ, \quad \tau_1\prime = 0.06s.$$

Nayfeh and Nasser [15] took $\tau_0 = 3K^*/\rho C^* \alpha^2$. The relaxation time $\tau_1$is considered to be of same order as that of $\tau_{0.}$ The thermal relaxation times with primes may be taken of same order as that of $\tau_{0.}$ The system of equations (7) is solved for amplitude ratios by using a computer program of Gauss elimination method for different angles of incidence of the incident longitudinal wave and incident transverse wave. The amplitude ratios for reflected longitudinal wave, reflected thermal wave, reflected transverse waves in medium $M_1$and corresponding refracted waves in medium M_2 are shown graphically in Figures 2 to13.

5.1. Incident Longitudinal Wave

The variations of the complex absolute values of the amplitude ratios of various reflected and refracted waves with angle of incidence varying from $0^o$to 90^o of incident longitudinal wave have been shown in Figs. 2 to 7.

Figure 2 shows the comparison of the variations of the amplitude ratios of reflected longitudinal wave with those without thermal and viscosity effects. The maximum effects of viscosity and thermal parameters are observed near grazing incidence.

Figure 3 depicts variations of amplitude ratios for reflected thermal wave. The amplitude ratios become larger in absence of viscosity. The amplitude ratios for reflected transverse wave become lesser in absence of thermal effects as shown in Figure 4. Thermal as well as viscous effects are observed on all refracted waves and are shown graphically in Figures 5 to 7.

5.2. Incident Transverse Wave

The variations of complex absolute values of amplitude ratios of various reflected and refracted waves with the angle of incidence are shown in Figs. 8 to 13 for incident transverse wave. In absence of viscosity, the incidence of transverse wave is not possible. The only effects we can observe on certain reflected wave is thermal effect. However if we look at the variations in the absence of thermal disturbances, the reflected and refracted thermal waves will disappear in Figures 9 and 12. Thermal effects on other reflected and refracted waves are not much considerable.

6. Conclusions

From above theoretical as well as numerical analysis following points were observed:

1. The presence of viscosity and thermal disturbances in a liquid half space play an important role in reflection and refraction phenomenon.

2. Due to the presence of viscosity, the existence of transverse wave in a thermally conducting medium is possible. The analysis presented in the paper for incidence of transverse wave is not possible in the absence of viscosity.

3. Viscous and thermal effects are observed significantly on all reflected and refracted waves for the incidence of longitudinal waves.

4. Thermal effects on reflected and refracted waves other than thermal waves, for the incidence of transverse waves are not much considerable, but reflected and refracted thermal waves are possible due to presence of thermal disturbances in liquid half spaces.

The model studied in the present paper may be helpful to experimental scientists/seismologists working in various fields such as oil exploration, earthquake estimation, and exploration of mineral ores present in the earth's crust.

References

[1] A.E. Green, K.A. Lindsay, K.A. Thermoelasticity, *J. Elasticity.*, **2** (1972), 1-7.

[2] H.W. Lord, Y. Shulman, A generalised dynamical theory of thermoelasticity, *J. Mech. Phys. Solids.*, **15** (1967), 299-309.

[3] H. Deresiewicz, Effect of boundaries on waves in a thermo-elastic solid: Reflection of plane waves from plane boundary, *J. Mech. Phys. Solids.,* **8** (1960), 164-172.

[4] A.N. Sinha, S.B. Sinha, Reflection of thermoelastic waves at a solid half space with thermal relaxation, *J. Phys. Earth.*, **22** (1974), 237-244.

[5] S.B. Sinha, K.A. Elsibai, Reflection of thermoelastic waves at a solid half-space with two thermal relaxation times, *J. Thermal Stresses.*, **19** (1996), 763-777.

[6] S.B. Sinha, K.A. Elsibai, Reflection and refraction of thermoelastic waves at an interface of two semi-infinite media with two thermal relaxation times, *J. Thermal Stresses.*, **20** (1997), 129-146.

[7] B. Singh, Reflection of plane sound wave from a micropolar generalized thermoelastic solid half space, *Journal of Sound and Vibration*, **235** (2000), 685-696.

[8] B. Singh, Reflection and reflection of plane waves at liquid/thermo-microstreach elastic solid interface, *Int. J. Eng. Sci.*, **39** (2001), 583-598.

[9] J.N. Sharma, V. Kumar, D. Chand, Reflection of generalized thermoelastic waves from the boundary of a half-space, *J. Thermal Stresses*, **26** (2003), 925-942.

[10] M.I.A. Othman, Y. Song, Reflection of plane waves from an elastic solid half-space under hydrostatic initial stress without energy dissipation, *International Journal Solids and Structures*, **44** (2007), 5651-5664.

[11] B. Singh, Effect of hydrostatic initial stresses on waves in a thermoelastic solid half-space, *Applied Mathematics and Computation*, **198** (2008), 494-505.

[12] B. Singh, Reflection of plane waves at the free surface of a monoclinic thermoelastic solid half-space, *European Journal of Mechanics, A-Solids*, **29** (2010), 911-916.

[13] B. Singh, Plane waves in a thermally conducting viscous liquid, *SADHANA, I.A. Sci.*, **29** (2004), 27-34.

[14] M. Fehler, Interaction of seismic waves with viscous liquid layer, *Bull. Am. Seismic Soc.*, **72** (1982), 55-72.

[15] A. Nayfeh, S.N. Nasser, Thermoelastic waves in solids with thermal relaxation, *Acta Mech.*, **12** (1971), 53-69.

Figures

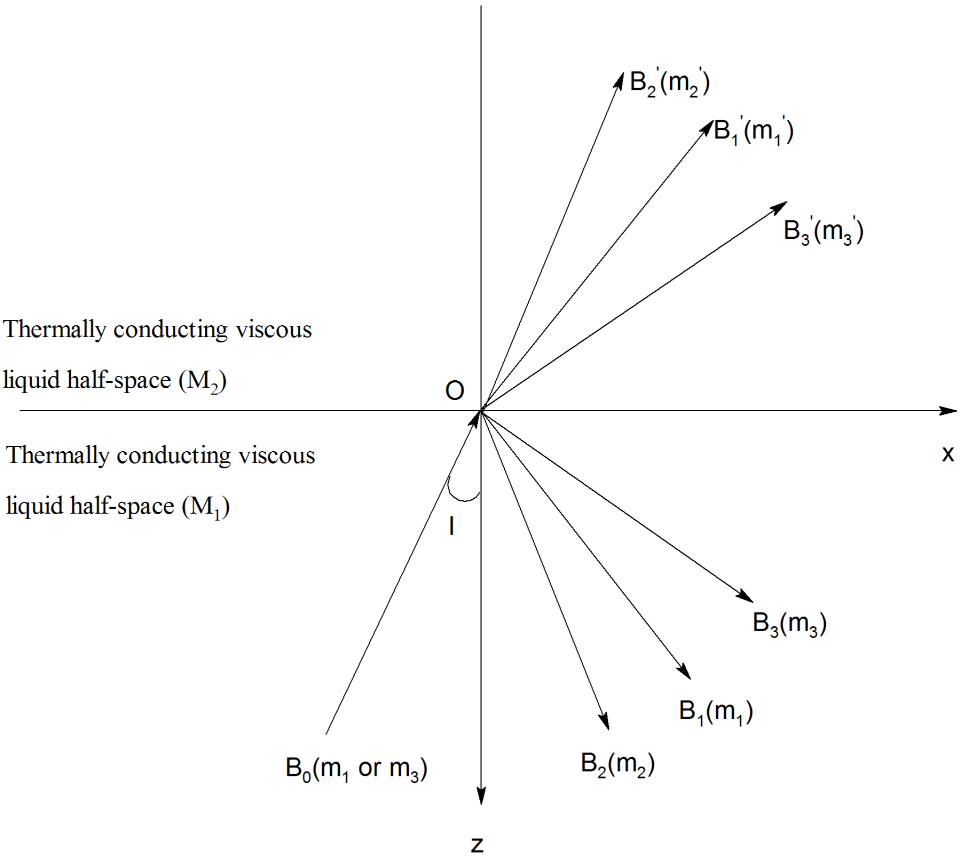

Figure 1: The complete geometry of the problem

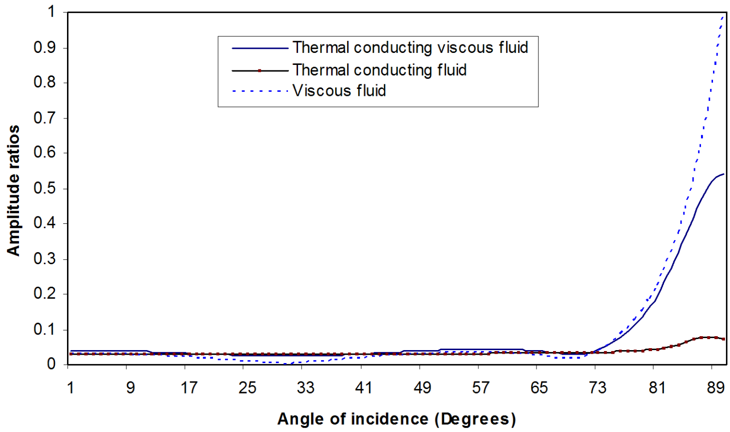

Fig. 2 Variations of amplitude ratios of reflected longitudinal waves versus angle of incidence of longitudinal wave.

Figure 2

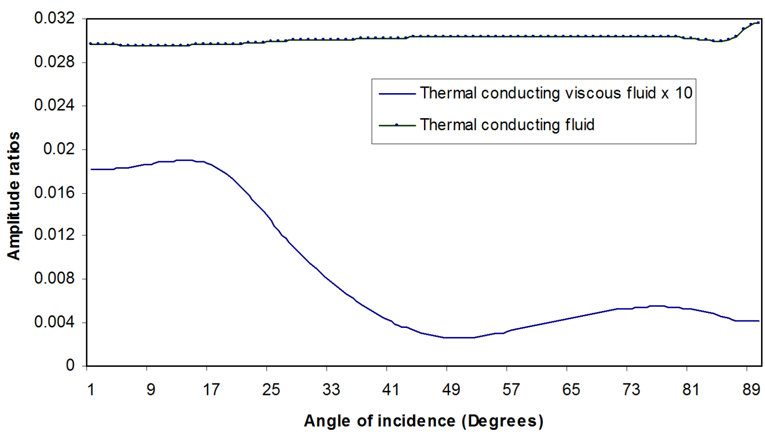

Fig. 3 Variations of amplitude ratios of reflected thermal waves versus angle of incidence of longitudinal wave.

Figure 3

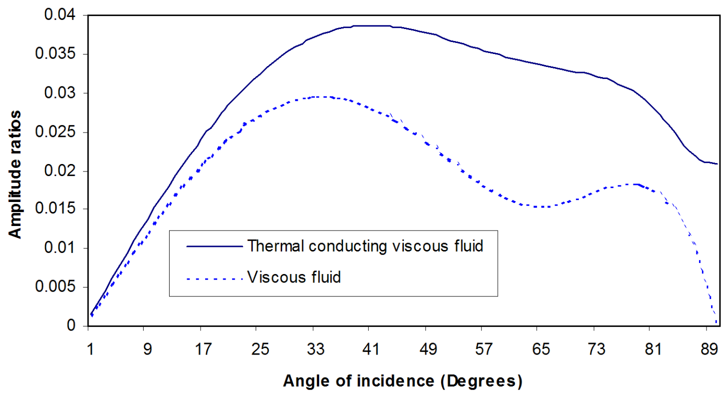

Fig. 4 Variations of amplitude ratios of reflected transverse waves versus angle of incidence of longitudinal wave.

Figure 4

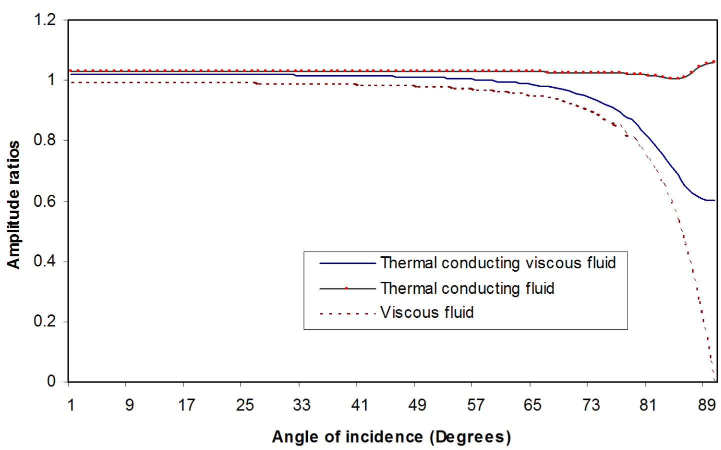

Fig. 5 Variations of amplitude ratios of refracted longitudinal waves versus angle of incidence of longitudinal wave.

Figure 5

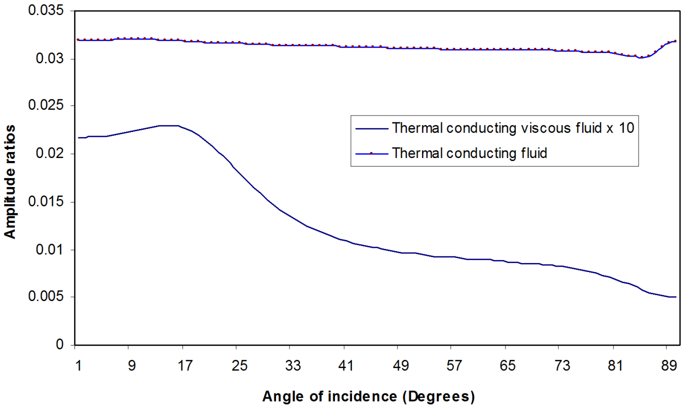

Fig. 6 Variations of amplitude ratios of refracted thermal waves versus angle of incidence of longitudinal wave.

Figure 6

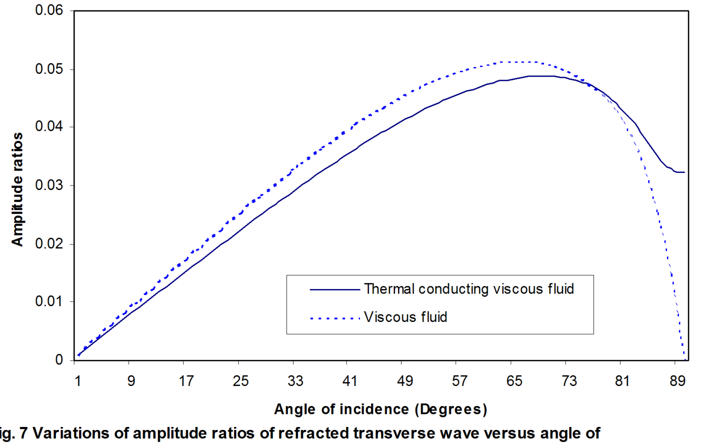

Fig. 7 Variations of amplitude ratios of refracted transverse wave versus angle of incidence of longitudinal wave.

Figure 7

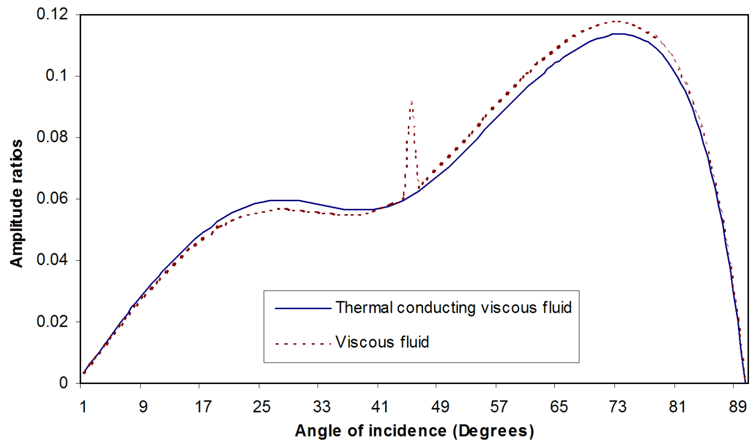

Fig 8 Variations of amplitude ratios of reflected longitudinal waves versus angle of incidence of transverse wave.

Figure 8

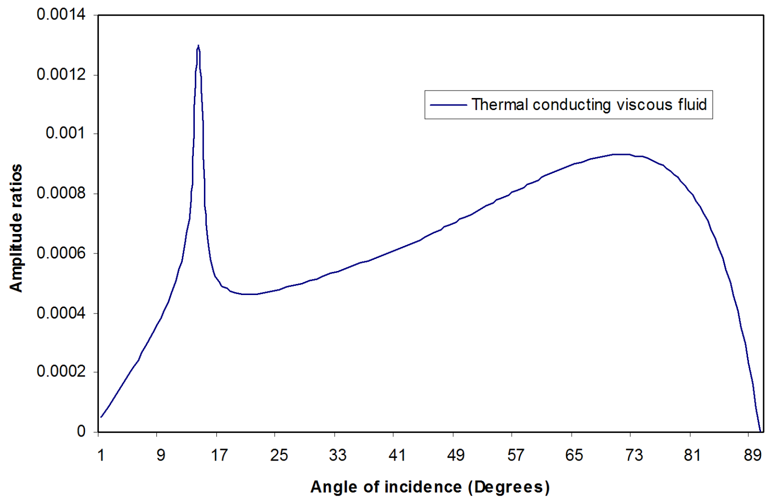

Fig. 9 Variations of amplitude ratios of reflected thermal wave versus angle of incidence of transverse wave.

Figure 9

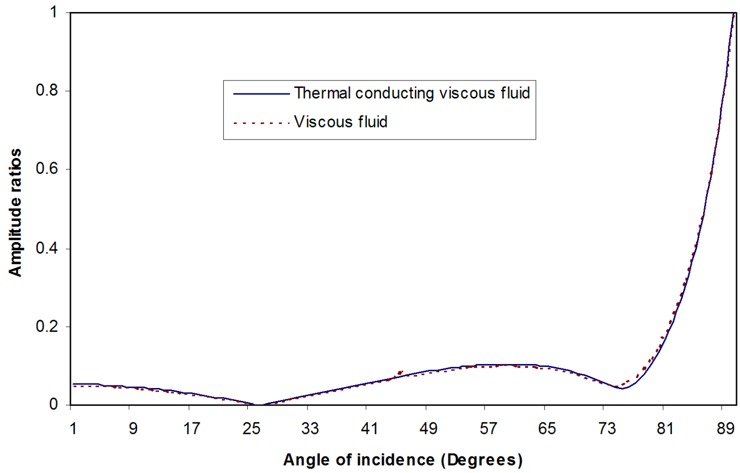

Fig. 10 Variations of amplitude ratios of reflected transverse waves versus angle of incidence of transverse wave.

Figure 10

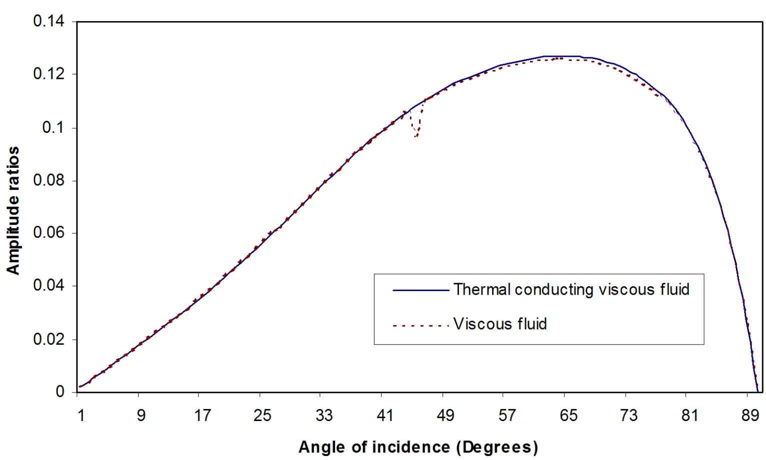

Fig. 11 Variations of amplitude ratios of refracted longitudinal waves versus angle of incidence of transverse wave.

Figure 11

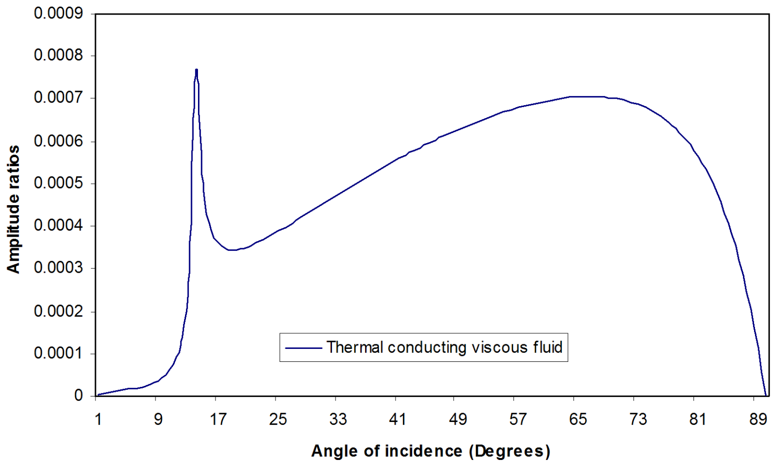

Fig. 12 Variations of amplitude ratios of refracted thermal wave versus angle of incidence of transverse wave.

Figure 12

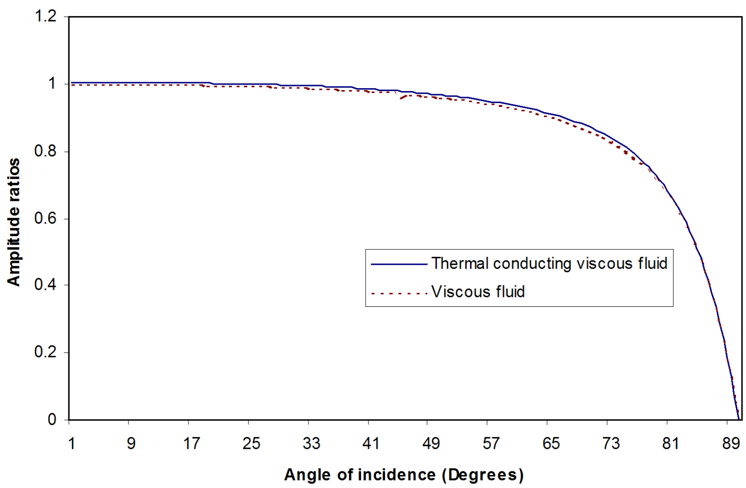

Fig. 13 Variations of amplitude ratios of refracted transverse waves versus angle of incidence of transverse wave.

Figure 13