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QUASILINEARIZATION OF DYNAMIC EQUATIONS ON TIME SCALES INVOLVING THE SUM OF THREE FUNCTIONS

Peiguang Wang[1 §], Yonghong Wu[3]
[1]College of Electronic and Information Engineering
Hebei University
Baoding, 071002, P.R. CHINA
[2]Department of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987, Perth, WA6845, AUSTRALIA

Abstract: In this paper, we present and discuss a method of quasilinearization, coupled with the method of upper and lower solutions for the solutions of a class of two-point boundary value problem of dynamic equations on time scales concerning the sum of three functions. A monotone iterative scheme whose elements converge rapidly to the unique solution of the problem is established, and the convergence is shown to be of order $k + 1$ ($k \geq 1$).

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1. Introduction

The study of dynamic systems on time scales has attracted intensive research worldwide as the study on time scales unifies the study of both discrete and continuous processes. The pioneering works in this field include those by Agarwal and Bohner [1], Kaymakçalan et al [2], Erbe and Hilger [3]. An interesting and robust method for constructing monotone approximate sequences that converge quadratically to solutions of nonlinear problems is quasilinearization. In some cases, it provides a constructive procedure for the solutions of nonlinear prob-

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§Correspondence author

lems involving convex/concave functions [9]-[11]. Now, the method has been applied and extended to dynamic systems on time scales to make it applicable to a large class of problems [7], [8]. Recently, Akin [3] discussed the method of upper and lower solutions to boundary value problem(BVP) on time scales. And, Eloe [4] developed a quasilinearization method for a BVP, concerning convex function, to obtain a sequence of approximate solutions that converge to the unique solution of the given system and the convergence is quadratic.

From the practical point of view, the natural question is whether it is possible to have a higher order convergence, which is useful and interesting, under a less restrictive condition on the forcing function. The answer is affirmative. Therefore, in this paper, we revisit the BVP given in [4] but extend the forcing function to the sum of three functions as defined in (2.1), and attempt to construct sequences not only converging uniformly to the unique solution of the BVP, but also possessing convergence rates higher than quadratic. Moreover, for the sake of self-containment, we should start at that the convergence is quadratic. Consequently, the paper is organized as follows. In Section 2, we introduce the definition of upper and lower solutions of the BVP, and their related fundamental properties. In Section 3, sufficient conditions for the method of quasilinearization are given for the BVP, and an example is added to illustrate the results obtained.

2. Preliminaries

Consider the boundary value problem (BVP)

$$\begin{aligned} &x^{\Delta^2}(t) = H(t, x^\sigma(t)) = f(t, x^\sigma(t)) + g(t, x^\sigma(t)) + h(t, x^\sigma(t)),\ t \in \mathbb{T}^{k^2},\\ &x(a) = A,\ x(b) = B, \end{aligned} \tag{2.1}$$

where $H(t, x(t))$ is continuous on $[a, b] \times R$. Let $\mathbb{T}$ be any time scales and $[a, b]$ be a subset of $\mathbb{T}$ such that $[a, b] = \{t \in \mathbb{T} : a \le t \le b\}$. For the details of basic notions connected to time scales, we refer reader to the reference [1]. Before we proceed further, we need to list the following known results relating to upper and lower solutions, see [4].

Definition 2.1. Let $\alpha(t)$, $\beta(t)$ be such that $\alpha^{\Delta^2}(t)$, $\beta^{\Delta^2}(t)$ are continuous on $\mathbb{T}^{k^2}$. We say α is a lower solution of the BVP (2.1), if

$$\begin{aligned} &\alpha^{\Delta^2}(t) \ge H(t, \alpha^\sigma(t)), \qquad t \in \mathbb{T}^{k^2},\\ &\alpha(a) \le A, \quad \alpha(b) \le B, \end{aligned}$$

and β is an upper solution of the BVP (2.1), if the reversed inequalities hold.

Theorem 2.1. *Assume that $\alpha(t)$, $\beta(t)$ are lower and upper solutions of the BVP (2.1), respectively, such that $\alpha(t) \leq \beta(t)$, $t \in \mathbb{T}$. Further, assume that H is continuous. Then, there exists a solution $x(t)$ of the BVP (2.1) such that $\alpha(t) \leq x(t) \leq \beta(t)$ for $t \in \mathbb{T}$.*

Theorem 2.2. *Assume that $H(t, x)$ is continuous and strictly increasing in x, then the solution of the BVP (2.1) is unique.*

Theorem 2.3. *Assume that:*

(i) $\alpha(t)$, $\beta(t)$ are lower and upper solutions of the BVP (2.1) on $\mathbb{T}$, respectively;

(ii)$H(t, x)$ is continuous and strictly increasing in x.

Then $\alpha(t) \leq \beta(t)$ on $\mathbb{T}$.

3. Main results

First, let Banach space $\mathbb{B}$ be set of continuous functions on $\mathbb{T}$ with a norm $\|.\|$, which denotes $\|x\| = \max_{t\in\mathbb{T}} |x(t)|$. In the following results, note that f_x, f_{xx} are the usual partial derivatives of f over the time scales R.

Theorem 3.1. *For the BVP (2.1), assume that*

(A1) $\alpha_0(t)$, $\beta_0(t)$ are lower and upper solutions of the BVP (2.1) on $\mathbb{T}$, respectively, with $\alpha_0(t) \leq \beta_0(t)$;

(A2) f, f_x, f_{xx}, g, g_x, g_{xx} are continuous on $\mathbb{T}^{k^2} \times R$, with $f_{xx} + \phi_{xx} \geq 0$, $g_{xx} + \psi_{xx} \leq 0$, where ϕ, ϕ_x, ϕ_{xx}, ψ, ψ_x, ψ_{xx} are also continuous on $\mathbb{T}^{k^2} \times R$ with $\phi_{xx} \geq 0$, $\psi_{xx} \leq 0$ and $h(t, x)$ satisfies $h(t, x) - h(t, y) \leq k(x - y)$, for $\alpha_0^\sigma \leq y \leq x \leq \beta_0^\sigma$ and $k > 0$. Further, $F_x(t, x) + G_x(t, y) - \phi_x(t, y) - \psi_x(t, x) + k \geq 0$, where $F = f + \phi$, $G = g + \psi$, for $\alpha_0^\sigma \leq y \leq x \leq \beta_0^\sigma$.

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, converging uniformly and quadratically on $\mathbb{T}$ to the unique solution of the problem (2.1).

Proof. In view of (A2) and the mean value theorem, we have

$$\begin{aligned} f(t, x) &\geq F(t, y) + F_x(t, y)(x - y) - \phi(t, x), \\ g(t, x) &\leq G(t, y) + G_x(t, y)(x - y) - \psi(t, x), \end{aligned}$$

for $t \in \mathbb{T}^{k^2}$ and x, $y \in R$ such that $x \geq y$. Now, we define

$$\begin{aligned} Z_1(t, x^\sigma; \alpha_0, \beta_0) = {} & F(t, \alpha_0^\sigma) + G(t, \alpha_0^\sigma) + h(t, x^\sigma) - \phi(t, \alpha_0^\sigma) - \psi(t, \alpha_0^\sigma) \\ & + [F_x(t, \beta_0^\sigma) + G_x(t, \alpha_0^\sigma) - \phi_x(t, \alpha_0^\sigma) - \psi_x(t, \beta_0^\sigma)](x^\sigma - \alpha_0^\sigma), \end{aligned} \tag{3.1}$$

$$Z_2(t,x^\sigma;\alpha_0,\beta_0)=F(t,\beta_0^\sigma)+G(t,\beta_0^\sigma)+h(t,x^\sigma)-\phi(t,\beta_0^\sigma)-\psi(t,\beta_0^\sigma)\\ +[F_x(t,\beta_0^\sigma)+G_x(t,\alpha_0^\sigma)-\phi_x(t,\alpha_0^\sigma)-\psi_x(t,\beta_0^\sigma)](x^\sigma-\beta_0^\sigma). \quad (3.2)$$

We shall also consider the following BVPs

$$\begin{aligned} &x^{\Delta^2}=Z_1(t,x^\sigma;\alpha_0,\beta_0), \qquad t\in\mathbb{T}^{k^2},\\ &x(a)=A, \quad x(b)=B, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} &x^{\Delta^2}=Z_2(t,x^\sigma;\alpha_0,\beta_0), \qquad t\in\mathbb{T}^{k^2},\\ &x(a)=A, \quad x(b)=B. \end{aligned} \quad (3.4)$$

Applying (A_2), (3.1) and the mean value theorem repeatedly, we get

$$\begin{aligned} &H(t,\beta_0^\sigma)-Z_1(t,\beta_0^\sigma;\alpha_0,\beta_0)\\ &\quad=F(t,\beta_0^\sigma)+G(t,\beta_0^\sigma)+h(t,\beta_0^\sigma)-\phi(t,\beta_0^\sigma)-\psi(t,\beta_0^\sigma)\\ &\quad\;-F(t,\alpha_0^\sigma)-G(t,\alpha_0^\sigma)-h(t,\beta_0^\sigma)+\phi(t,\alpha_0^\sigma)+\psi(t,\alpha_0^\sigma)\\ &\quad\;-[F_x(t,\beta_0^\sigma)+G_x(t,\alpha_0^\sigma)-\phi_x(t,\alpha_0^\sigma)-\psi_x(t,\beta_0^\sigma)](\beta_0^\sigma-\alpha_0^\sigma)\\ &\quad=[F_x(t,c_1)+G_x(t,c_2)-\phi_x(t,c_3)-\psi_x(t,c_4)\\ &\quad\;-F_x(t,\beta_0^\sigma)-G_x(t,\alpha_0^\sigma)+\phi_x(t,\alpha_0^\sigma)+\psi_x(t,\beta_0^\sigma)](\beta_0^\sigma-\alpha_0^\sigma)\\ &\quad=[F_{xx}(t,c_5)(c_1-\beta_0^\sigma)+G_{xx}(t,c_6)(c_2-\alpha_0^\sigma)\\ &\quad\;-\phi_{xx}(t,c_7)(c_3-\alpha_0^\sigma)-\psi_{xx}(t,c_8)(c_4-\beta_0^\sigma)](\beta_0^\sigma-\alpha_0^\sigma)\\ &\quad\le 0, \end{aligned} \quad (3.5)$$

where $\alpha_0^\sigma\le c_1\le c_5\le\beta_0^\sigma$, $\alpha_0^\sigma\le c_6\le c_2\le\beta_0^\sigma$, $\alpha_0^\sigma\le c_7\le c_3\le\beta_0^\sigma$, and $\alpha_0^\sigma\le c_4\le c_8\le\beta_0^\sigma$.

Thus, the assumption (A_1), (3.1) and (3.5) yield

$$\alpha_0^{\Delta^2}\ge H(t,\alpha_0^\sigma)=Z_1(t,\alpha_0^\sigma;\alpha_0,\beta_0),$$

$$\beta_0^{\Delta^2}\le H(t,\beta_0^\sigma)\le Z_1(t,\beta_0^\sigma;\alpha_0,\beta_0).$$

Hence, by Theorem 2.1, there exists a solution $\alpha_1(t)$ of the BVP (3.3) such that

$$\alpha_0(t)\le\alpha_1(t)\le\beta_0(t), \qquad t\in\mathbb{T}.$$

Similarly, we arrive at

$$\begin{aligned} &H(t,\alpha_0^\sigma)-Z_1(t,\alpha_0^\sigma;\alpha_0,\beta_0)\\ &\quad=F(t,\alpha_0^\sigma)+G(t,\alpha_0^\sigma)+h(t,\alpha_0^\sigma)-\phi(t,\alpha_0^\sigma)-\psi(t,\alpha_0^\sigma)\\ &\quad\;-F(t,\beta_0^\sigma)-G(t,\beta_0^\sigma)-h(t,\alpha_0^\sigma)+\phi(t,\beta_0^\sigma)+\psi(t,\beta_0^\sigma)\\ &\quad\;-[F_x(t,\beta_0^\sigma)+G_x(t,\alpha_0^\sigma)-\phi_x(t,\alpha_0^\sigma)-\psi_x(t,\beta_0^\sigma)](\alpha_0^\sigma-\beta_0^\sigma)\\ &\quad=[F_x(t,c_1)+G_x(t,c_2)-\phi_x(t,c_3)-\psi_x(t,c_4)\\ &\quad\;-F_x(t,\beta_0^\sigma)-G_x(t,\alpha_0^\sigma)+\phi_x(t,\alpha_0^\sigma)+\psi_x(t,\beta_0^\sigma)](\alpha_0^\sigma-\beta_0^\sigma)\\ &\quad=[F_{xx}(t,c_5)(c_1-\beta_0^\sigma)+G_{xx}(t,c_6)(c_2-\alpha_0^\sigma)\\ &\quad\;-\phi_{xx}(t,c_7)(c_3-\alpha_0^\sigma)-\psi_{xx}(t,c_8)(c_4-\beta_0^\sigma)](\alpha_0^\sigma-\beta_0^\sigma)\\ &\quad\ge 0, \end{aligned} \quad (3.6)$$

where $\alpha_0^\sigma \le c_1 \le c_5 \le \beta_0^\sigma$, $\alpha_0^\sigma \le c_6 \le c_2 \le \beta_0^\sigma$, $\alpha_0^\sigma \le c_7 \le c_3 \le \beta_0^\sigma$, and $\alpha_0^\sigma \le c_4 \le c_8 \le \beta_0^\sigma$.

Then, in view of (A_1), (3.2) and (3.6), it is clear that

$$\alpha_0^{\Delta^2} \ge H(t,\alpha_0^\sigma) = Z_2(t,\alpha_0^\sigma;\alpha_0,\beta_0),$$

$$\beta_0^{\Delta^2} \le H(t,\beta_0^\sigma) \le Z_2(t,\beta_0^\sigma;\alpha_0,\beta_0).$$

Therefore, as before, there exists a solution $\beta_1(t)$ of the BVP (3.4) such that

$$\alpha_0(t) \le \beta_1(t) \le \beta_0(t), \qquad t \in \mathbb{T}.$$

Now since $\alpha_1^{\Delta^2} = Z_1(t,\alpha_0^\sigma;\alpha_0,\beta_0)$, we get that

$$\begin{aligned}
\alpha_1^{\Delta^2} &= F(t,\alpha_0^\sigma) + G(t,\alpha_0^\sigma) + h(t,\alpha_1^\sigma) - \phi(t,\alpha_0^\sigma) - \psi(t,\alpha_0^\sigma)\\
&\quad + [F_x(t,\beta_0^\sigma) + G_x(t,\alpha_0^\sigma) - \phi_x(t,\alpha_0^\sigma) - \psi_x(t,\beta_0^\sigma)](\alpha_1^\sigma - \alpha_0^\sigma)\\
&= H(t,\alpha_1^\sigma) - F(t,\alpha_1^\sigma) - G(t,\alpha_1^\sigma) - h(t,\alpha_1^\sigma) + \phi(t,\alpha_1^\sigma) + \psi(t,\alpha_1^\sigma)\\
&\quad + F(t,\alpha_0^\sigma) + G(t,\alpha_0^\sigma) + h(t,\alpha_1^\sigma) - \phi(t,\alpha_0^\sigma) - \psi(t,\alpha_0^\sigma)\\
&\quad + [F_x(t,\beta_0^\sigma) + G_x(t,\alpha_0^\sigma) - \phi_x(t,\alpha_0^\sigma) - \psi_x(t,\beta_0^\sigma)](\alpha_1^\sigma - \alpha_0^\sigma)\\
&= H(t,\alpha_1^\sigma) + [-F_x(t,c_1) - G_x(t,c_2) + \phi_x(t,c_3) + \psi_x(t,c_4)\\
&\quad + F_x(t,\beta_0^\sigma) + G_x(t,\alpha_0^\sigma) - \phi_x(t,\alpha_0^\sigma) - \psi_x(t,\beta_0^\sigma)](\alpha_1^\sigma - \alpha_0^\sigma)\\
&= H(t,\alpha_1^\sigma) + [F_{xx}(t,c_5)(\beta_0^\sigma - c_1) - G_{xx}(t,c_6)(c_2 - \alpha_0^\sigma)\\
&\quad + \phi_{xx}(t,c_7)(c_3 - \alpha_0^\sigma) - \psi_{xx}(t,c_8)(\beta_0^\sigma - c_4)](\alpha_1^\sigma - \alpha_0^\sigma)\\
&\ge H(t,\alpha_1^\sigma),
\end{aligned}$$

where $\alpha_0^\sigma \le c_1 \le c_5 \le \beta_0^\sigma$, $\alpha_0^\sigma \le c_6 \le c_2 \le \beta_0^\sigma$, $\alpha_0^\sigma \le c_7 \le c_3 \le \beta_0^\sigma$, and $\alpha_0^\sigma \le c_4 \le c_8 \le \beta_0^\sigma$, because of the mean value theorem, (A_2) and $\alpha_1 \ge \alpha_0$.

Similarly, we obtain

$$\begin{aligned}
\beta_1^{\Delta^2} &= F(t,\beta_0^\sigma) + G(t,\beta_0^\sigma) + h(t,\beta_1^\sigma) - \phi(t,\beta_0^\sigma) - \psi(t,\beta_0^\sigma)\\
&\quad + [F_x(t,\beta_0^\sigma) + G_x(t,\alpha_0^\sigma) - \phi_x(t,\alpha_0^\sigma) - \psi_x(t,\beta_0^\sigma)](\beta_1^\sigma - \beta_0^\sigma)\\
&= H(t,\beta_1^\sigma) + [-F_x(t,c_1) - G_x(t,c_2) + \phi_x(t,c_3) + \psi_x(t,c_4)\\
&\quad + F_x(t,\beta_0^\sigma) + G_x(t,\alpha_0^\sigma) - \phi_x(t,\alpha_0^\sigma) - \psi_x(t,\beta_0^\sigma)](\beta_1^\sigma - \beta_0^\sigma)\\
&= H(t,\beta_1^\sigma) + [F_{xx}(t,c_5)(\beta_0^\sigma - c_1) - G_{xx}(t,c_6)(c_2 - \alpha_0^\sigma)\\
&\quad + \phi_{xx}(t,c_7)(c_3 - \alpha_0^\sigma) - \psi_{xx}(t,c_8)(\beta_0^\sigma - c_4)](\beta_1^\sigma - \beta_0^\sigma)\\
&\le H(t,\beta_1^\sigma),
\end{aligned}$$

and $\beta_1 \le \beta_0$.

We can conclude from the above estimates that α_1 and β_1 are lower and upper solutions, respectively, for the BVP (2.1), and it then follows from Theorem 2.3 that $\alpha_1(t) \le \beta_1(t)$, $t \in \mathbb{T}$. Consequently, these results yield

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \qquad t \in \mathbb{T}.$$

Continuing this process by induction, we have sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t), \qquad t \in \mathbb{T}, \qquad n = 0, 1, 2, \ldots,$$

where for each n, α_{n+1} is a solution of the BVP

$$\begin{aligned} &x^{\Delta^2} = Z_1(t, x^\sigma; \alpha_n, \beta_n), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B, \end{aligned}$$

and β_{n+1} is a solution of the BVP

$$\begin{aligned} &x^{\Delta^2} = Z_2(t, x^\sigma; \alpha_n, \beta_n), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B. \end{aligned}$$

Since $\mathbb{T}$ is compact, it follows that the convergence of each sequence, $\{\alpha_n\}$ or $\{\beta_n\}$, is uniform. We shall show that each sequence $\{\alpha_n\}$, $\{\beta_n\}$ converges to the unique solution of the BVP (2.1). In Erbe and Peterson [6], the Green's function $G(t, s)$, associated with the BVP (2.1), has been constructed by

$$G(t, s) = \begin{cases} (a - t)(b - \sigma(s))/(b - a), & t \le s. \\ (a - \sigma(s))(b - t)/(b - a), & \sigma(s) \le t. \end{cases}$$

And $x(t)$ is a solution of the BVP (2.1), if and only if, $x(t)$ is continuous on $\mathbb{T}$ and

$$x(t) = \int_a^{\rho(b)} G(t, s) H(s, x^\sigma(s)) \Delta s, \qquad t \in \mathbb{T}.$$

Now define

$$\alpha_{n+1}(t) = \int_a^{\rho(b)} G(t, s) Z_1(s, \alpha_{n+1}^\sigma; \alpha_n, \beta_n) \Delta s, \qquad t \in \mathbb{T}.$$

Note that $\{\alpha_n\}$ converges monotonically and uniformly to some function α and

$$Z_1(s, \alpha_{n+1}^\sigma; \alpha_n, \beta_n) \to H(s, \alpha^\sigma),$$

where the convergence is uniform on $\mathbb{T}$.

Then, we have desired result

$$\alpha(t) = \int_a^{\rho(b)} G(t, s) H(s, \alpha^\sigma(s)) \Delta s, \qquad t \in \mathbb{T}.$$

A similar discussion applies to $\{\beta_n\}$. Hence both $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly on $\mathbb{T}$ to the unique solution of the problem (2.1).

Now, we should show that the convergence of each sequence, $\{\alpha_n\}$ or $\{\beta_n\}$, is quadratic. For this purpose, we define $p_n = x - \alpha_n$, $q_n = \beta_n - x$, $t \in \mathbb{T}$, where x denotes the unique solution of the BVP (2.1), and note that $p_n \geq 0$ and $q_n \geq 0$. Applying (A2) and the mean value theorem, we can find c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 such that $\alpha_n^\sigma \leq c_1, c_2, c_3, c_4 \leq x^\sigma$, $c_1 \leq c_5 \leq \beta_n^\sigma$, $\alpha_n^\sigma \leq c_6 \leq c_2$, $\alpha_n^\sigma \leq c_7 \leq c_3$, $c_4 \leq c_8 \leq \beta_n^\sigma$ and

$$\begin{aligned}
p_{n+1}^{\Delta^2} &= x^{\Delta^2} - \alpha_{n+1}^{\Delta^2} \\
&= F(t,x^\sigma) + G(t,x^\sigma) + h(t,x^\sigma) - \phi(t,x^\sigma) - \psi(t,x^\sigma) \\
&\quad - F(t,\alpha_n^\sigma) - G(t,\alpha_n^\sigma) - h(t,\alpha_{n+1}^\sigma) + \phi(t,\alpha_n^\sigma) + \psi(t,\alpha_n^\sigma) \\
&\quad - [F_x(t,\beta_n^\sigma) + G_x(t,\alpha_n^\sigma) - \phi_x(t,\alpha_n^\sigma) - \psi_x(t,\beta_n^\sigma)](\alpha_{n+1}^\sigma - \alpha_n^\sigma) \\
&= [F_x(t,c_1) + G_x(t,c_2) - \phi_x(t,c_3) - \psi_x(t,c_4)](x^\sigma - \alpha^\sigma) \\
&\quad - [F_x(t,\beta_n^\sigma) + G_x(t,\alpha_n^\sigma) - \phi_x(t,\alpha_n^\sigma) - \psi_x(t,\beta_n^\sigma)](x^\sigma - \alpha_n^\sigma) \\
&\quad + [k + F_x(t,\beta_n^\sigma) + G_x(t,\alpha_n^\sigma) - \phi_x(t,\alpha_n^\sigma) - \psi_x(t,\beta_0^\sigma)](x^\sigma - \alpha_{n+1}^\sigma) \\
&= [-F_{xx}(t,c_5)(\beta_n^\sigma - c_1) + G_{xx}(t,c_6)(c_2 - \alpha_n^\sigma) \\
&\qquad\quad - \phi_{xx}(t,c_7)(c_3 - \alpha_n^\sigma) + \psi_{xx}(t,c_8)(\beta_n^\sigma - c_4)](x^\sigma - \alpha_n^\sigma) \\
&\quad + [k + F_x(t,\beta_n^\sigma) + G_x(t,\alpha_n^\sigma) - \phi_x(t,\alpha_n^\sigma) - \psi_x(t,\beta_n^\sigma)](x^\sigma - \alpha_{n+1}^\sigma) \\
&\geq -[(M_1 + M_4)p_n q_n + (M_2 + M_3)p_n^2],
\end{aligned}$$

where $|F_{xx}| \leq M_1$, $|G_{xx}| \leq M_2$, $|\phi_{xx}| \leq M_3$, $|\psi_{xx}| \leq M_4$.

From [6], clearly,

$$p_{n+1}(t) \leq \int_a^{\rho(b)} G(t,s) p_{n+1}^{\Delta^2} \Delta s, \qquad t \in \mathbb{T}.$$

and $G(t,s) \leq 0$. Taking the maximum over $\mathbb{T}$, we obtain

$$\|p_{n+1}\| \leq C_1 \|p_n\| \|q_n\| + C_2 \|p_n\|^2, \qquad t \in \mathbb{T}, \tag{3.7}$$

where C_1 and C_2 provide a bound on $(M_1 + M_4) \int_a^{\rho(b)} G(t,s)\Delta s$ and $(M_2 + M_3) \int_a^{\rho(b)} G(t,s)\Delta s$, respectively.

Similar quadratic convergence result can be drawn for $\{\beta_n\}$, and we have

$$\|q_{n+1}\| \leq C_1 \|q_n\|^2 + C_2 \|p_n\| \|q_n\|, \qquad t \in \mathbb{T}. \tag{3.8}$$

Hence combing (3.7) and (3.8), it is easy to conclude that

$$\|p_{n+1}\| + \|q_{n+1}\| \leq C[\|p_n\| + \|q_n\|]^2,$$

which means

$$\|x - \alpha_{n+1}\| + \|\beta_{n+1} - x\| \leq C[\|x - \alpha_n\| + \|\beta_n - x\|]^2,$$

where C is an appropriate positive constant.

Throughout the following theorem, we use the $f^{(i)}(t,x)$ as the usual ith order partial derivative of f with respect to x.

Theorem 3.2. *For the BVP (2.1), assume that*

(B1) $\alpha_0(t)$, $\beta_0(t)$ are lower and upper solutions of the BVP (2.1) on $\mathbb{T}$, respectively, such that $\alpha_0(t) \le \beta_0(t)$;

(B2) $f^{(i)}(t,x)$, $g^{(i)}(t,x)$, $\phi^{(i)}(t,x)$ and $\psi^{(i)}(t,x)(i = 0,1,2,\cdots,k+1)$ exist and are continuous on $\mathbb{T}^{k^2} \times R$, satisfying $F^{k+1}(t,x) = f^{(k+1)}(t,x) + \phi^{(k+1)}(t,x) \ge 0$ and $G^{(k+1)}(t,x) = g^{(k+1)}(t,x) + \psi^{(k+1)}(t,x) \le 0$.

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, converging uniformly on $\mathbb{T}$ to the unique solution of the problem (2.1). Moreover, the convergence is order $k+1 (k > 1)$.

Proof. Define Z_1^* and Z_2^*, for $k > 1$,

$$\begin{aligned} Z_1^*(t,x^\sigma;\alpha_0,\beta_0) = &\sum_{i=0}^{k-1} \tfrac{1}{i!} F^{(i)}(t,\alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i + \sum_{i=0}^{k-1} \tfrac{1}{i!} G^{(i)}(t,\alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i \\ &- \sum_{i=0}^{k-1} \tfrac{1}{i!} \phi^{(i)}(t,\alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i - \sum_{i=0}^{k-1} \tfrac{1}{i!} \psi^{(i)}(t,\alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i + h(t,x^\sigma) \\ &+ \tfrac{1}{k!}[F^{(k)}(t,\beta_0^\sigma) + G^{(k)}(t,\alpha_0^\sigma) - \phi^{(k)}(t,\alpha_0^\sigma) - \psi^{(k)}(t,\beta_0^\sigma)](x^\sigma - \alpha_0^\sigma)^k, \end{aligned}$$

$$\begin{aligned} Z_2^*(t,x^\sigma;\alpha_0,\beta_0) = &\sum_{i=0}^{k-1} \tfrac{1}{i!} F^{(i)}(t,\beta_0^\sigma)(x^\sigma - \beta_0^\sigma)^i + \sum_{i=0}^{k-1} \tfrac{1}{i!} G^{(i)}(t,\beta_0^\sigma)(x^\sigma - \beta_0^\sigma)^i \\ &- \sum_{i=0}^{k-1} \tfrac{1}{i!} \phi^{(i)}(t,\beta_0^\sigma)(x^\sigma - \beta_0^\sigma)^i - \sum_{i=0}^{k-1} \tfrac{1}{i!} \psi^{(i)}(t,\beta_0^\sigma)(x^\sigma - \beta_0^\sigma)^i + h(t,x^\sigma) \\ &+ \begin{cases} \tfrac{1}{k!}[F^{(k)}(t,\beta_0^\sigma) + G^{(k)}(t,\alpha_0^\sigma) \\ \quad - \phi^{(k)}(t,\alpha_0^\sigma) - \psi^{(k)}(t,\beta_0^\sigma)](x^\sigma - \beta_0^\sigma)^k, & k \text{ is odd.} \\ \tfrac{1}{k!}[F^{(k)}(t,\alpha_0^\sigma) + G^{(k)}(t,\beta_0^\sigma) \\ \quad - \phi^{(k)}(t,\beta_0^\sigma) - \psi^{(k)}(t,\alpha_0^\sigma)](x^\sigma - \beta_0^\sigma)^k, & k \text{ is even.} \end{cases} \end{aligned}$$

We shall consider the following BVPs

$$\begin{aligned} &x^{\Delta^2} = Z_1^*(t,x^\sigma;\alpha_0,\beta_0), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &x^{\Delta^2} = Z_2^*(t,x^\sigma;\alpha_0,\beta_0), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B. \end{aligned} \tag{3.10}$$

using the same procedure as that used in the previous theorem, we can find that there exist solutions, $\alpha_1(t)$ of the BVP (3.9) and $\beta_1(t)$ of the BVP (3.10), such that $\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t)$, $t \in \mathbb{T}$. Again, using the procedure successively, we can obtain a monotone sequence satisfying

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t), \qquad t \in \mathbb{T},$$

where α_n, β_n are, respectively, a solution of the BVP

$$\begin{aligned} &x^{\Delta^2} = Z_1^*(t, x^\sigma; \alpha_{n-1}, \beta_{n-1}), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B, \end{aligned}$$

and a solution of the BVP

$$\begin{aligned} &x^{\Delta^2} = Z_2^*(t, x^\sigma; \alpha_{n-1}, \beta_{n-1}), \qquad t \in \mathbb{T}^{k^2}, \\ &x(a) = A, \quad x(b) = B. \end{aligned}$$

We can now claim that $\{\alpha_n\}$, $\{\beta_n\}$ converge on $\mathbb{T}$ to the unique solution, $x(t)$ of the BVP (2.1). The details are omitted to avoid repetition. Finally, to show the convergence of order $k+1$, we set $p_n = x - \alpha_n$, $q_n = \beta_n - x$, $t \in \mathbb{T}$. By the generalized mean value theorem and (B2), there exists $\alpha_n^\sigma \leq c_1, c_2, c_3, c_4 \leq \alpha_{n+1}^\sigma$, such that

$$\begin{aligned} p_{n+1}^{\Delta^2} &= x^{\Delta^2} - \alpha_{n+1}^{\Delta^2} \\ &= F(t,x^\sigma) + G(t,x^\sigma) + h(t,x^\sigma) - \phi(t,x^\sigma) - \psi(t,x^\sigma) \\ &\quad - \sum_{i=0}^{k-1} \tfrac{1}{i!} F^{(i)}(t,\alpha_n^\sigma)(\alpha_{n+1}^\sigma - \alpha_n^\sigma)^i - \sum_{i=0}^{k-1} \tfrac{1}{i!} G^{(i)}(t,\alpha_n^\sigma)(\alpha_{n+1}^\sigma - \alpha_n^\sigma)^i \\ &\quad + \sum_{i=0}^{k-1} \tfrac{1}{i!} \phi^{(i)}(t,\alpha_n^\sigma)(\alpha_{n+1}^\sigma - \alpha_n^\sigma)^i + \sum_{i=0}^{k-1} \tfrac{1}{i!} \psi^{(i)}(t,\alpha_n^\sigma)(\alpha_{n+1}^\sigma - \alpha_n^\sigma)^i - h(t,\alpha_{n+1}^\sigma) \\ &\quad - \tfrac{1}{k!}[F^{(k)}(t,\beta_n^\sigma) + G^{(k)}(t,\alpha_n^\sigma) - \phi^{(k)}(t,\alpha_n^\sigma) - \psi^{(k)}(t,\beta_n^\sigma)](\alpha_{n+1}^\sigma - \alpha_n^\sigma)^k \\ &= F(t,x^\sigma) + G(t,x^\sigma) + h(t,x^\sigma) - \phi(t,x^\sigma) - \psi(t,x^\sigma) \\ &\quad - F(t,\alpha_{n+1}^\sigma) - G(t,\alpha_{n+1}^\sigma) - h(t,\alpha_{n+1}^\sigma) + \phi(t,\alpha_{n+1}^\sigma) + \psi(t,\alpha_{n+1}^\sigma) \\ &\quad + \tfrac{1}{k!}[F^{(k)}(t,c_1) + G^{(k)}(t,c_2) - \phi^{(k)}(t,c_3) - \psi^{(k)}(t,c_4) \\ &\quad - F^{(k)}(t,\beta_n^\sigma) - G^{(k)}(t,\alpha_n^\sigma) + \phi^{(k)}(t,\alpha_n^\sigma) + \psi^{(k)}(t,\beta_n^\sigma)](\alpha_{n+1}^\sigma - \alpha_n^\sigma)^k, \end{aligned}$$

Using the monotone nature of H and $\alpha_{n+1} \leq x$, we can conclude

$$\begin{aligned} p_{n+1}^{\Delta^2} &\geq \tfrac{1}{k!}[-F^{(k+1)}(t,c_5)(\beta_n^\sigma - c_1) + G^{(k+1)}(t,c_6)(c_2 - \alpha_n^\sigma) \\ &\qquad - \phi^{(k+1)}(t,c_7)(c_3 - \alpha_n^\sigma) + \psi^{(k+1)}(t,c_8)(\beta_n^\sigma - c_4)](\alpha_{n+1}^\sigma - \alpha_n^\sigma)^k \\ &\geq -[(M_1 + M_4)(p_n)^k q_n + (M_2 + M_3)(p_n)^{k+1}], \end{aligned}$$

where $c_1 \leq c_5 \leq \beta_n^\sigma$, $\alpha_n^\sigma \leq c_6 \leq c_2$, $\alpha_n^\sigma \leq c_7 \leq c_3$, $c_4 \leq c_8 \leq \beta_n^\sigma$ and $\frac{1}{k!}|F^{(k+1)}| \leq M_1$, $\frac{1}{k!}|G^{(k+1)}| \leq M_2$, $\frac{1}{k!}|\phi^{(k+1)}| \leq M_3$, $\frac{1}{k!}|\psi^{(k+1)}| \leq M_4$.

In view of [6], clearly,

$$\|p_{n+1}\| \leq \int_a^{\rho(b)} G(t,s)[(M_1+M_4)(p_n)^k q_n + (M_2+M_3)(p_n)^{k+1}]\Delta s, \qquad t \in \mathbb{T},$$

which, on taking maximum over $t \in \mathbb{T}$, gives

$$\|p_{n+1}\| \leq L_1\|p_n\|^k\|q_n\| + L_2\|p_n\|^{k+1}, \qquad t \in \mathbb{T}, \tag{3.11}$$

where L_1 and L_2 provide a bound on $(M_1+M_4)\int_a^{\rho(b)} G(t,s)\Delta s$ and $(M_2+M_3)\int_a^{\rho(b)} G(t,s)\Delta s$, respectively.

Proceeding as far the discussion of p_{n+1}, we have

$$\|q_{n+1}\| \leq L_1\|q_n\|^{k+1} + L_2\|p_n\|\|q_n\|^k, \qquad t \in \mathbb{T}. \tag{3.12}$$

Hence from (3.11) and (3.12), it can be concluded that

$$\|p_{n+1}\| + \|q_{n+1}\| \leq L[\|p_n\| + \|q_n\|]^{k+1},$$

that is

$$\|x-\alpha_{n+1}\| + \|\beta_{n+1}-x\| \leq L[\|x-\alpha_n\| + \|\beta_n - x\|]^{k+1},$$

where L is an appropriate positive constant.

Remark 3.1. The monotone character on

$$f^{(k+1)}(t,x)+\phi^{(k+1)}(t,x), \quad g^{(k+1)}(t,x)+\psi^{(k+1)}(t,x),$$
$$\phi^{(k+1)}(t,x), \quad \psi^{(k+1)}(t,x)$$

can be weakened. Setting

$$-M_1 \leq F^{(k+1)}(t,x) \leq N_1, \quad -M_2 \leq G^{(k+1)}(t,x) \leq N_2,$$
$$-M_3 \leq \phi^{(k+1)}(t,x) \leq N_3, \quad -M_4 \leq \psi^{(k+1)}(t,x) \leq N_4,$$

$t \in \mathbb{T}^k \times R$, for some constants M_i, $N_i > 0$, i=1, 2, 3, 4. We also arrive at Theorem 3.2.

Remark 3.2. If $g(t,x) = h(t,x) = 0$, then we have quadratic convergence as in [4]. On the other hand, if $f(t,x) = g(t,x) = 0$, then we get linear convergence.

Example 3.1. Let $\mathbb{T} = R$, and consider the following BVP

$$\begin{aligned} &x^{\Delta^2}(t) = (t+2)\sin x^\sigma - \tfrac{1}{100}x^\sigma e^t, \qquad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \\ &x(-\tfrac{\pi}{2}) = 0, \quad x(\tfrac{\pi}{2}) = 1, \end{aligned}$$

where $f(t,x) = (t+2)\sin x$, $h(t,x) = -\frac{1}{100}xe^t$, $\phi(t,x) = 9x^2$, $k = \frac{1}{100}$ and $F(t,x) = (t+2)\sin x + 9x^2$. Observe that $\alpha_0 = 0$, $\beta_0 = 1$ are lower and upper solutions for the BVP, respectively. All assumptions in Theorem 3.1 and Theorem 3.2 are satisfied. Hence, Theorem 3.1 and Theorem 3.2 are applicable to the BVP.

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