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# PRIME GAP DIAGONALS AND GOLDBACH'S CONJECTURE

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**Abstract:** A display of the doubly infinite set of prime doublets (any two odd primes, equal or unequal) is illustrated. An argument for Goldbach's conjecture is introduced that implies that Goldbach's conjecture is valid, providing there are no missing evens in the display. Properties of prime gap diagonals and that of a fundamental symmetry of the display are argued that described that imply location of missing evens. The notion that an even is incompatible with its lead prime is introduced. Evens are partitioned according to the lead prime of their respective rows and a component of the hypothesis is established for the respective two sets. A translation group relevant to the prime-gap cts. A translation group is discussed relevant to the prime-gap diagonal. Numerical examples are included. As the prime number sequence has been shown to be quasi chaotic, a mathematical proof of Goldbach's conjecture does not exist. One may, however, construct a good argument for its *validity*. Namely, consistent within the limits set by this property.

**AMS Subject Classification:** 26A33
**Key Words:** prime doublets, missing evens, prime-gap diagonals, translation group, incompatible even

## 1. Introduction

C. Goldbach's conjecture (as modified by L. Euler, 1742) states that any even integer $\geq 6$, may be written as the sum of two primes. A closely allied theorem states that if Goldbach's conjecture is valid, then for every positive integer, $n$, two primes $q$ and $r$ exist, such that,

$$\phi(q) + \phi(r) = 2n. \quad (1)$$

(The Euler function $\phi(m)$ equals the number of positive integers $\leq m$, relatively

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prime to m). Erdös (1945) conjectured that this statement holds for $q$ and $r$ not prime. A related theorem proved by I. M. Vinogradov (1937), states that any 'sufficiently large' odd integer may be written as the sum of three primes, was likewise conjectured by Goldbach (proof of Goldbach's first conjecture implies Goldbach's second conjecture). Another form of this conjecture is presented in the present work in terms of prime differences. A number of theorems and statements related to this conjecture may be found in the collection compiled by M. Yuan [3] and an extensive work by J. M. Deshouillers et al. (1988). In a recent work, it was established that the prime sequence is quasi-chaotic (Liboff and Wong,1998). Namely, a histogram of the intervals of nearest-neighbor primes was noted to very roughly follow the Wigner distribution [6], [7]. Here it was discovered (numerically) that the most frequent interval between successive primes has the value six. In another study, a 'quasi-entropic' property of the prime sequence was established by Liboff and Wong(1998). In the present work we are concerned with an argument for Goldbach's conjecture. The concept of a translation group (Liboff,2004) is discussed with regard to As it has been established that the prime number sequence is quasi chaotic (Liboff and Wong, 1998). Here it was discovered that a histogram the most frequent interval was noted to very roughly follow the Wigner distribution. (Haake, 2000; Blumel and Reinhart, 1977).

**Terminology.** An argument related to a theorem results in the theorem being *valid*. A mathematical proof related to a theorem results in the theorem being *proved*.

## 2. Enumeration of Prime Doublets

**Definition.** A prime doublet is any two equal or unequal odd primes, $(p_i, p_j)$. There are three classes of prime doublets: (a) Consecutive prime pairs, e.g., $(23, 29), (31, 37) \cdots$, (b) Pairs that are not consecutive e.g., $(349, 359)$, $(11, 19) \cdots$, (c) 'Diagonal' prime doublets, $(3, 3), (5, 5) \cdots$.

The display of evens and related prime doublets includes rows of denumerable infinite extent. If $p_k$ is the lead prime of the $(k)$-$row$, then $p_{k+1}$ is the lead prime of the $(k+1)$-$row$, where $p_k$ is the $k^{th}$ odd prime. The $q^{th}$ even in the $k - row$ is equal to the sum of the lead prime and the $q^{th}$ prime of the row. Every prime doublet in a row contains the first prime of the row. A 'diagonal' doublet contains two equal primes and a 'diagonal even' is the sum of these primes. Every lead prime in a row corresponds to a diagonal even. Group I of prime doublets are shown below together with related summed evens in adjoining parentheses. Group II of larger prime doublets have their respective evens in parentheses beneath primes. When speaking of 'an even in a row,' which is clear for group I, for group II, denotes the even beneath the related prime.

Group I

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3(6) | 5(8) | 7(10) | 11(14) | 13(16) | 17(20) | 19(22) | 23(26) | 29(32)··· |
| 5(10) | 7(12) | 11(16) | 13(18) | 17(22) | 19(24) | 23(28) | 29(34) | 31(36)··· |
| 7(14) | 11(18) | 13(20) | 17(24) | 19(26) | 23(30) | 29(36) | 31(38) | 37(42)··· |
| 11(22) | 13(24) | 17(28) | 19(30) | 23(34) | 29(40) | 31(42) | 37(48) | 41(52)··· |
| 13(26) | 17(30) | 19(32) | 23(36) | 29(42) | 31(44) | 37(50) | 41(54) | 43(56)··· |
| 17(34) | 19(36) | 23(40) | 29(46) | 31(48) | 37(54) | 41(58) | 43(60) | 47(64)··· |
| 19(38) | 23(42) | 29(48) | 31(50) | 37(56) | 41(60) | 43(62) | 47(64) | 51(70)··· |
| 23(46) | 29(52) | 31(54) | 37(60) | 41(64) | 43(66) | 47(70) | 51(74) | 53(76)··· |
| | | ⋮ | | | | | | |

(2a)

Group II

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1249 | 1259 | 1277 | 1279 | 1283 | 1289 | 1291 | 1297··· |
| (2498) | (2508) | (2526) | (2528) | (2532) | (2538) | (2540) | (2546)··· |
| 1259 | 1277 | 1279 | 1283 | 1289 | 1291 | 1297 | 1301··· |
| (2518) | (2536) | (2538) | (2542) | (2548) | (2550) | (2556) | (2560)··· |
| 1277 | 1279 | 1283 | 1289 | 1291 | 1297 | 1301 | 1303··· |
| (2554) | (2556) | (2560) | (2566) | (2568) | (2574) | (2578) | (2580)··· |
| 1279 | 1283 | 1289 | 1291 | 1297 | 1301 | 1303 | 1307··· |
| (2558) | (2562) | (2568) | (2570) | (2576) | (2580) | (2582) | (2586)··· |
| 1283 | 1289 | 1291 | 1297 | 1301 | 1303 | 1307 | 1319··· |
| (2566) | (2572) | (2574) | (2580) | (2584) | (2586) | (2590) | (2602)··· |
| 1289 | 1291 | 1297 | 1301 | 1303 | 1307 | 1319 | 1321··· |
| (2578) | (2580) | (2586) | (2590) | (2592) | (2596) | (2608) | (2610)··· |
| 1291 | 1297 | 1301 | 1303 | 1307 | 1319 | 1321 | 1327··· |
| (2582) | (2588) | (2592) | (2594) | (2598) | (2610) | (2612) | (2618)··· |
| 1297 | 1301 | 1303 | 1307 | 1319 | 1321 | 1327 | 1361··· |
| (2594) | (2598) | (2600) | (2604) | (2616) | (2618) | (2624) | (2658)··· |
| 1301 | 1303 | 1307 | 1319 | 1321 | 1327 | 1361 | 1367··· |
| (2602) | (2604) | (2608) | (2620) | (2622) | (2628) | (2662) | (2668)··· |
| 1303 | 1307 | 1319 | 1321 | 1327 | 1361 | 1367 | 1373··· |
| (2606) | (2610) | (2622) | (2624) | (2630) | (2664) | (2670) | (2676)··· |
| 1307 | 1319 | 1321 | 1327 | 1361 | 1367 | 1373 | 1381··· |
| (2614) | (2626) | (2628) | (2634) | (2668) | (2674) | (2680) | (2688)··· |
| 1319 | 1321 | 1327 | 1361 | 1367 | 1373 | 1381 | 1399··· |
| (2638) | (2640) | (2646) | (2680) | (2686) | (2692) | (2700) | (2718)··· |
| | | ⋮ | | | | | |

(2b)

Groups I and II are components of an $\aleph_0 \times \aleph_0$ matrix of prime doublets with all odd primes included in the display. By 'display' we mean the extension of Group I to all prime doublets. Group II is a subset of this display. In this subset, evens in parenthesis lie beneath related primes The first row in the display, $(3, p_k)$, includes all odd primes $\{p_k\}, k \geq 3$. The second row includes all odd primes $k \geq 5$, etc. The display is such that all primes in the $n^{th}$ row are present in the $(n+1)^{th}$ row missing the lead prime of the $n^{th}$ row. All prime doublets are included in the display. Each site in the display corresponds to a satisfied Goldbach relation.

**Definition.** A *normal doublet* is two unequal primes. A *diagonal prime doublet* is not a normal doublet. The prime doublets in the first row of the display include the (3,3) diagonal doublet and all normal doublets that include the prime, 3. In addition to the diagonal doublet, $(p_k, p_k)$, the $k^{th}$ row of the display contains all normal doublets $(p_k, p_{k+i}), i = 1, 2, 3, \cdots$, where (as noted) $p_k$ is the $k^{th}$ odd prime.

**Definition.** Even intervals in the set $\mathcal{E}_1$ have values $\lesssim 4$. All other evens exist in the set $\mathcal{E}_2$.

**Lemma I.** *The primes in the $k^{th}$ row of the display (2) are all included in any $k' < k$ row.*

*Proof.* Primes in the $k^{th}$ row are given by

$$p_k, p_{k+1}, \cdots \equiv P_k$$

Primes in the $(k-1)^{th}$ row are given by

$$p_{k-1}, P_k$$

which establishes the lemma.

**Lemma II.** (Diagonal Prime Mapping) Any string of consecutive primes in the $k^{th}$-*row* may be mapped onto its image in any preceding $(k-j)^{th} row$, $(0 < j < k)$ through diagonal transfer. This mapping includes prime gaps.

*Proof.* A diagonal up and to the right through the lead prime of a prime string in a given row, intersects sites with the same prime string.

**Argument I 1.** If it can be established that there are no missing evens in the prime-doublet display, then Goldbach's hypothesis is valid.

*Proof.* As the prime-doublet display contains all prime doublets, and each site in the display is a valid Goldbach relation, it follows that if it can be shown that all evens are included in the display, Goldbach's conjecture is valid.

### 3. Prime-Gap Diagonal

In the following, a *prime gap* refers to the interval $(p_n, p_{n+1},\ p_{n+1} - p_n > 2)$.

As noted in Lemma II, the display (2) includes a diagonal symmetry according to which any prime remains constant along a diagonal up and to the right of the starting prime. This sequence is called a *prime diagonal*. The evens in a prime gap change as the gap is mapped to earlier prime rows.

**Definition.** (Shift-Prime Mapping) Choice of a prime near a missing even and mapping that prime to a different domain, through diagonal transfer.

**Definition.** (Not Compatible) An even $e_m$ is missing from the row with lead prime $p_m^0$ iff a prime $p_c$ does not exist such that $e_m = p_m^0 + p_c$. In this event one says that $e_m$ is *not compatible* with its lead prime.

**Definition.** *An even gap* is a gap in the evens (an interval between two successive evens in a given row that differ by more than two) and is void of evens.

(Note that an even gap is a property of the display, whereas a gap in the primes $[(p_n, p_{n+1}; p_{n+1} - p_n \gg 2\ )]$ is a property of the primes).

**Definition.** A *missing even* is an even that is not present in a finite monotonic sequence of evens in a given row of the display.

**Lemma III.** *A prime $p_0$ is present in rows with lead primes $p'_0 \leq p_0$ and not in rows with lead primes, $p'_0 > p_0$.*

*Proof.* The primes in a row with lead prime $p_0$ contains only primes $p_k \geq p_0$. We note the following properties:

(i). The lead prime of the $k^{th}$ row and the $k^{th}$ term in the first row are each equal to the $k^{th}$ prime in the odd-prime sequence.

(ii). In a given row, the progression of evens is a strongly monotonic sequence with the difference between successive elements equal to the difference between corresponding successive primes.

(iii). The lead prime of the $p_{k+1}$ row is equal to the second prime in the $p_k$ row.

As an example of the preceding analysis, consider the prime gap (1307-1319) in the (1249-1297) row-interval, containing four evens: 2606, 2608, 2610, 2612. All of these fall near the diagonal of 1319, and are located by the shift prime operation.

**Additive $\Delta p$ Property.** We observe the following additive property of the increments of the lead primes in the display (2). Namely

$$\delta p_{s,k} = \sum_{i=s}^{k-1} \Delta p_{i,i+1} \tag{3}$$

where $\delta p_{sk}$ is the net increment in the lead primes from the $s$ to the $k$- *row*, $s < k$, and $\Delta p_{i,i+1}$ is the increment between lead primes, $(p_i, p_{i+1})$.

## 4. Small Primes. Diagonal Property

In our first study, we discuss evens, $e \in \mathcal{E}_1$. Note that all primes are contained in $\mathcal{E}_1$.

**Theorem 2.** *The difference between successive evens on a diagonal equals the difference between the lead primes of the respective successive rows. That is,*

$$e_i - e_j = p_i - p_j \tag{4a}$$

*where $e_i > e_j$ and $p_i > p_j$ refers to lead primes.*

*Proof.* Consider the adjacent rows in the display (2).

$$\begin{gathered} p_n + p_n,\ p_n + p_{n+1},\ p_n + p_{n+2},\ p_n + p_{n+3}, \cdots \\ p_{n+1} + p_{n+1},\ p_{n+1} + p_{n+2},\ p_{n+1} + p_{n+3},\ p_{n+1} + p_{n+4}, \cdots \\ p_{n+2} + p_{n+2},\ p_{n+2} + p_{n+3},\ p_{n+2} + p_{n+4},\ p_{n+2} + p_{n+5}, \cdots \end{gathered} \tag{4b}$$

Every entry in (4b) is an even number.

Consider the difference between successive evens of a diagonal on the $n+1$ and the $n$ rows, respectively

$$p_n + p_{n+2} - (p_{n+1} + p_{n+2}) = p_n - p_{n+1} \tag{4c}$$

which we recognize to be the difference between the lead primes of the successive rows.

**Theorem 3.** *A missing even in $\mathcal{E}_1$ is located on the diagonal through a prime near the even or on a closely shifted diagonal. [The small uncertainty in this statement is due in part to the quasi-chaotic property of the primes (Liboff and Wong, 1998)].*

*Proof.* Consider that the even, $e_m$ is missing from the $p_n$ diagonal. One examines the diagonal with lead prime $p_{n\pm\Delta}$, $\Delta \lesssim 4$. The reason that $e_m$ is found in this domain is that in $\mathcal{E}_1$, $e_m$ differs from elements of its environment by the said values of $\Delta$.

In a domain free of prime gaps, both evens and primes have intervals between successive terms that are equal. It follows that in such domains, the preceding argument applies equally to $\mathcal{E}_2$.

An example of these rules is given by the evens, 44,46, that are missing in the 11-row in the (42-48) interval. The even 46 is on the diagonal through 31, and 46 is on this diagonal shifted one unit to the left through 29.

Another example addresses the missing evens 2552, 2554, the second of which is a diagonal even. The even 2552 is on the diagonal of 1303 at the intersection with the 1249 row.

In $\mathcal{E}_2$, increments between evens are far larger than 4, and, as discussed below, other arguments must be brought into play.

## 5. Large Primes

The prime-number theorem (LeVeque,1977) states that $\pi(p)$, the number of primes less than the prime $p$, obeys the relation

$$\frac{\pi(p)}{li(p)} \to 1 \tag{5a}$$

in the large prime domain, where

$$li(p) \equiv \int_2^p \frac{dt}{\ln t} \,. \tag{5b}$$

In this same limit, the derivative

$$li'(p) = \frac{1}{\ln p} \tag{5c}$$

represents the related approximate density of primes, that grows vanishingly small in the limit of large primes. Thus we recover the known result that gaps between primes grow arbitrarily large. For example, with $P(N)$ representing the $Nth$ prime and, $M$, representing a million, one finds

$$P(50M) = 982451653, \tag{6a}$$

$$P(50M + 1) = P(50M) + 54, \tag{6b}$$

$$P(50M + 2) = P(50M + 1) + 30, \tag{6c}$$

$$P(50M + 3) = P(50M + 2) + 72. \tag{6d}$$

**Prime Gaps.** In the following numerical example, the prime gap (48623, 48647), includes thirteen evens. Here is a list of these evens with corresponding related prime increments.The first prime of the gap, 48623, is labeled $\delta$:

$$\begin{aligned} 48626 &= 3 + \delta, \\ 48628 &= 5 + \delta, \\ 48634 &= 11 + \delta, \\ 48636 &= 13 + \delta, \\ 48640 &= 17 + \delta, \\ 48642 &= 19 + \delta, \\ 48646 &= 23 + \delta, \\ 48630 &= 11 + 48619, \\ 48632 &= 13 + 48619, \\ 48638 &= 19 + 48619, \\ 48654 &= 13 + 48641, \\ 48660 &= 13 + 48647, \\ 48644 &= 51 + 48593. \end{aligned} \tag{7}$$

All the prime increments of this list lie in $\mathcal{E}_1$ The related evens are found in the eight rows: (3,5,11,13,17,19,23,51).

## 6. Criterion

**Lemma IV.** *A domain exists in the large prime domain, free of gaps, in which primes are relatively close together.*

*Proof.* We recall Fermat's Little Theorem relevant to the prime $p$: $a^p = a(mod\ p)$ so that

$$a^{p_1} = a + kp_1, \tag{8a}$$

$$a^{p_2} = a + k'p_2, \tag{8b}$$

where $a > k > 0$ are integers. Subtracting we obtain

$$a^{p_2} - a^{p_1} = k'p_2 - kp_1, \tag{8c}$$

$$a^{p_2} - a^{p_1} \approx k(p_2 - p_1). \tag{8d}$$

As $p_1$ and $p_2$ are both in the large prime domain, it is assumed in the last equation that $k \approx k'$. To establish this property, we set

$$a = \ln y.$$

With (8d) we obtain,

$$a(p_2 - p_1) = k(p_2 - p_1).$$

As $a > k$, we conclude that $p_2 \approx p_1$.

**Theorem 4.** *In the domain $\mathcal{E}_2$, a missing even may be found with the method of diagonal mapping.*

*Proof.* For such cases, we chose the prime diagonal rule and examine a prime close to the missing even and consider the difference between that prime and the missing even. For example, consider the missing even 2552. Note that the prime 2549 differs from the missing even by 3. Examining the 3 - row gives the missing even, 2549+3=2552. That is, for any large missing even, choose a prime $p$ that differs from it by a relatively small prime, $\Delta$. Then the missing even exits in the $\Delta$ row.

This process is valid because of the property that a p-row contains all primes greater than p, so that in the small $p$ domain, rows contain all large primes.

Again, it is noted, that this small element of uncertainty is due, in part, to the quasi-chaotic property of the primes [Liboff & Wong,1998].

In the event that the missing even falls in the domain of a prime gap, examination reveals the bounds of the gap and, as the evens are successive, one knows the evens in the gap.

Consider, for example, the large prime gaps shown in (6). We recall

$$P(50M+1) = P(50M) + 54.$$

The sequence of evens in the gap between the two primes is given by the 26 inclusive numbers

$$982451653, \cdots 54, \cdots 56, \cdots 58, \cdots, \quad \cdots 106,\ 9824516107.$$

To find these evens in the display, the arguments of Theorem 4 apply.

Consider an even $e$,that is missing in the display (2) with one of the following properties:

1. $e \in \mathcal{E}_1$. Then its location is known.

2. $e \in \mathcal{E}_2$. Then one examines the prime diagonal through a prime near $e$ to give its location.

3. In the event that an even is missing in a prime gap, the gap may be mapped back to the domain of small primes, where the relevant prime doublet is found as in item 2 ( all evens in the gap are known).

Having described methods for discovering missing evens over all rows of the display, we conclude that all evens are accounted for in the prime doublet display. In accord with Theorem I, we may conclude that this theorem is a good argument for the validity of Goldbach's conjecture.

## 7. Prime Diagonal Translation Group

Consider an element, $(p, e)$, in the prime-doublet display (2). The operation $T_j(p, e)$ translates this point through the diagonal down and to the left through the starting point $(p, e)$ at common values of $p$ through the interval, $j$. A unit jump in $j$ occurs for a displacement between adjacent rows in (2). The value $j = 1$ is assigned to the first row of the display. It follows that

$$T_j(p, e) + T_{j_1}(p, e_1) = T_{(j+j_1)}(p, e_2). \tag{10}$$

This relation defines the operation of the group. The identity element of this translation group is

$$E = T_0(p, e), \tag{11}$$

as

$$T_0(p, e) + T_j(p, e) = T_j(p, e). \tag{12}$$

The element $T_{-j}(p, e)$ displaces $(p, e)$ in the direction opposite to that of $T_j(p, e)$ through the interval $j$. The inverse of $T_j$ is $T_{-j}$. Namely

$$T_j + T_{-j} = T_0. \tag{13}$$

Thus, three key ingredients of a group are included in this description (Liboff, 2004), *viz.*, the bi-product operation, the inverse and the identity element. An example of the operation (10) is given by

$$T_2(13, 18) + T_1(13, 16) = T_3(13, 20). \tag{14}$$

Consider that a net displacement in a translation is beyond the starting doublet in a row. First note that the evens along a diagonal grow as the elements of a row increase. Namely, evens are obtained by adding consecutive primes to the starting prime. Thus, a diagonal attached to a given prime may be extended to the left of the starting doublet. Consider the diagonal of the prime, 11 that begins with the doublet (11,18). Extending the diagonal beyond the starting doublet (11,22) gives the doublets: (11,18), (11, 22), (11,24)$\cdots$ that appear on the $extended 11 - diagonal$.

## 8. Conclusions

Resulting from a recent work in which it was shown that the prime number sequence is quasi chaotic, it was concluded that a mathematical proof of Goldbach's conjecture does not exist. A display of the doubly infinite set of prime doublets (any two odd primes, equal or unequal) and their related evens, was illustrated. A theorem was introduced that implies Goldbach's conjecture is valid providing there are no missing evens in the display. Primes were partitioned into sets of small and large primes. Properties of a prime-gap diagonal were described that include location of missing evens. This property was employed in establishing Goldbach's conjecture in the small prime domain. To prove the conjecture for the large prime domain, a mapping from the large to the small prime domains was introduced. A translation group relevant to the prime-doublet display was described. Numerical examples were included.

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