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BASES IN THE SPACE OF VECTOR VALUED ANALYTIC DIRICHLET SERIES

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Abstract: The space X of all functions represented by vector valued Dirichlet series and analytic in a half plane is considered in this paper. The space is endowed with a certain topology under which X become a Frechet space. On this space the form of linear continuous operator F from X to X is characterized. We have also given a characterization of proper bases.

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1. Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, (\sigma, t \text{ are real variables}), \tag{1.1}$$

where $a_n's$ belong to a commutative Banach algebra E with identity element ω with $||\omega|| = 1$ and $\lambda_n's \in R$ satisfying the condition

$$0 < \lambda_1 < \lambda_2 < \lambda_3 ... < \lambda_n ..., \quad \lambda_n \to \infty \text{ as } n \to \infty. \tag{1.2}$$

Let $\sigma_c(f)$ and $\sigma_a(f)$ be the abscissa of convergence and abscissa of absolute convergence respectively of f. If the sequence $\{\lambda_n\}$ satisfies

$$\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0 \tag{1.3}$$

then by [1] for each $f \in X$,

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$$\sigma_c(f) = \sigma_a(f) = -\lim_{n\to\infty} \sup \frac{\log ||a_n||}{\lambda_n}.$$

Suppose that the vector valued Dirichlet series given by (1.1) converges absolutely in a left half plane $\sigma < l$. Then the given series represents vector valued analytic functions in the half plane $\sigma < l$ and the series is also called vector valued analytic Dirichlet series.

B.L. Srivastava [1] defined the growth properties such as order, type, lower order, lower type etc. of the vector valued analytic Dirichlet series taking E to be a Banach space. He also obtained the coefficient characterizations of order and type.

By giving different topologies on the set of analytic functions defined by Dirichlet series of one complex variable, Kamthan and Gautam [2] obtained various topological properties. In this paper we shall consider the space of vector valued analytic Dirichlet series and obtain these properties.

Let X be the class of functions f represented by (1.1) satisfying

$$\lim_{n\to\infty} \sup \frac{\log ||a_n||}{\lambda_n} = -l \quad , \tag{1.4}$$

where l is a given positive number. For each $f \in X$, let us define

$$||f||_\sigma = \sum_{n=1}^{\infty} ||a_n|| e^{\sigma\lambda_n} \, , \qquad \text{for } \sigma < l.$$

Thus in view of (1.4) $||f||_\sigma$ is clearly well defined and for each $\sigma < l$ introduces a norm on X. We denote by $X(\sigma)$, the space X equipped with the norm $||...||_\sigma$. Let ρ be the topology generated by the family of norms $\{||f||_\sigma : \sigma < l\}$ which is equivalent to the topology generated by the invariant metric λ, where

$$\lambda\,(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||f-g||_{\sigma_n}}{1+||f-g||_{\sigma_n}} \, ,$$

where $\{\sigma_n\}$ is a sequence such that $\sigma_1 < \sigma_2 < ... < \sigma_n < ...;\ \sigma_n \to l$ as $n \to \infty$.

Throughout this paper we shall assume that the space X is equipped with the topology generated by the metric λ. Now we give some definitions.

A sequence $\{\alpha_n\} \subseteq X$ is said to be linearly independent if for any sequence $\{c_n\}$ of complex numbers for which $\sum_{n=1}^{\infty} c_n\alpha_n$ converges in X , $\sum_{n=1}^{\infty} c_n\alpha_n = 0$ implies that $c_n = 0\ \forall n$. A subspace X_0 of X is said to be spanned by a sequence $\{\alpha_n\} \subseteq X$ if X_0 consists of all linear combinations $\sum_{n=1}^{\infty} c_n\alpha_n$ such that $\sum_{n=1}^{\infty} c_n\alpha_n$ converges in X. A sequence $\{\alpha_n\} \subseteq X$ which is linearly independent and spans a closed subspace X_0 of X is said to be a base in X_0. In particular, if $e_n \in X$, $e_n(s) = \omega\, e^{s\lambda_n}$, $n \geq 1$, then $\{e_n\}$ is a base in X. A sequence $\{\alpha_n\} \subseteq X$ will be called a 'proper base' if it is a base and it satisfies the following condition:

"for all sequences $\{a_n\} \subseteq E$, convergence of $\sum_{n=1}^{\infty} a_n\, \alpha_n$ in X implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X".

2. Main Results

We shall prove the following result.

Theorem 2.1. *The space X is a Frechet space.*

Proof. Here, as defined above, X is a normed linear metric space. For showing that X is a Frechet space, we need to show that X is complete. Let $\{f_\alpha\}$ be a Cauchy sequence in X. Hence it is a Cauchy sequence in $X(\sigma)$ for each real $\sigma < l$. Therefore, for any given $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon, \sigma)$ such that

$$||f_\alpha - f_\beta||_\sigma < \varepsilon \qquad \forall \alpha, \beta \geq n_0 \,.$$

Denoting by $f_\alpha(s) = \sum_{n=1}^{\infty} a_n^{(\alpha)} e^{s.\lambda_n}$, $f_\beta(s) = \sum_{n=1}^{\infty} a_n^{(\beta)} e^{s.\lambda_n}$, we have therefore

$$\sum_{n=1}^{\infty} ||a_n^{(\alpha)} - a_n^{(\beta)}||e^{\sigma\lambda_n} < \varepsilon \quad \forall \alpha, \beta \geq n_0. \tag{2.1}$$

Therefore for each fixed $n = 1, 2, ...,$ $\left\{a_n^{(\alpha)}\right\}$ is a Cauchy sequence in the Banach space E. Hence there exists a sequence $\{a_n\} \subseteq E$ such that

$$\lim_{\alpha\to\infty} a_n^{(\alpha)} = a_n \;, \quad n \geq 1.$$

Now letting $\beta \to \infty$ in (2.1), we have for $\alpha \geq n_0$,

$$\sum_{n=1}^{\infty} ||a_n^{(\alpha)} - a_n||e^{\sigma\lambda_n} \leq \varepsilon. \tag{2.2}$$

Let us denote by $f = \sum_{n=1}^{\infty} a_n e_n$. Now it remains to show that $f \in X$. We choose σ_i such that $l < \sigma_i + \varepsilon$. From (2.2) we have

$$\sum_{n=1}^{\infty} ||a_n^{(\alpha)} - a_n||e^{\sigma_i\,\lambda_n} \leq \varepsilon \qquad \alpha \geq n_1, \tag{2.3}$$

where $n_1 = n_1(\varepsilon, \sigma_i)$. Keeping α as fixed in (2.3) and in view of (1.4) we observe that

$$||a_n^{(\alpha)}|| \le e^{(-l+\varepsilon)\lambda_n} \qquad \forall n \ge n_2 \,,$$

where $n_2 = n_2(\varepsilon, \alpha)$. Then

$$\begin{aligned} & ||a_n|| \le ||a_n^{(\alpha)} - a_n|| + ||a_n^{(\alpha)}|| & \\ \Rightarrow \quad & ||a_n|| \le \varepsilon e^{-\sigma_i \lambda_n} + e^{(-l+\varepsilon)\lambda_n} & \forall n \ge M = \max(n_1, n_2) \\ & < 2e^{(-l+\varepsilon)\lambda_n}. & \end{aligned}$$

Thus

$$\lim_{n\to\infty} \sup \frac{\log ||a_n||}{\lambda_n} \le -l \quad .$$

Thus $\sum_{n=1}^{\infty} a_n e_n \in X$. Therefore $f_\alpha \to f \in X$. Hence X is complete. This proves Theorem 2.1. □

Theorem 2.2. *A necessary and sufficient condition for the linear transformation* $F : X \to X$ *with* $F(e_n) = \alpha_n \in X$, $n = 1, 2, ...,$ to be continuous *is that for each* $\sigma < l$

$$\lim_{n\to\infty} \sup \frac{\log ||\alpha_n||_\sigma}{\lambda_n} < l \,. \tag{2.4}$$

Proof. Let F be a continuous linear transformation from X into X with $F(e_n) = \alpha_n$, $n = 1, 2, ...$ Then for any given σ, there exists a σ_1 $(\sigma, \sigma_1 < l)$ and a finite constant K such that

$$||F(e_n)||_\sigma \le K||e_n||_{\sigma_1}$$

$$\Rightarrow ||\alpha_n|| \le K e^{\sigma_1 \lambda_n} \quad n \ge 1$$

$$\Rightarrow \lim_{n\to\infty} \sup \frac{\log ||\alpha_n||_\sigma}{\lambda_n} \le \sigma_1 < l \,.$$

Conversely, let the sequence $\{\alpha_n\}$ satisfy (2.4) and let $\alpha(s) = \sum_{n=1}^{\infty} a_n e_n \in X$. Then there exists an $\varepsilon > 0$ such that

$$\frac{\log ||\alpha_n||_\sigma}{\lambda_n} \le l - \varepsilon \qquad \texttt{for all } n \ge n_1(\varepsilon).$$

Further, for a given $\eta > 0$ such that $\eta < \varepsilon$,

$$||a_n|| \le e^{(-l+\eta)\lambda_n} \qquad \text{for all } n \ge n_2(\eta) \,.$$

Hence

$$||a_n||.||\alpha_n||_\sigma \leq e^{(-l+\eta)\,\lambda_n} e^{(l-\varepsilon)\,\lambda_n} \qquad \text{for all } n \geq \max(n_1, n_2)$$
$$= e^{(\eta-\varepsilon)\,\lambda_n}\,.$$

Hence the series $\sum_{n=1}^{\infty} ||a_n||\,||\alpha_n||_\sigma$ is convergent. As $\sigma < l$, therefore $\sum_{n=1}^{\infty} a_n\alpha_n$ is convergent in X. Hence there exists a linear transformation $F : X \to X$ such that $F(\alpha) = \sum_{n=1}^{\infty} a_n\alpha_n$ and $F(e_n) = \alpha_n, n = 1, 2...$, for each $\alpha_n \in X$. Now we prove the continuity of F. Given $\sigma < l$, there exists $\eta > 0$ such that

$$\frac{\log ||\alpha_n||_\sigma}{\lambda_n} \leq l - \eta, \qquad \texttt{for all} \quad n \geq N$$
$$\Rightarrow\ ||\alpha_n||_\sigma \leq\ e^{(l-\eta)\,\lambda_n}\,, \qquad \texttt{for all} \quad n \geq N$$
$$\Rightarrow\ ||\alpha_n||_\sigma \leq\ Ke^{(l-\eta)\,\lambda_n}\,, \qquad \texttt{for all} \quad n \geq 1.$$

Now,

$$||F(\alpha)||\ \leq\ K\sum_{n=1}^{\infty} ||a_n||e^{(l-\eta)\,\lambda_n} = K||\alpha||_{l-\eta}\,.$$

Hence $F : \{X, ||...,\sigma\} \to \{X, ||..., l-\eta\}$ is continuous. Since $\sigma < l$ is arbitrary, it shows that F is continuous. This proves Theorem 2.2. □

We now give the characterization of proper bases. First we prove

Lemma 2.1. *Let $\{a_n\} \subseteq E$ and $\{\alpha_n\} \subset X$ be given sequences. The following three conditions are equivalent:*

(i) Convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X implies the convergence of $\sum_{n=1}^{\infty} a_n\,\alpha_n$ in X.

(ii) The convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X implies that $\lim_{n\to\infty} a_n\,\alpha_n = 0$ in X.

(iii) $\limsup_{n\to\infty} \frac{\log ||\alpha_n||_\sigma}{\lambda_n} < l\,, \qquad \text{for all } \sigma < l.$

Proof. First suppose that (i) holds. Then for any sequence $\{a_n\}$, where a_n's belong to Banach algebra E, $\sum_{n=1}^{\infty} a_n e_n$ converges in X implies that $\sum_{n=1}^{\infty} a_n\,\alpha_n$ converges in X which in turn implies that $a_n\alpha_n \to 0$ as $n \to \infty$. Hence (i)⇒(ii). Now we assume that (ii) is true but (iii) is false. This implies that for some $\sigma_1 < l$,

$$\limsup_{n\to\infty} \frac{\log ||\alpha_n||_{\sigma_1}}{\lambda_n} \geq l\,.$$

Hence there exists a sequence $\{n_k\}$ of positive integers, such that

$$\limsup_{k\to\infty} \frac{\log ||\alpha_{n_k}||_{\sigma_1}}{\lambda_{n_k}} \geq l -\ k^{-1}\,, \qquad \forall\, n_k\,,\, k = 1, 2, \dots\,.$$

We define a sequence $\{a_n\} \subseteq E,$ as

$$a_n = \begin{cases} \omega e^{-(l-k^{-1})\lambda_{n_k}} & n = n_k\ , \text{ for } k = 1, 2, 3... \\ 0 & n \neq n_k\ . \end{cases}$$

Then, we have

$$||a_{n_k}||.e^{\sigma\lambda_{n_k}} = e^{-(l-k^{-1})\lambda_{n_k}} e^{\sigma.\lambda_{n_k}} = e^{-(l-\sigma-k^{-1})\lambda_{n_k}}.$$

For large k, $(l - \sigma - k^{-1}) > 0$. Hence $\sum_{k=1}^{\infty} ||a_{n_k}||.e^{\sigma.\lambda_{n_k}}$ converges in X for all $\sigma < l$.

On the other hand, for all $k = 1, 2, 3...$,

$$||a_{n_k}||.\, ||\alpha_{n_k}||_{\sigma_1} \geq e^{-(l-k^{-1})\lambda_{n_k}}.e^{(l-k^{-1})\lambda_{n_k}} = 1.$$

Therefore the sequence $\{a_n\, \alpha_n\}$ does not tend to zero as $n \to \infty$ and this contradicts (ii). Hence (ii) $\Rightarrow$(iii). Lastly we show that (iii) $\Rightarrow$(i).

In course of the proof of Theorem 2.2 above, we have already proved that if (iii) holds then there exists a linear continuous transformation $F : X \to X$ with $F(e_n) = \alpha_n \in X$, $n = 1, 2....$ By continuity of F,

$$\begin{aligned} F(\sum_{n=1}^{\infty} a_n e_n) &= F(\lim_{n\to\infty} \sum_{k=1}^{n} a_k e_k) \\ &= \lim_{n\to\infty} \{\sum_{k=1}^{n} a_k F(e_k)\} = \sum_{n=1}^{\infty} a_n\, \alpha_n. \end{aligned}$$

Thus the proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $\{a_n\} \subseteq E$ and $\{\alpha_n\} \subseteq X$. The following three properties are equivalent:*

(a) $\lim_{n\to\infty} (a_n\alpha_n) = 0$ *in X* implies that $\sum_{n=1}^{\infty} a_n e_n$ *converges in X.*

(b) Convergence of $\sum_{n=1}^{\infty}(a_n\alpha_n)$ in X implies that $\sum_{n=1}^{\infty} a_n e_n$ converges in X.

(c) $\lim_{\sigma\to l} \left\{ \lim_{n\to\infty} \inf \frac{\log ||\alpha_n||_\sigma}{\lambda_n} \right\} \geq l$.

Proof. Obviously (a) $\Rightarrow$(b). We now prove that (b) $\Rightarrow$ (c). To prove this, we suppose that (b) holds but (c) does not hold. Therefore

$$\lim_{\sigma\to l} \left\{ \lim_{n\to\infty} \inf \frac{\log ||\alpha_n||_\sigma}{\lambda_n} \right\} < l.$$

Since $||...||_\sigma$ increases as σ increases, this implies that for each $\sigma < l$,

$$\lim_{n\to\infty} \inf \frac{\log ||\alpha_n||_\sigma}{\lambda_n} < l\ , \quad \text{for all } \sigma < l\,.$$

Hence if $\eta > 0$ is a fixed small positive number, then for each $r > 0$, we can find a positive number n_r such that $\forall r$, we have $n_{r+1} > n_r$ and

$$\begin{aligned} &\frac{\log ||\alpha_{n_r}||_\sigma}{\lambda_{n_r}} < l - \eta \\ \Rightarrow\ \ &||\alpha_{n_r}||_\sigma \le e^{(l-\eta)\lambda_{n_r}}\,. \end{aligned} \tag{2.5}$$

Now we choose a positive number $\eta_1 < \eta$, and define a sequence $\{a_n\} \subseteq E$ as

$$a_n = \begin{cases} \omega e^{-(l-\eta_1)\lambda_{n_r}} & n = n_r\ , \text{ for } r = 1, 2, 3... \\ 0 & n \neq n_r. \end{cases}$$

Then, for any $\sigma < l$

$$\sum_{n=1}^{\infty} ||a_n||.||\alpha_n||_\sigma = \sum_{r=1}^{\infty} ||a_{n_r}||.||\alpha_{n_r}||_\sigma. \tag{2.6}$$

Omit from the above series those finite number of terms, which correspond to those number n_r for which $1/r > \eta_1$. The remainder of the series in (2.6) is dominated by $\sum_{r=1}^{\infty} ||a_{n_r}||.||\alpha_{n_r}||_\sigma$. Now by (2.5) and (2.6), we find that

$$\sum_{n=1}^{\infty} ||a_n||.||\alpha_n||_\sigma \le \sum_{r=1}^{\infty} ||a_{n_r}||.||\alpha_{n_r}||_\sigma \le \sum_{r=1}^{\infty} e^{(l-\eta)\lambda_{n_r}}.e^{-(l-\eta_1)\lambda_{n_r}} = \sum_{r=1}^{\infty} e^{(\eta_1-\eta)\lambda_{n_r}}.$$

Since $\eta_1 < \eta$, above series is convergent. For this sequence $\{a_n\}$ as defined above, $\sum_{n=1}^{\infty} ||a_n||\alpha_n$ converges in $X\,(\sigma)$ for each $\sigma < l$ and hence converges in X. But we have,

$$\sum_{n=1}^{\infty} ||a_n||.\, e^{\sigma.\lambda_n} = \sum_{r=1}^{\infty} ||a_{n_r}||\,.e^{\sigma.\lambda_{n_r}} = \sum_{r=1}^{\infty} e^{-(l-\eta_1)\lambda_{n_r}}\,.e^{\sigma.\lambda_{n_r}} = \sum_{r=1}^{\infty} e^{(\sigma+\eta_1-l)\lambda_{n_r}}.$$

Now given η_1 choose $\sigma < l$ such that $\sigma + \eta_1 > l$, then the above series is divergent for this σ. Hence $\sum_{n=1}^{\infty} a_n e_n$ does not converge in X and this is a contradiction. Therefore (b) $\Rightarrow$ (c).

Now we prove that (c) $\Rightarrow$ (a). We assume (c) is true but (a) does not hold. Then there exists a sequences $\{a_n\}$, where a_n's belongs to Banach space E, for which $||a_n||\alpha_n \to 0$ in X, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in X.This implies that

$$\lim_{n\to\infty} \sup \frac{\log ||a_n||}{\lambda_n} > -l.$$

Hence there exists a positive number ε and a sequence $\{n_k\}$ of positive integers such that

$$\frac{\log ||a_{n_k}||}{\lambda_{n_k}} \geq e^{(l-\varepsilon)\lambda_{n_k}} \quad . \tag{2.7}$$

We choose a positive number η such that $\eta < \varepsilon/2$, by assumption we can find a positive number $\sigma = \sigma(\eta)$ such that

$$\lim_{n\to\infty} \inf \frac{\log ||\alpha_n||_\sigma}{\lambda_n \log \lambda_n} \geq l - \eta \ .$$

Hence there exists $N = N(\eta)$, such that

$$\frac{\log ||\alpha_n||_\sigma}{\lambda_n} \geq l - 2\eta\,, \qquad \forall\, n \geq N. \tag{2.8}$$

Therefore,

$$\begin{aligned} ||a_{n_k}||.||\alpha_{n_k}||_\sigma &\geq e^{(-l+\varepsilon)\lambda_{n_k}}.e^{(l-2\eta)\lambda_{n_k}}\ , \\ &= e^{(\varepsilon-2\eta)\lambda_{n_k}} \to \infty \quad \text{as } k \to \infty\,, \quad \text{since } \varepsilon > 2\eta. \end{aligned}$$

Thus $\{||a_n||\ \alpha_n\}$ does not tend to zero in $X(\sigma)$ for the σ chosen above and this is a contradiction. Thus (c) $\Rightarrow$ (a) is proved. This completes the proof of Lemma 2.2. □

Now combining Lemma 2.1 and Lemma 2.2 above, we get the following

Theorem 2.3. *A base $\{\alpha_n\}$ in a closed subspace X_0 of X is proper if and only if it satisfies the conditions (iii) and (c).*

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