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APPROXIMATE AMENABILITY OF CENTERS OF BANACH ALGEBRAS

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Abstract: Let G be a locally compact topological group. In this paper we study approximate amenability of the center of $L^1(G)$. We show that if the center of $L^1(G)$ is approximately amenable then G is amenable.

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1. Introduction

The concept of amenability for the Banach algebras introduced by Johnson in 1972 in [9], has proved to be of enormous importance in Banach algebra theory. The notion of approximate amenability of Banach algebras was introduced by Ghahramani and Loy in [4].

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to X$ is a linear map which satisfies

$$D(ab) = a.D(b) + D(a).b,$$

for all $a, b \in \mathcal{A}$. The derivation δ is said to be inner if there exists $x \in X$ such that $\delta(a) = \delta_x(a) = a.x - x.a$ for all $a \in \mathcal{A}$.

The Banach algebra $\mathcal{A}$ is approximate amenable if for every Banach $\mathcal{A}$-bimodule X and every bounded derivation $D : \mathcal{A} \to X^*$ there exists a net (D_α) of inner derivations such that

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$$\lim_{\alpha} D_\alpha(a) = D(a)$$

for all $a \in \mathcal{A}$. An approximate diagonal for $\mathcal{A}$ is a net (M_i) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that, for each $a \in \mathcal{A}$,

$$a.M_i - M_i.a \to 0 \qquad \text{and} \qquad a\pi(M_i) \to a.$$

The Banach algebra $\mathcal{A}$ is said to be pseudo-amenable if it has an approximate diagonal.

Let a linear subspace $S^1(G)$ of $L^1(G)$ denoted the Segal algebra. Let G and H locally compact groups, $S^1(G)$ and $S^1(H)$ are Segal algebras on G and H, respectively. Then there is a unique continuous linear map $T : S^1(G)\widehat{\otimes}S^1(H) \longrightarrow S^1(G\times H)$, such that $T(f\otimes g) = f\otimes g$ for all $f \in S^1(G)$ and $g \in S^1(H)$. Further T is a isometric $*-$homomorphism (see [3]).

The range of T contains $\chi_{E\times F}$ for each $E \times F$ in Borel set of $G \times H$ and linear span of the collection of these functions is dense in $S^1(G \times H)$. Let $f \in S^1(G)$ and $g \in S^1(H)$, identify $f \otimes g \in S^1(G)\widehat{\otimes}S^1(H)$ with element of $S^1(G \times H)$ given by

$$(f \otimes g)(x, y) = f(x)g(y) \qquad (x \in G, y \in H).$$

A Segal algebra on a group G has an approximate identity but it is only amenable when $L^1(G)$ and G are amenable [9]. When G is amenable and a Segal algebra on G has a central approximate identity, then it is pseudo-amenable [7]. This paper considers some results about amenability and approximate amenability of the centers of $L^1(G)$ and $S^1(G)$, which we denote with $ZL^1(G)$ and $ZS^1(G)$, respectively. We denote the set of all continuous definite functions on G by $P(G)$.

2. Center of $L^1(G)$

The center of $L^1(G)$ is the following set

$$ZL^1(G) = \{f \in L^1(G) : f * g = g * f, \quad \text{for all} \quad g \in L^1(G)\}.$$

In general case, when G is compact, $ZL^1(G)$ is not amenable. In fact, it fails to be amenable whenever G is either non-abelian and connected (Section 1.4, [1]), or when G is a product of infinitely many non-abelian finite groups (Section 1.5, [1]). From [13], $ZL^1(G) \neq 0$ if and only if $G \in [IN]$, then $ZS^1(G) \neq 0$ if and only if $G \in [IN]$ (Theorem 1, [11]) and also the center of $S^1(G)$ is dense in the center of $L^1(G)$ (Theorem 2, [11]).

If G is a compact topological group, then $ZL^1(G)$ has a bounded approximate identity in the center of $S^1(G)$, so we have the following Theorem.

Theorem 2.1. *Let G be a compact group. Then $ZL^1(G)$ is approximately amenable.*

Proof. By Theorem 1 of [11], there exists a net (f_ν) in $ZS^1(G)$ such that (f_ν) is a bounded approximate identity, and therefore $2(f_\nu \otimes f_\nu) \subseteq ZS^1(G)\widehat{\otimes} ZS^1(G)$. Now set $M_\nu = 2(f_\nu \otimes f_\nu), F_\nu = f_\nu$ and $G_\nu = f_\nu$.

Thus for every $f \in ZL^1(G)$ we have

$$f.M_\nu - M_\nu.f + F_\nu \otimes a - a \otimes G_\nu = f.M_\nu - M_\nu.f + f_\nu \otimes f - f \otimes f_\nu$$

$$\begin{aligned} &= 2(f.f_\nu \otimes f_\nu - f_\nu \otimes f_\nu.f) + f_\nu \otimes f - f \otimes f_\nu \\ &= f.f_\nu \otimes f_\nu - f_\nu \otimes f_\nu.f + (f.f_\nu - f) \otimes f_\nu + f_\nu \otimes (f - f.f_\nu) \\ &\longrightarrow 0 \end{aligned}$$

also we have

$$\pi(M_\nu).f - F_\nu.f - G_\nu.f = 2f_\nu^2 f - f_\nu.f - f_\nu.f \to 0.$$

Therefore according to Proposition 2.6 of [4], $ZL^1(G)$ is approximately contractible and by Theorem 2.1 of [5], $ZL^1(G)$ is approximately amenable. □

By $C(G)$ and $C_c(G)$, we denote the continuous functions and continuous functions with compact support, respectively. For a closed subgroup H of G, G/H has a quasi-invariant measure (see [8] and [15]). For $f \in L^1(G)$ we define

$$T_H f(sH) = \int_H f(sx)dx \qquad (sH \in G/H),$$

T_H is a linear operator operator of $L^1(G)$ onto $L^1(G/H)$ with

$$\|T_H f\| \le \|f\|_1, \quad f \in L^1(G) \quad \text{and} \quad T_H(C(G)) = C(K/H).$$

If H is a closed normal subgroup of G, then $S^1(G/H)$ is a Segal algebra. By restriction of T_H on $ZL^1(G)$ and $ZS^1(G)$ we can conclude the following statements

$$T_H(ZL^1(G)) = ZL^1(G/H) \quad \textit{and} \quad T_H(ZS^1(G)) = ZS^1(G/H).$$

Corollary 2.2. *Let H be a closed normal subgroup of locally compact group G. If $ZL^1(G)$ is approximately amenable, then $ZL^1(G/H)$ is approximately amenable.*

Proof. The mapping T_N from $ZL^1(G)$ into $ZL^1(G/H)$ is a surjective quotient map. According to Proposition 2.2 of [4], $ZL^1(G/H)$ is approximately amenable. □

In the above corollary, we can replace Segal algebra $S^1(G)$ and result is will be true.

Theorem 2.3. *Let G be a locally compact group and let $ZL^1(G)$ is approximately amenable. Then G is amenable.*

Proof. Proof is similar to the proof of Theorem 3.2 of [4]. $L^\infty(G)$ is $M(G)$-bimodule by the general action defined by

$$\langle f, \mu.\varphi\rangle = \langle f * \mu, \varphi\rangle \qquad \textit{and} \qquad \varphi.\mu = \mu(G)\varphi,$$

for all $f \in L^1(G), \mu \in M(G)$ and $\varphi \in L^\infty(G)$. Let $\delta_e = n \in L^\infty(G)^*$ so that $\langle 1, n\rangle = 1$, and define $\Delta : M(G) \longrightarrow L^\infty(G)^*$, by $\Delta(\mu) = \mu.n - n.\mu$. Let m_G be the left Haar measure on G. Then for each $f \in L^1(G)$ we have

$$\begin{aligned}\langle 1, \Delta f\rangle &= \langle 1, f.n - n.f\rangle = \langle 1, f.n\rangle - \langle 1, n.f\rangle \\ &= \langle \int_G f(g)dm_G(g), n\rangle - \langle f * 1, n\rangle = 0\end{aligned}$$

We know that $\mathbb{C}1$ is a closed submodule of $L^\infty(G)$. Let $X = L^\infty(G)/\mathbb{C}1$, so $X^* \cong \{m \in L^\infty(G)^* : \langle 1, m\rangle = 0\}$. Since for each $f \in L^1(G)$, $\langle 1, \Delta f\rangle = 0$, thus $\Delta(ZL^1(G)) \subset X^*$. Let $D : ZL^1(G) \longrightarrow X^*$ be a continuous derivation such that $Df = \Delta f$, for all $f \in ZL^1(G)$.

Since $ZL^1(G)$ is approximately amenable, thus there exists a net $(\xi_\alpha) \subseteq X^*$ such that for all $f \in ZL^1(G)$, we have

$$Df = \lim_\alpha (f.\xi_\alpha - \xi_\alpha.f).$$

Let $\eta_\alpha = n - \xi_\alpha$. Then $\eta_\alpha \in L^\infty(G)^*$ and

$$\langle 1, \eta_\alpha\rangle = \langle 1, n - \xi_\alpha\rangle = \langle 1, n\rangle - \langle 1, \xi_\alpha\rangle = 1 - 0 = 1 \qquad (1)$$

Let $f \in ZL^1(G) \cap P(G)$, then

$$\begin{aligned}\Delta(\delta_g) &= \delta_g.n - n.\delta_g = \langle \delta_g, \Delta\rangle = \langle \delta_g.f, \Delta\rangle = \Delta(\delta_g * f) - \delta_g\Delta(f) \\ &= \lim_\alpha((\delta_g * f).\xi_\alpha - \xi_\alpha.(\delta_g * f) - \delta_g(f.\xi_\alpha - \xi_\alpha.f)) \\ &= \lim_\alpha(\delta_g * f * \xi_\alpha - \xi_\alpha - \delta_g * f * \xi_\alpha + \delta_g.\xi_\alpha) \\ &= \lim_\alpha(\delta_g.\xi_\alpha - \xi_\alpha)\end{aligned}$$

So,

$$\lim_\alpha \delta_g.(n - \xi_\alpha) - (n - \xi_\alpha) = 0 \qquad (g \in G)$$

According to (1), and by taking a suitable subnet $(\xi_\beta) \subseteq (\xi_\alpha)$, $\liminf_\beta \|n - \xi_\beta\| > 0$. With setting $\eta_\beta = \frac{n-\xi_\beta}{\|n-\xi_\beta\|}$, we have $\|\eta_\beta\| = 1$ and $\delta_g.\eta_\beta - \eta_\beta \to 0$ for all $g \in G$.

Since $L^\infty(G)$ is a unital C^*-algebra, so we have $\delta_g.|\eta_\beta| - |\eta_\beta| \to 0$ in norm for all $g \in G$. Let η be a w^*-cluster point of $|\eta_\beta|$. Then

$$\langle 1, \eta \rangle = w^* - \lim_\beta \langle 1, |\eta_\beta| \rangle = \|\eta_\beta\| = 1.$$

Therefore proof is complete. □

Corollary 2.4. *Let G be a locally compact group. If $ZL^1(G)$ is approximately amenable, then $L^1(G)$ is amenable.*

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