International Journal of Pure and Applied Mathematics
Volume 70 No. 7 2011, 901-914

SUB-DIFFERENTIAL APPROACH FOR SOLVING THE MULTI-OBJECTIVE OPTIMIZATION PROBLEM, APPROXIMATION BY THE INTERIOR-POINT ALGORITHM

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Abstract: Using the sub-differential concept in the sense of convex analysis, we give a characterization of the solution of the multi-objective optimization problem. Formulations of our initial problem in various practical forms are explored. Some numerical examples, illustrating the theoretical study and showing the interest of the used approach, are discussed.

AMS Subject Classification: 90C29, 90C51, 46N10
Key Words: multi-objective optimization problem, sub-differential approach, convex analysis, interior-point algorithm

1. Introduction

Increasingly, the decision to problems arising from life require consideration of multiple conflicting objectives and sometimes even contradictory. In such cases, there is no single optimum. Solving real world problems has led to the development of multi-criteria optimization (resp. multi-objective, multilevel, or multi-criteria programming), as it best reflects the various conflicting criteria that prohibit an "ideal" solution (optimal for each decision maker under each objective considered separately), because the main difficulty that we encounter in mono objective optimization comes from the fact that modelling a problem

Received: February 21, 2011
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with just one equation can be a very difficult task. The goal of modelling the problem using one equation can introduce a bias during the modelling phase.

The general mathematical model of a multi-objective optimization problem is usually specified by, [1] and [3]

$$\min_{x\in X} \sum_{i=1}^{k} \lambda_i f_i(x), \quad \sum_{i=1}^{k} \lambda_i = 1, \ \ \lambda_i \geq 0.$$

This problem is sometimes solved in the form of Multi-attribute utility function of Keeney-Raiffa, [6]

$$\min_{x\in X} C \prod_{i=1}^{k} \Big(k_i u_i\big(f_i(x)\big) + 1\Big), \quad 0 \leq k_i \leq 1,$$

where u_i is a strictly non-decreasing function and $C > 0$ is a convenient constant. Many other forms of the above problem can be found in [3].

In this work, we will be interested by solving the above problem in the general form

$$\min_{x\in X} \varphi\big(f_1(x), .., f_k(x)\big), \tag{1}$$

where φ is the set of objective functions and X denotes the set of constraints, that is,

$$X = \{x \in \mathbb{R}^n / g_i(x) \leq 0, i = 1, .., m\}.$$

If there is equality constraints, that is,

$$X = \{x \in \mathbb{R}^n / u_i(x) \leq 0, i = 1, .., p; \ v_j(x) = 0, j = 1, .., k; \ p + k = m\},$$

or in the matrix form

$$X = \{x \in \mathbb{R}^n / Ax \leq b\},$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$, or as well known, with small processing

$$X = \{x \in \mathbb{R}^n / Ax = b, x \geq 0\}.$$

Now, let us consider the next scheme

$$\begin{array}{ccccc} & \phi & & \varphi & \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & (f_1(x), .., f_k(x)) & \longrightarrow & \varphi(f_1(x), .., f_k(x)), \end{array}$$

and let us set $F(x) = \varphi \circ \phi(x)$. Then our initial problem (1) is reduced to the following equivalent one

$$\min_{x\in X} F(x) \tag{2}$$

It is well-known that the solutions set of (2) is nonempty if, for example, one of the following statements holds:

• X is a nonempty compact subset of $\mathbb{R}^n$ and F is lower semi-continuous on X.

• X is a nonempty closed convex subset of $\mathbb{R}^n$, F is a convex lower semi-continuous functional on X and, X is bounded or F is coercive in the sense $F(x) \to +\infty$ when $\|x\| \to +\infty$.

Throughout the following, we assume that the problem (2) has many solutions.

Our fundamental goal here is to explore the sub-differential approach, in the sense of convex analysis, in the aim to characterize the solutions of (2) in a practical form. This characterization allows us to formulate the set solutions of (2) in three directions:

First, we write (2) in a linear programming form simple for practical purposes as well as for theoretical viewpoint.

Secondly, we characterize the solutions of (2) as fixed points of an explicit map having a geometric character.

Thirdly, we apply the known interior-point algorithm to our problem posed in the above linear form.

The above formulations will be illustrated by numerical examples showing the interest of the related approach.

2. Background Material and Preliminary Results

This section is devoted to introduce some basic definitions and results that will be needed throughout this paper. Let $\mathbb{R}^n$ be the finite dimensional space of real vectors. For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ we set

$$\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i,$$

the standard inner product of $\mathbb{R}^n$. Given a functional $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the notation $f^* : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ refers to the conjugate of f defined by

$$\forall y \in \mathbb{R}^n \qquad f^*(y) = \sup_{x\in\mathbb{R}^n} \Big\{\langle x, y\rangle - f(x)\Big\}.$$

If we denote by $\Gamma_0(\mathbb{R}^n)$ the cone of lower semi-continuous convex functionals from $\mathbb{R}^n$ into $\mathbb{R}\cup\{+\infty\}$ not identically equal to $+\infty$, we recall that $f^* \in \Gamma_0(\mathbb{R}^n)$

and, $f^{**} = f$ if and only if $f \in \Gamma_0(\mathbb{R}^n)$. As important example of functional taking the value $+\infty$ is Ψ_A, namely the indicator functional of the set $A \subset \mathbb{R}^n$, defined by $\Psi_A(x) = 0$ if $x \in A$ and $\Psi_A(x) = +\infty$ else. It is known that, $\Psi_A \in \Gamma_0(\mathbb{R}^n)$ if and only if A is a nonempty closed convex subset of $\mathbb{R}^n$. Further, Ψ_A^* conjugate of Ψ_A, is given by

$$\forall y \in \mathbb{R}^n \quad \Psi_A^*(y) = \sup_{x \in A} \langle x, y \rangle.$$

A special interest of the indicator functional is to reduce a minimization problem with constraints to a problem without constraints as shown by the following equivalence

$$\left(\inf_{x \in A} f(x) = f(x_0),\ x_0 \in A \right) \Longleftrightarrow \left(\inf_{x \in \mathbb{R}^n} (f + \Psi_A)(x) = f(x_0),\ x_0 \in A \right).$$

If we denote by *dom* f the effective domain of f defined by *dom* $f = \{x \in \mathbb{R}^n, \quad f(x) < +\infty\}$, the sub-differential of f at $x \in$ *dom* f is the (possibly empty) subset of $\mathbb{R}^n$ defined by

$$\partial f(x) = \{y \in \mathbb{R}^n;\ \forall z \in \mathbb{R}^n \ \ f(z) \geq f(x) + \langle y, z - x \rangle\}.$$

It is well-known that, for all $x \in$ *dom* f one has

$$y \in \partial f(x) \iff f(x) + f^*(y) = \langle x, y \rangle.$$

Recall that if f is convex and Gâteaux-differentiable (in short G-differentiable) at $x \in \mathbb{R}^n$, with gradient $Df(x)$, then $\partial f(x) = \{Df(x)\}$. Inversely, if f is convex and $\partial f(x)$ is a singleton then f is G-differentiable with $\partial f(x) = \{Df(x)\}$. The sub-differential concept stems its importance in the fact that it characterizes the solution of a minimization problem in the sense

$$\left(\inf_{x \in A} f(x) = f(x_0),\ x_0 \in A \right) \Longleftrightarrow 0 \in \partial(f + \Psi_A)(x_0).$$

With this it is well known that if f is convex and A is a nonempty closed convex with $int(dom\ f) \cap A \neq \emptyset$ (or $dom\ f \cap int(A) \neq \emptyset$) then $\partial(f + \Psi_A)(x_0) = \partial f(x_0) + \partial \Psi_A(x_0)$ and thus

$$\left(\inf_{x \in A} f(x) = f(x_0),\ x_0 \in A \right) \Longleftrightarrow 0 \in \partial f(x_0) + \partial \Psi_A(x_0).$$

For further details about the above notions and results, we refer the reader to [9] for instance.

3. Sub-Differential Approach

As already pointed out of, this section is devoted to characterize the solutions of the problem (2) via the sub-differential approach. We first notice that if the functional F is G-differentiable, with φ and ϕ also G-differentiable, then we have

$$DF(x) = \nabla\varphi(\phi(x))\nabla\phi(x) = \nabla\varphi(f_1(x),..,f_k(x))\nabla(f_1(x),..,f_k(x)),$$

where

$$\nabla(f_1(x),..,f_k(x)) = \begin{pmatrix} \dfrac{\partial f_1(x)}{\partial x_1} & \dfrac{\partial f_1(x)}{\partial x_2} & \cdots\cdots & \dfrac{\partial f_1(x)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_k(x)}{\partial x_1} & \dfrac{\partial f_k(x)}{\partial x_2} & \cdots\cdots & \dfrac{\partial f_k(x)}{\partial x_n} \end{pmatrix}.$$

Now, we are in position to recite our first main result which, under convenient hypothesis, gives a simple characterization of a solution of our problem. Precisely, we may state the following.

Theorem 1. *Assume that X is a nonempty closed convex subset of $\mathbb{R}^n$ and F is a convex and G-differentiable function on X. Then, x_0 is a solution of (2) if and only if one of the two following assertions holds:*

$$\inf_{x\in X} \langle DF(x_0), x\rangle = \langle DF(x_0), x_0\rangle. \tag{3}$$

$$\forall x \in X \quad \langle DF(x_0), x - x_0\rangle \geq 0. \tag{4}$$

Proof. First, it is easy to see that (3) and (4) are equivalent. Let x_0 be a solution of (2), i.e.

$$\inf_{x\in X} F(x) = F(x_0),$$

or equivalently

$$\inf_{x\in\mathbb{R}^n} (F + \Psi_X)(x) = F(x_0).$$

According to the above, with the fact that F is G-differentiable, this is equivalent to

$$0 \in \partial(F + \Psi_X)(x_0) = \partial F(x_0) + \partial\Psi_X(x_0) = DF(x_0) + \partial\Psi_X(x_0),$$

and so

$$-DF(x_0) \in \partial\Psi_X(x_0),$$

or again, since $x_0 \in X$,

$$\Psi_X^*(-DF(x_0)) = -\langle DF(x_0), x_0\rangle.$$

This, with the definition of Ψ_X^*, yields

$$\sup_{x\in X} \langle -DF(x_0), x\rangle = -\langle DF(x_0), x_0\rangle,$$

so proving the desired result. □

Corollary 2. *Let $X = B(a,r) = \{x \in \mathbb{R}^n, \quad \|x-a\| \le r\}$ be the closed ball of center a and radius $r > 0$. Then, x_0 is a solution of (2) if and only if*

$$r\|DF(x_0)\|_* = -\langle DF(x_0), a - x_0\rangle,$$

where $\|.\|_*$ denotes the dual norm of $\|.\|$.

Proof. Theorem 1, with a simple manipulation, yields

$$\sup_{x\in B(0,1)} \langle -DF(x_0), a + rx\rangle = -\langle DF(x_0), x_0\rangle,$$

or equivalently,

$$\sup_{x\in B(0,1)} r\langle -DF(x_0), x\rangle = \langle DF(x_0), a - x_0\rangle.$$

Summarizing, x_0 is a solution of (2) if and only if

$$r\|DF(x_0)\|_* = \langle DF(x_0), a - x_0\rangle,$$

which completes the proof. □

Remark 3. *Three classical norms of $\mathbb{R}^n$ are well-known*

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \ \ \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}, \ \ \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

It is not hard to see that $\|.\|_1$ and $\|\|_\infty$ are mutually dual, and $\|.\|_2$ is self-dual. Taking $X = B_1(0,1) := \{x \in \mathbb{R}^n, \quad \|x\|_1 \le 1\}$ in the above corollary, x_0 is a solution of (2) if and only if

$$\max_{1\le i\le n} |D_iF(x_0)| = -\langle D_iF(x_0), x_0\rangle,$$

where $\big(D_iF(x_0)\big)_{i=1}^n$ are the coordinates of $DF(x_0)$. See the example below for explicit computations.

Example 4. *Let* $n = 3$, $k = 2$,

$$\min_{x \in X} F(x) = f_1(x) - f_2(x),$$

where

$$f_1(x) = x_1^2 + x_2^2 + x_3^2\,,\; f_2(x) = x_1 - x_3 + 1,$$

and

$$X = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 /\, |x_1| + |x_2| + |x_3| \leq 1\}.$$

Then the solution satisfies

$$\max\{|2x_1 - 1|,\ |2x_2|,\ |2x_3 + 1|\} = -\big((2x_1 - 1)x_1 + 2x_2^2 + (2x_3 + 1)x_3\big).$$

Corollary 5. *Let* X *b a nonempty convex compact subset of* $\mathbb{R}^n$ *and let* $F : X \longrightarrow \mathbb{R}$ *be a convex function. Then,* x_0 *is a solution of (2) if and only if*

$$\inf_{x \in \mathcal{E}xt(X)} \langle DF(x_0), x\rangle = \langle DF(x_0), x_0\rangle,$$

where the notation $\mathcal{E}xt(X)$ *refers to the set of all extremal points of* X.

Proof. According to the above theorem, it is clearly equivalent to say that, x_0 is a solution of (2) if and only if

$$\sup_{x \in X} \langle -DF(x_0), x\rangle = -\langle DF(x_0), x_0\rangle.$$

Since X is convex and compact then, by Krein-Milman Theorem [9], X coincides with the closed convex hull of the set of all its extremal points, namely $X = \overline{co}\ \mathcal{E}xt(X)$. Furthermore, the function $x \longmapsto \langle -DF(x_0), x\rangle$ is continuous and linear (so convex). It follows that we can write

$$\sup_{x \in X} \langle -DF(x_0), x\rangle = \sup_{x \in \mathcal{E}xt(X)} \langle -DF(x_0), x\rangle.$$

The desired result follows after a simple reduction, so completes the proof. □

Remark 6. *If* X *is a convex polyhedron of* $\mathbb{R}^n$ *then the set of its extremal points is finite and reduced to the set of its vertex points. Let us get* $\mathcal{E}xt(X) = \{a_1, a_2, ..., a_r\}$ *with* $a_i \in \mathbb{R}^n$ *for* $i = 1, 2, ..., r$. *By virtue of Corollary 2, a solution* x_0 *of (2) is characterized, in this example, by the simple relationship*

$$\min_{1 \leq i \leq r} \langle DF(x_0), a_i\rangle = \langle DF(x_0), x_0\rangle.$$

This implies that, there exists an extremal point a_{i_0} of X such that

$$\langle DF(x_0), a_{i_0}\rangle = \langle DF(x_0), x_0\rangle,$$

that is to say, $DF(x_0)$ and $x_0 - a_{i_0}$ are orthogonal and thus two situations are possible:

(i) $DF(x_0) = 0$ i.e. x_0 is a critical point of F, or, $x_0 = a_{i_0}$ i.e. a_{i_0} is a solution of (2).

(ii) $DF(x_0)$ and $x_0 - a_{i_0}$ are linearly independent.

Corollary 7. *With the same assumption as in Theorem 1, x_0 is a solution of (2) if and only if x_0 is a fixed point of the map*

$$h(x) = Proj_X\big(x - DF(x)\big) := arg\min_{y\in X} \|y - x + DF(x)\|,$$

where $Proj_X(x)$ denotes the projection of x on the closed convex set X.

Proof. According to the above theorem, x_0 is a solution of (2) if and only if x_0

$$\forall x \in X \qquad \langle DF(x_0), x - x_0\rangle \geq 0.$$

Another formulation of this variational inequality takes the form of a subset inclusion given by, [8]

$$0 \in DF(x_0) + N_X(x_0) \tag{5}$$

where the notation $N_X(x)$ refers to the normal cone of X at a point x defined as follows

$$N_X(x) = \begin{cases} \emptyset & \text{if } x \notin X; \\ \{y;\ y^T(c - x) \leq 0, \forall c \in X\} & \text{if } x \in X. \end{cases}$$

Following [4], formulation (5) can also be rewritten as a fixed point of the next problem: find x_0 such that

$$x_0 = Proj_X(x_0 - DF(x_0)).$$

The proof of the corollary is completed. □

Example 8. *Let $n = 2$, $k = 4$,*

$$\min_{x\in X} F(x) = f_1(x) + f_2(x)f_3(x) - f_4(x),$$

where

$$f_1(x) = x_1^2 + x_2^2\ ,\ f_2(x) = x_1\ ,\ f_3(x) = x_2 + 1\ ,\ f_4(x) = x_2 - 5,$$

and

$$X = \{x = (x_1, x_2) \in \mathbb{R}^2 /\ 20 \le x_1 \le 90;\ 50 \le x_2 \le 150\}.$$

Clearly, the set of extremal points of X is given by

$$\mathcal{E}xt(X) = \{(20, 50),\ (20, 150),\ (90, 50),\ (90, 150)\}.$$

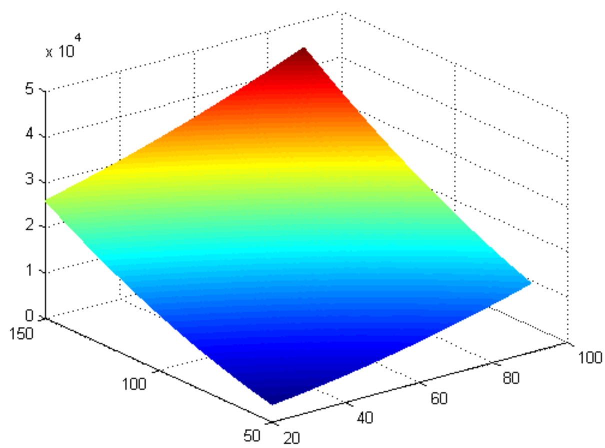

Figure 1: Graph of $F(x)$

Let $x_0 = (90, 50)$ *for which* $DF(x_0) = \begin{pmatrix} 231 \\ 189 \end{pmatrix}$, *and the problem is reduced to*

$$\inf_{x \in \mathcal{E}xt(X)} \langle DF(x_0), x \rangle = \langle DF(x_0), x_0 \rangle,$$

Then the optimum point is $(20, 50)$.

4. Numerical Approach: Interior-Point Algorithm

Consider again our initial problem (2). as we have seen in the above section, the set solutions of such problem has been characterized by (3). In the present section, we will apply a numerical method, namely the interior-point algorithm, for approaching the set solutions of (3) (and so that of (2). For further details

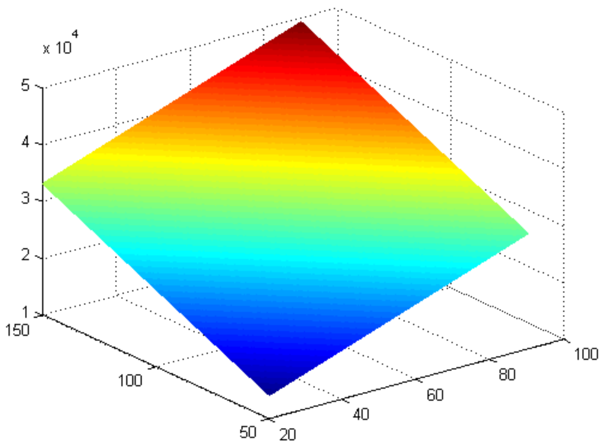

Figure 2: graph of $\langle DF(x_0), x\rangle$

about the principle of the interior-point approach, with related applications, the reader is referred to [5] for instance.

Let then write the linear minimization problem (3) according to said standard form,

$$(P)\begin{cases} \inf\ DF(x_0)^T x, \\ Ax = b, \\ x \geq 0, \end{cases}$$

where its admissible set is denoted by

$$X = \{x \in \mathbb{R}^n / Ax = b, x \geq 0\}.$$

In a first part, by the Lagrangian dualization, we obtain the maximization problem, usually called the dual problem of (P), defined as follows

$$(D)\begin{cases} \sup\ b^T y, \\ A^T y + z = DF(x_0), \\ z \geq 0, \end{cases}$$

with its admissible set

$$Y = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n / A^T y + z = DF(x_0), z \geq 0\}.$$

In a second part, to the primal problem (P) we associate another problem, called the penalized problem with a logarithmic barrier, given by

$$\begin{cases} \inf\ DF(x_0)^T x - \mu \sum_j ln\ x_j, \\ Ax = b, \\ x > 0, \end{cases}$$

where μ is a positive real number.

Using the principle of Karush-Kuhn-Tucker-condition [7], the optimality conditions of (P) say that x is a solution of this problem if and only if there exists a pair $(y, z) \in \mathbb{R}^m \times \mathbb{R}^n$ such that we have

$$\begin{cases} A^T y + z = DF(x_0),\ z \geq 0, \\ Ax = b,\ x \geq 0, \\ x^T z = \mu. \end{cases} \tag{6}$$

They are also the optimality conditions of the dual problem: (y, z) is a solution of (D) if and only if there exists $x \in \mathbb{R}^n$ such that (6) holds true.

To solve (6) using a Newton step request to have a correct number of equations, i.e.

$$\begin{cases} A^T y + z = DF(x_0),\ z \geq 0, \\ Ax = b,\ x \geq 0, \\ Xz = \mu e, \end{cases} \tag{7}$$

where $X = diag(x_1, ..., x_n)$ and $e = (1, ..., 1)$.

When $\mu = 0$ we find the optimality conditions of the original linear primal-dual problem.

The solutions $x(\mu)$ of the problem (7) describe the central path, application image $\mu \to s_\mu = (x_\mu, y_\mu, z_\mu)$ where $z_\mu = \mu X_\mu^{-1} e$ and (x_μ, y_μ) is a solution of the primal-dual problem. The point $x(\mu)$ is the analytic center of the primal polyhedron. When μ goes to 0, the central path converges, intuitively, to the central record of all optimal solutions of the linear program.

The interior-point algorithm consists to find a solution following an elusive goal along the central path, using a Newton step.

The Newton step $d = (dx, dy, dz)$ on (7) is well defined in s if $x > 0$ and $s > 0$, with

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} -A^T - z + DF(x_0) \\ -Ax + b \\ \mu e - Xz \end{pmatrix}, \tag{8}$$

where $S = diag(s_1, ..., s_n)$ and μ is defined by $\mu = \sigma\widetilde{\mu}(s)$, with $\sigma \in]0,1[$ and $\widetilde{\mu}(s)$ is the "closest" point to s in the central path, i.e.

$$\widetilde{\mu}(s) = \frac{x^T z}{n}.$$

So, we will content ourselves with a primal-dual solution (x, y, z) approximating the type predictor-corrector whose quality is satisfactory when in the vicinity of the central path. This neighborhood is the KKT-conditions where we replace the nonlinear constraints by the constraint lower aggregate.

Is considered close to the central path, the set defined by

$$\Lambda(\theta) := \{s \,/\, \|Xz - \widetilde{\mu}(s)e\|_2 \leq \theta\, \widetilde{\mu}(s),\ \ \widetilde{\mu}(s) > 0\}$$

The algorithm uses two neighborhoods of this type: $\Lambda(\frac{1}{4})$ and $\Lambda(\frac{1}{2})$.

We now are in position to state a description of an iteration of the algorithm summarized as follows:

Step 1 : *Prediction phase*

- Newton direction d, solution of (8), where $\mu = 0$,
- Take the biggest step $\alpha \in]0,1]$ such that $s + \alpha d \in \Lambda(\theta)$,
- Through iterated $s' = s + \alpha d$.

Step 2 : *Correction phase*

- Newton direction d', solution of (8) in $s = s'$, where $\mu = \widetilde{\mu}(s')$,
- New iterated $s_+ = s' + d'$ (unit step).

Remark 9. *As pointed out of in [5], the algorithm converges i.e.* $\widetilde{\mu}(s_k) \to 0$ *linearly, and for all* $\varepsilon > 0$ *there exists* $K = O(n^{\frac{1}{2}} log(\varepsilon^{-1}))$ *as* $\widetilde{\mu}(s_k) < \varepsilon\widetilde{\mu}(s_0)$, *when* $k \geq K$.

Finally, we end this section by illustrating our above study with numerical examples showing the interest of our approach.

Example 10. *Let us consider again the statement of Example 8. Executing a Matlab program for the interior-point algorithm, we find as solution the point* $(20, 50)$ *even from the third iteration. The objective function value is* 3865.

Example 11. *Let* $n = 3$, $k = 2$,

$$\min_{x \in X} F(x) = f_1(x) + 2f_2(x),$$

with

$$f_1(x) = x_1 - 3x_2 + 2\ ,\ f_2(x) = 2x_3 + x_2 - 3,$$

and

$$\begin{aligned} X &= \{x \in \mathbb{R}^3 /\ x_1 + 2x_2 = 5,\ x_1 + 3x_3 = 6,\ \textit{and}\ x_1, x_2, x_3 \geq 0\} \\ &= \{x \in \mathbb{R}^n,\ \ Ax = b,\ x \geq 0\}, \end{aligned}$$

with

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \qquad \textit{and} \qquad b = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Following the above, our problem is here reduced to the following

$$\begin{cases} \min \begin{pmatrix} 1 & -1 & 4 \end{pmatrix} x, \\ Ax = b, \\ x \geq 0, \end{cases}$$

Using a Matlab program for the interior-point algorithm, we find $(1, 2, 1.667)$ *as solution at* 6 *iterations, with the objective function value* 1.66666872811617.

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