International Journal of Pure and Applied Mathematics
Volume 70 No. 7 2011, 951-962

ON MULTILINEAR POLYNOMIALS IN PRIME RINGS WITH DERIVATIONS

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Abstract: Let R be a prime ring with extended centroid C and characteristic different from 2, d a nonzero derivation of R, $f(x_1,\dots,x_n)$ a nonzero multilinear polynomial over C and I a nonzero right ideal of R. If the mapping $x \mapsto [d^2(x), d(x)]$ is centralizing on $\{f(x_1,\dots,x_n)|x_1,\dots,x_n \in I\}$, then one of the following holds:

(1) d is inner derivation induced by an element $a \in Q$ such that $aI = (0) = a^2I$;

(2) $[[f(x_1,\dots,x_n), x_{n+1}]x_{n+2}] = 0$ is an identity for I.

AMS Subject Classification: 16W25, 16R50, 16N60
Key Words: prime ring, derivation, extended centroid, Martindale quotient ring

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Throughout the paper R will denote a prime ring with center $Z(R)$ and extended centroid C. For given $x, y \in R$, let $[x, y]_0 = x, [x, y]_1 = [x, y] = xy - yx$ and inductively for $k > 1$, $[x, y]_k = [[x, y]_{k-1}, y]$.

Let $S \subseteq R$. A mapping $F : R \to R$ is called centralizing on S if $[F(x), x] \in Z(R)$ holds for all $x \in S$. In particular, if $[F(x), x] = 0$ holds for all $x \in S$, then the mapping F is said to be commuting on S.

Received: April 3, 2011

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The study of centralizing and commuting mappings was initiated by Posner in [18]. Posner's result [18] states that existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. This result have been generalized by a number of authors in literature by several ways. In[12] Lanski generalized the Posner's result to a Lie ideal. More precisely Lanski proved that if L is a noncommutative Lie ideal of R such that a nonzero derivation d of R centralizing on L, then char $R = 2$ and R satisfies $s_4(x_1, x_2, x_3, x_4)$, the standard identity. Note that a noncommutative Lie ideal of R contains all the commutators $[x_1, x_2]$ for x_1, x_2 in some nonzero ideal of R (see [12, Lemma 2 (i), (ii)]). So, it is natural to study the situation when a nonzero derivation d centralizing on the set $\{[x_1, x_2]|x_1, x_2 \in I\}$ or more general case $\{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in I$ where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C and I is a nonzero two sided ideal of R. In [13] Lee and Lee proved that if a nonzero derivation d is centralizing on $\{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in I\}$, where I is a nonzero ideal of R, then $f(x_1, \ldots, x_n)$ is central-valued on R, except when char $R = 2$ and R satisfies $s_4(x_1, x_2, x_3, x_4)$. Recently, in [6], we proved that if char $(R) \neq 2$ and d is a nonzero derivation of R such that d^2 is centralizing on $\{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in \rho\}$, where ρ is a nonzero right ideal of R, then $\rho C = eRC$ for some idempotent element e in the socle of RC and either $f(x_1, \ldots, x_n)$ is central in $eRCe$ or $eRCe$ satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$ unless d is an inner derivation induced by $b \in Q$ such that $b^2 = 0$.

From this line of investigation, in the present paper, our aim is to study the case when d is a nonzero derivation of R such that the mapping $x \mapsto [d^2(x), d(x)]$ is centralizing on $\{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in \rho\}$, where ρ is a nonzero right ideal of R.

First, we fix some notations concerning quotient rings. Q denotes the two-sided Martindale's quotient ring of R and C denotes the center of Q, which is called the extended centroid of R. Note that Q is also a prime ring with C a field. We will make a frequent use of the following notation: $f(x_1, \ldots, x_n) = x_1x_2 \ldots x_n + \sum_{I \neq \sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$ for some $\alpha_\sigma \in C$ where S_n is the permutation group over n elements and we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write

$$d(f(x_1, \ldots, x_n) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)$$

and

$$d^2(f(x_1,\ldots,x_n)) = f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n)$$
$$+ 2\sum_{i<j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n) + \sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n).$$

1. The Case for $I = R$

First we study the case of matrix ring.

Lemma 1.1. *Let F be a field of characteristic $\neq 2$ and $R = M_k(F)$, the $k\times k$ matrix algebra over the field F. Suppose that $a \in R$ and $f(x_1,\ldots,x_n)$ is a multi-linear polynomial over F such that*

$$[[[a,[a,f(x)]],[a,f(x)]],f(x)] \in Z(R)$$

for all $x = (x_1,\ldots,x_n) \in R^n$. Then either $a \in F\cdot I_k$ or $f(x_1,\ldots,x_n)$ is central-valued on R.

Proof. Let $a = (a_{ij})_{k\times k}$. If $k = 1$, the result holds trivially. So assume that $k \geq 2$. We assume further that char $(F) \neq 2$ and $f(x_1,\ldots,x_n)$ is not central-valued on R.

Since $f(x_1,\ldots,x_n)$ is not central on R, by [16, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r = (r_1,\ldots,r_n)$ in R such that $f(r_1,\ldots,r_n) = \gamma e_{ij}$ with $0 \neq \gamma \in F$ and $i \neq j$. Since the set $f(R) = \{f(x_1,\ldots,x_n), x_i \in R\}$ is invariant under the action of all inner automorphisms of R, $f(r) = \gamma e_{ij}$ holds for any $i \neq j$. Thus

$$\begin{aligned}&[[[a,[a,f(r_1,\ldots,r_n)]],[a,f(r_1,\ldots,r_n)]],f(r_1,\ldots,r_n)]\\&=[[[a,[a,\gamma e_{ij}]],[a,\gamma e_{ij}]],\gamma e_{ij}]\\&=[[a^2\gamma e_{ij}-2a\gamma e_{ij}a+\gamma e_{ij}a^2,a\gamma e_{ij}-\gamma e_{ij}a],\gamma e_{ij}]\in Z(R).\end{aligned}$$

Commuting both sides with e_{ij} we obtain

$$\begin{aligned}0 &= [[a^2\gamma e_{ij}-2a\gamma e_{ij}a+\gamma e_{ij}a^2,a\gamma e_{ij}-\gamma e_{ij}a,\gamma e_{ij}],e_{ij}]\\ &= -8\gamma^3 a_{ji}^3 e_{ij}.\end{aligned}$$

This implies that $a_{ji} = 0$ for any $i \neq j$. Thus a is a diagonal matrix. Now for any F-automorphism θ of R, a^θ enjoy the same property as a does, namely,

$$[[[a^\theta,[a^\theta,f(x_1,\ldots,x_n)]],[a^\theta,f(x_1,\ldots,x_n)]],f(x_1,\ldots,x_n)] \in Z(R)$$

for all $x_1, \ldots, x_n \in R$. Hence, a^θ must be diagonal. Write, $a = \sum_{i=0}^{k} a_{ii}e_{ii}$; then for $s \neq t$, we have

$$(1 + e_{ts})a(1 - e_{ts}) = \sum_{i=0}^{k} a_{ii}e_{ii} + (a_{ss} - a_{tt})e_{ts}$$

diagonal. Hence, $a_{ss} = a_{tt}$ and so a is a scalar matrix that is $a \in F \cdot I_k$.

Theorem 1.3. *Let R be a prime ring of characteristic different from 2 and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C. Suppose that d is a nonzero derivation of R such that*

$$[[d^2(f(x_1, \ldots, x_n)), d(f(x_1, \ldots, x_n))], f(x_1, \ldots, x_n)] \in Z(R)$$

for all $x_1, \ldots, x_n \in R$. Then $f(x_1, \ldots, x_n)$ is central-valued on R.

Proof. Let $f(x_1, \ldots, x_n)$ be noncentral-valued on R. Assume first that d is Q-inner, i.e., $d(x) = [a, x]$ for all $x \in R$ and for some $a \in Q$. Since d is nonzero, $a \notin C$. Thus R satisfies the generalized polynomial identity

$$g(x_1, \ldots, x_{n+1}) = [[[[a, [a, f(x_1, \ldots, x_n)]], [a, f(x_1, \ldots, x_n)]], f(x_1, \ldots, x_n)], x_{n+1}].$$

Since $f(x_1, \ldots, x_n)$ is noncentral-valued on R and $a \notin C$, this is a nontrivial GPI. By Chuang [4] this GPI is also satisfied by Q. Since by [7, Theorem 2.5 and 3.5], Q and $Q \otimes_C \overline{C}$ are centrally closed and satisfy the same generalized polynomial identities, we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite, where $\overline{C}$ is the algebraic closure of C. By Martindale's theorem [17], R is a primitive ring having nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [10, p. 75], R is isomorphic to a dense ring of linear transformations on a vector space V over C. Assume first that V is finite dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$ with $k = \dim_C V$. By Lemma 1.1, we have $f(x_1, \ldots, x_n)$ is central-valued on R.

Assume next that V is infinite dimensional over C. Since V is infinite dimensional over C then as in lemma 2 in [19], the set $f(R)$ is dense on R and so from

$$[[[a, [a, f(r_1, \ldots, r_n)]], [a, f(r_1, \ldots, r_n)]], f(r_1, \ldots, r_n)] \in Z(R)$$

for all $r_1, \ldots, r_n \in R$, we have

$$[[[a, [a, r]], [a, r]], r] \in Z(R) \tag{1}$$

for all $r \in R$. Let e be an idempotent element of H. Replacing r with $er(1-e)$ in (1), we obtain

$$[[[a,[a,er(1-e)]],[a,er(1-e)]],er(1-e)] \in Z(R).$$

Commuting both sides with $er(1-e)$, we obtain

$$\begin{aligned} 0 &= [[[[a,[a,er(1-e)]],[a,er(1-e)]],er(1-e)],er(1-e)] \\ &= -8(er(1-e)a)^3 er(1-e). \end{aligned}$$

This implies that $0 = -8((1-e)aer)^5$ for all $r \in R$. Since char $R \neq 2$, by [8], it follows that $(1-e)aer = 0$ for any $r \in R$, implying $(1-e)ae = 0$. Similarly, replacing r with $(1-e)re$ in (1), we shall get $ea(1-e) = 0$. Thus for any idempotent $e \in H$, we have $(1-e)ae = 0 = ea(1-e)$ that is $[a,e] = 0$. Therefore, $[a,E] = 0$, where E is the additive subgroup generated by all idempotents of H. Since E is non central Lie ideal of H, this implies $a \in C$ (see [3, Lemma 2]), a contradiction.

Assume next that d is not Q-inner. Then R satisfies the differential identity

$$\begin{aligned} [[f^{d^2}(r_1,\ldots,r_n) + 2\sum_i f^d(r_1,\ldots,d(r_i),\ldots,r_n) \\ + 2\sum_{i<j} f(r_1,\ldots,d(r_i),\ldots,d(r_j),\ldots,r_n) \\ + \sum_i f(r_1,\ldots,d^2(r_i),\ldots,r_n), f^d(r_1,\ldots,r_n) \\ + \sum_i f(r_1,\ldots,d(r_i),\ldots,r_n)], f(r_1,\ldots,r_n)] \in Z(R). \end{aligned}$$

By Kharchenko's [11] theorem, R satisfies the polynomial identity

$$\begin{aligned} [[f^{d^2}(r_1,\ldots,r_n) + 2\sum_i f^d(r_1,\ldots,x_i,\ldots,r_n) \\ + 2\sum_{i<j} f(r_1,\ldots,x_i,\ldots,x_j,\ldots,r_n) \\ + \sum_i f(r_1,\ldots,y_i,\ldots,r_n), f^d(r_1,\ldots,r_n) \\ + \sum_i f(r_1,\ldots,x_i,\ldots,r_n)], f(r_1,\ldots,r_n)] \in Z(R). \end{aligned}$$

In particular, R satisfies blended component

$$[[\sum_i f(r_1,\dots,y_i,\dots,r_n),\sum_i f(r_1,\dots,x_i,\dots,r_n)],f(r_1,\dots,r_n)]\in Z(R).$$

In particular for $x_1=r_1$ and $x_2=\cdots=x_n=0$ we get R satisfies

$$[[\sum_i f(r_1,\dots,y_i,\dots,r_n),f(r_1,\dots,r_n)],f(r_1,\dots,r_n)]\in Z(R).$$

Let b be a noncentral element of R. Then replacing y_i with $[b,r_i]$, we have that

$$[\sum_{i=0}^{n} f(r_1,\dots,[b,r_i],\dots,r_n),f(r_1,\dots,r_n)]_2\in Z(R)$$

which gives,

$$[b,f(r_1,\dots,r_n)]_3=0$$

for all $r_1,\dots,r_n\in R$ implying $f(r_1,\dots,r_n)$ is central-valued on R by [13, Theorem].

2. The Case for *I*

We need the following lemmas.

Lemma 2.1. *Let I be a nonzero right ideal of R and d a derivation of R. Then the following conditions are equivalent:*

(i) d is an inner derivation induced by some $b\in Q$ such that $bI=0$;

(ii) $d(I)I=0$.

For its proof, we refer to [9].

Lemma 2.2. *Let R be a prime ring with char $(R)\neq 2$. If*

$$[[d^2(f(x_1,\dots,x_n)),d(f(x_1,\dots,x_n))],f(x_1,\dots,x_n)]\in Z(R)$$

for all $x_1,\dots,x_n\in I$, then either R satisfies a nontrivial generalized polynomial identity or d is an inner derivation induced by $a\in Q$ such that $aI=(0)=a^2I$.

Proof. Suppose, on the contrary, that R does not satisfy any nontrivial generalized polynomial identity. Then R must be noncommutative, otherwise R satisfies trivially a nontrivial GPI. Now we consider the following two cases:

Case I. Suppose that d is Q-inner derivation induced by an element $a \in Q$. Then for any $u \in I$,

$$[[a,[a,f(ux_1,\dots,ux_n)]],[a,f(ux_1,\dots,ux_n)],f(ux_1,\dots,ux_n),ux_{n+1}]$$

is a GPI for R, so it is the zero element in $Q *_C C\{x_1,x_2,\dots,x_{n+1}\}$. Denote $l_Q(I)$ the left annihilator of I in Q. Suppose first that $\{1,a,a^2\}$ are linearly C-independent modulo $l_Q(I)$, that is, $(\alpha a^2+\beta a+\gamma)I=0$ if and only if $\alpha=\beta=\gamma=0$. Since R is not a GPI-ring, a fortiori it cannot be a PI-ring. Thus, by [14, Lemma 3] there exists $u_0 \in I$ such that $\{a^2u_0, au_0, u_0\}$ are linearly C-independent. In this case, we have

$$\begin{aligned}&[a^2f(u_0x_1,\dots,u_0x_n)-2af(u_0x_1,\dots,u_0x_n)a+f(u_0x_1,\dots,u_0x_n)a^2,\\ &af(u_0x_1,\dots,u_0x_n)-f(u_0x_1,\dots,u_0x_n)a,f(u_0x_1,\dots,u_0x_n),u_0x_{n+1}]=0. \quad (2)\end{aligned}$$

In the expansion of it, it follows that
$a^2f(u_0x_1,\dots,u_0x_n)(af(u_0x_1,\dots,u_0x_n))f(u_0x_1,\dots,u_0x_n)u_0x_{n+1}$ appears non-trivially, a contradiction.

Therefore, $\{1,a,a^2\}$ are linearly C-dependent modulo $l_R(\rho)$, that is, there exist $\alpha,\beta,\gamma \in C$, not all zero, such that $(\alpha a^2+\beta a+\gamma)I=0$. Suppose that $\alpha=0$. Then $\beta \neq 0$, otherwise $\gamma=0$. Thus by $(\beta a+\gamma)I=0$, we have that $(a+\beta^{-1}\gamma)I=0$. Since a and $a+\beta^{-1}\gamma$ induce the same inner derivation, we may replace a by $a+\beta^{-1}\gamma$ in the basic hypothesis. Therefore, in any case we may suppose $aI=0$ and then by Lemma 2.1, $d(I)I=0$.

Next suppose that $\alpha \neq 0$. In this case there exist $\lambda,\mu \in C$ such that $a^2u_0=\lambda au_0+\mu u_0$ for all $u_0 \in I$. Choose $u_0 \in I$ such that au_0 and u_0 are linearly C-independent, otherwise we have again $aI=0$ and hence by Lemma 2.1, $d(I)I=0$. Replacing a^2u_0 with $\lambda au_0+\mu u_0$ in (2), we get, R satisfies

$$\begin{aligned}&[(\lambda a+\mu)f(u_0x_1,\dots,u_0x_n)-2af(u_0x_1,\dots,u_0x_n)a+f(u_0x_1,\dots,u_0x_n)a^2,\\ &af(u_0x_1,\dots,u_0x_n)-f(u_0x_1,\dots,u_0x_n)a,f(u_0x_1,\dots,u_0x_n),u_0x_{n+1}]=0.\end{aligned}$$

Since au_0 and u_0 are linearly C-independent,

$$\begin{gathered}\{(\lambda af(u_0x_1,\dots,u_0x_n)-2af(u_0x_1,\dots,u_0x_n)a)\\ (af(u_0x_1,\dots,u_0x_n)-f(u_0x_1,\dots,u_0x_n)a)\\ -af(u_0x_1,\dots,u_0x_n)((\lambda a+\mu)f(u_0x_1,\dots,u_0x_n)-2af(u_0x_1,\dots,u_0x_n)a\\ +f(u_0x_1,\dots,u_0x_n)a^2)\}f(u_0x_1,\dots,u_0x_n)u_0x_{n+1}=0.\end{gathered}$$

In above expression again replacing a^2u_0 with $\lambda au_0+\mu u_0$, we get that R satisfies

$$\begin{gathered}\{\lambda(af(u_0x_1,\dots,u_0x_n))^2-\lambda af(u_0x_1,\dots,u_0x_n)^2a\\ -2af(u_0x_1,\dots,u_0x_n)(\lambda a+\mu)f(u_0x_1,\dots,u_0x_n)-2a(f(u_0x_1,\dots,u_0x_n)a)^2\\ -af(u_0x_1,\dots,u_0x_n)(\lambda a+\mu)f(u_0x_1,\dots,u_0x_n)\\ +2(af(u_0x_1,\dots,u_0x_n))^2a-af(u_0x_1,\dots,u_0x_n)^2(\lambda a+\mu)\}\\ .f(u_0x_1,\dots,u_0x_n)u_0x_{n+1}=0,\end{gathered}$$

that is,

$$\begin{aligned}\{(-2\lambda)(af(u_0x_1,\dots,u_0x_n))^2-2\lambda af(u_0x_1,\dots,u_0x_n)^2a&\\ -4\mu af(u_0x_1,\dots,u_0x_n)^2\}f(u_0x_1,\dots,u_0x_n)u_0x_{n+1}&=0.\end{aligned}$$

Since char $(R)\neq 2$, we have

$$\begin{aligned}\{-\lambda(af(u_0x_1,\dots,u_0x_n))^2-\lambda af(u_0x_1,\dots,u_0x_n)^2a-2\mu af(u_0x_1,\dots,u_0x_n)^2\}&\\ \times f(u_0x_1,\dots,u_0x_n)u_0x_{n+1}=0&\end{aligned}$$

a trivial GPI for R. Since au_0 and u_0 are linearly C-independent, above relation yields $\lambda=0=\mu$. Therefore, $a^2u_0=0$ for all $u_0\in I$, i.e., $a^2I=(0)$.

Case II. Next suppose that d is an outer derivation. If for all $u\in I$, $d(u)\in uC$, then $[d(u),u]=0$ which implies that R is commutative (see [2]), a contradiction. Therefore there exists $u\in I$ such that $d(u)\notin uC$ i.e., u and $d(u)$ are linearly C-independent. By our assumption we have that R satisfies

$$\begin{gathered}[f^{d^2}(ux_1,\dots,ux_n)+2\sum_i f^d(ux_1,\dots,d(u)x_i+ud(x_i),\dots,ux_n)\\ +2\sum_{i<j}f(ux_1,\dots,d(u)x_i+ud(x_i),\dots,d(u)x_j+ud(x_j),\dots,ux_n)\\ +\sum_i f(ux_1,\dots,d^2(u)x_i+2d(u)d(x_i)+ud^2(x_i),\dots,ux_n),f^d(ux_1,\dots,ux_n)\\ +\sum_i f(ux_1,\dots,d(u)x_i+ud(x_i),\dots,ux_n),f(ux_1,\dots,ux_n)]\in Z(R).\end{gathered}$$

By Kharchenko's theorem [11],

$$\begin{gathered}[f^{d^2}(ux_1,\dots,ux_n)+2\sum_i f^d(ux_1,\dots,d(u)x_i+ur_i,\dots,ux_n)\\ +2\sum_{i<j}f(ux_1,\dots,d(u)x_i+ur_i,\dots,d(u)x_j+ur_j,\dots,ux_n)\\ +\sum_i f(ux_1,\dots,d^2(u)x_i+2d(u)r_i+us_i,\dots,ux_n),f^d(ux_1,\dots,ux_n)\end{gathered}$$

$$+\sum_i f(ux_1,\dots,d(u)x_i+ur_i,\dots,ux_n),f(ux_1,\dots,ux_n)]\in Z(R) \qquad (3)$$

for all $x_1,\dots,x_n,r_1,\dots,r_n,s_1,\dots,s_n\in R$. In particular, for $r_1=r_2=\ldots=r_n=0$,

$$\begin{aligned}[][f^{d^2}(ux_1,\dots,ux_n)+2\sum_i f^d(ux_1,\dots,d(u)x_i,\dots,ux_n)\\ +2\sum_{i<j} f(ux_1,\dots,d(u)x_i,\dots,d(u)x_j,\dots,ux_n)\\ +\sum_i f(ux_1,\dots,d^2(u)x_i+us_i,\dots,ux_n),f^d(ux_1,\dots,ux_n)\\ +\sum_i f(ux_1,\dots,d(u)x_i,\dots,ux_n),f(ux_1,\dots,ux_n)]\in Z(R) \qquad (4)\end{aligned}$$

which is a nontrivial GPI for R, because u and $d(u)$ are linearly C-independent, a contradiction.

Theorem 2.3. *Let R be a prime ring with extended centroid C and characteristic different from 2, d a nonzero derivation of R, $f(x_1,\dots,x_n)$ a nonzero multilinear polynomial over C and I a nonzero right ideal of R. If*

$$[[d^2(f(x_1,\dots,x_n)),d(f(x_1,\dots,x_n))],f(x_1,\dots,x_n)]\in Z(R)$$

for all $x_1,\dots,x_n\in I$, then one of the following holds:

(1) d is inner derivation induced by an element $a\in Q$ such that $aI=(0)=a^2I$;

(2) $[[f(x_1,\dots,x_n),x_{n+1}]x_{n+2}]=0$ is an identity for I.

Proof. Suppose that d is not Q-inner derivation induced by an element $a\in Q$ such that $aI=(0)$ and $a^2I=(0)$, otherwise conclusion (1) is obtained. Then by Lemma 2.1, $d(I)I\neq(0)$. Also let $[f(I),I]I=0$, that is, $[f(x_1,\dots,x_n),x_{n+1}]x_{n+2}=0$ is not an identity for I, otherwise conclusion (2) is obtained. Then by Lemma 2.2, R is a prime GPI-ring and so is Q (see [1] and [4]). Since Q is centrally closed over C, it follows from [17] that Q is a primitive ring with $H=Soc(Q)\neq 0$. Then $[f(IH),IH]IH\neq 0$. For otherwise, $[f(IQ),IQ]IQ=0$ by [1] and [4], a contradiction. Also, $d(IH)IH\neq 0$, otherwise $d(IQ)IQ=0$.

Choose $c_1,c_2,b_1,\dots,b_{n+2}\in IH$ such that $[f(b_1,\dots,b_n),b_{n+1}]b_{n+2}\neq 0$ and $d(c_1)c_2\neq 0$. Let $b\in IH$. Since H is a regular ring, there exists $e^2=e\in H$ such that $eH=bH+b_1H+\cdots+b_{n+2}H+c_1H+c_2H$. Then $e\in\rho H$ and $b=eb$, $b_i=eb_i$ for $i=1,\dots,n+2$, $c_j=ec_j$ for $j=1,2$. Then, we have $f(eHe)=f(eH)e\neq 0$. By [15, Theorem 2], since I and IQ satisfy the same differential identities, we may also assume that $[d^2(f(x_1,\dots,x_n)), d(f(x_1,\dots,x_n)), f(x_1,\dots,x_n), x_{n+1}]$ is an identity for IQ. In particular,

$$[d^2(f(x_1,\dots,x_n)), d(f(x_1,\dots,x_n)), f(x_1,\dots,x_n), x_{n+1}]$$

is an identity for IH and so for eH. It follows that, for all $r_1,\dots,r_n,r_{n+1}\in H$, $0=[d^2(f(er_1,\dots,er_n)), d(f(er_1,\dots,er_n)), f(er_1,\dots,er_n), r_{n+1}]$. Since $f(eHe) \neq 0$, we may write $f(x_1,\dots,x_n)=t(x_1,\dots,x_{n-1})x_n+h(x_1,\dots,x_n)$, where x_n never appears as last variable in any monomials of h and $t(eHe)\neq 0$. Let $r\in H$. Then replacing r_n with $r(1-e)$ we have

$$\begin{aligned}0=[&d^2(t(er_1,\dots,er_{n-1})er(1-e)), d(t(er_1,\dots,er_{n-1})er(1-e)),\\ &t(er_1,\dots,er_{n-1})er(1-e), er_{n+1}(1-e)].\end{aligned} \tag{5}$$

Now we know the fact that $d(x(1-e))e=-x(1-e)d(e)$, $(1-e)d(ex)=(1-e)d(e)ex$ and thus

$$\begin{aligned}(1-e)d^2(ex(1-e))e &= (1-e)d\Big\{d(e)ex(1-e)+ed(ex(1-e))\Big\}e\\ &= (1-e)d(e)d(ex(1-e))e+(1-e)d(e)d(ex(1-e))e\\ &= -2(1-e)d(e)ex(1-e)d(e).\end{aligned}$$

Using these facts, we get from (5) that

$$\begin{aligned}0=&-t(er_1,\dots,er_{n-1})er(1-e)[d^2(t(er_1,\dots,er_{n-1})er(1-e)),\\ &d(t(er_1,\dots,er_{n-1})er(1-e))]er_{n+1}(1-e)\\ &-er_{n+1}(1-e)[d^2(t(er_1,\dots,er_{n-1})er(1-e)),\\ &d(t(er_1,\dots,er_{n-1})er(1-e))]t(er_1,\dots,er_{n-1})er(1-e)\\ =&-t(er_1,\dots,er_{n-1})er(1-e)\{4(d(e)t(er_1,\dots,er_{n-1})er(1-e))^2d(e)\}er_{n+1}(1-e)\\ &-er_{n+1}(1-e)\{4(d(e)t(er_1,\dots,er_{n-1})er(1-e))^2d(e)\}t(er_1,\dots,er_{n-1})er(1-e).\end{aligned}$$

Replacing r_{n+1} with $t(er_1,\dots,er_{n-1})er$ in above relation, we get

$$0=-8\{t(er_1,\dots,er_{n-1})er(1-e)d(e)\}^3t(er_1,\dots,er_{n-1})er(1-e)$$

which implies

$$0=-8\{(1-e)d(e)t(er_1,\dots,er_{n-1})er\}^5$$

for all $r\in H$. Since char $(R)\neq 2$, by [8], $(1-e)d(e)t(er_1,\dots,er_{n-1})eH=0$ which implies $(1-e)d(e)t(er_1e,\dots,er_{n-1}e)=0$ for all $r_1,\dots,r_{n-1}\in H$. Since

eHe is a simple Artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [5, Lemma 2], $(1 - e)d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq IH$ and $d(b) = d(eb) \in d(eH) \subseteq IH$. This means that $d(IH) \subseteq IH$. It is easily seen that $d(l_H(IH)) \subseteq l_H(IH)$ holds and so d naturally induces a derivation δ on the prime ring $\overline{IH} = \frac{IH}{IH \cap l_H(IH)}$ defined by $\delta(\overline{x}) = \overline{d(x)}$ for $x \in IH$, where $l_H(IH)$ denotes the left annihilator of IH in H. Thus by assumption we have $[[[\delta^2(f(x_1, \ldots, x_n)), \delta(f(x_1, \ldots, x_n))], f(x_1, \ldots, x_n)], x_{n+1}]$ is a differential identity for $\overline{IH}$. By Theorem 1.3, either $\delta(\overline{IH}) = 0$ or $f(x_1, \ldots, x_n)$ is central-valued on $\overline{IH}$. If $\delta(\overline{IH}) = 0$, then $d(IH)IH = 0$, contradicting the choices of c_1, c_2. If $f(x_1, \ldots, x_n)$ is central-valued on $\overline{IH}$, then $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for IH, which contradicts with the choices of $b_1, \ldots, b_{n+2}$. Thus the proof of the theorem is complete.

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