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ON QUASI-(E, F)-CONVEX FUNCTIONS AND QUASI-(E, F)-CONVEX PROGRAMMING

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Abstract: In this paper, a new class of generalized convex functions, called quasi-(E,F)-convex functions, is introduced. Some important properties of it are developed. Further, we introduce quasi-(E,F)-convex programmings and investigate their optimal properties.

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1. Introduction

Convexity and generalized convexity play a significant role in optimization theory. The research on convexity and generalized convexity is popular for many years. For example, Finetti [1] first studied quasi-convexity in 1949. Mangassrian [2] brought forward pseudo-convexity in 1965. Youness [3] introduced E-convex sets and E-convex functions, discussed some of their basic properties and established some optimal results of E-convex programming, which is based on the effect of an operator E on the sets in 1999. By extending E-convexity, Jian [4] introduced (E, F)-convexity, which is based on the effect of two point-to-set maps E and F. Based on [4], Jian, Hu, Tang and Zheng [6] further introduced semi-(E, F)-convex functions and semi-(E, F)-convex programming. In addition, there are some new achievements in recent years (see [7]-[9]).

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In this paper, motivated by the idea in [4], we first introduce a new class of generalized convex function, i.e., quasi-(E, F)-convex functions, and discuss some important properties. And then we introduce quasi-(E,F)-convex programmings and investigate their optimal properties. For convenience, we first restate some definitions in [4] as follows.

Definition 1.1. A set $M \subseteq R^n$ is said to be an (E,F)-convex set, if there exist two point-to-set maps E, F : $R^n \to 2^{R^n}$ such that

$$\lambda E(x) + (1-\lambda)F(y) \subseteq M, \ \forall\ x, y \in M, \ \forall\ \lambda \in [0,1].$$

Definition 1.2. A map f : $R^n \to R$ is said to be an (E,F)-convex function on a set $M \subseteq R^n$ if there exist two point-to-set maps E, F : $R^n \to 2^{R^n}$ such that M is an (E,F)-convex set and

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \lambda f(\bar{x}) + (1-\lambda) f(\bar{y}),$$
$$\forall\ \bar{x} \in E(x), \ \forall\ \bar{y} \in F(y), \ \forall\ x, y \in M, \ \forall\ \lambda \in [0,1].$$

Definition 1.3. A point-to-set map E: $R^n \to 2^{R^n}$ is said to be a preservative map of set $M \subseteq R^n$ if $E(M) \subseteq M$, i.e., $E(x) \subseteq M$, $\forall\ x \in M$.

Definition 1.4. Given f: $R^n \to R$, $M \subseteq R^n$. A map E: $R^n \to 2^{R^n}$ is said to be a contractive map of function f on M, if

$$\sup f(E(x)) \leq f(x), \ \forall\ x \in M.$$

A map E : $R^n \to 2^{R^n}$ is said to be an expanding map of f on M if

$$\inf f(E(x)) \geq f(x), \ \forall\ x \in M.$$

Theorem 1.5. [4] *A set $M \subseteq R^n$ is a convex set if and only if M is an (E,F)-convex set for all preservative maps E and F of set M.*

2. Quasi-(E,F)-Convex Functions and their Properties

In this section, we present the definition of quasi-(E,F)-convex functions and discuss some main properties of quasi-(E,F)-convex functions.

Definition 2.1. A map $f : R^n \to R$ is said to be a quasi-(E,F)-convex function on a set $M \subseteq R^n$ if the two point-to-set maps E,F : $R^n \to 2^{R^n}$ ensure that M is an (E,F)-convex set and

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{f(\bar{x}), f(\bar{y})\},$$
$$\forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y),\ \forall\ x, y \in M,\ \forall\ \lambda \in [0,1].$$

Furthermore, if the inequality above is strict for $\bar{x} \neq \bar{y}$ and $\lambda \in (0,1)$, then f is called strictly quasi-(E,F)-convex on M.

The following proposition holds true clearly from the above definition.

Proposition 2.2. *Each (E,F)-convex function is also a quasi-(E,F)-convex function.*

Theorem 2.3. *Let $M \subseteq R^n$ be a convex set. Then a map $f : R^n \to R$ is a quasi-convex function on M if and only if f is a quasi-(E,F)-convex function on M for any preservative maps E, F of set M.*

Proof. Assume that f is quasi-convex on M, and $E, F : R^n \to 2^{R^n}$. Since E, F are preservative maps of M, we have $E(M) \subseteq M, F(M) \subseteq M$. One knows that M is (E, F)-convex set from Theorem 1.5. So we have

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{f(\bar{x}), f(\bar{y})\},\ \forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y),\ \forall\ x,\ y \in M,$$

therefore f is a quasi-(E, F)-convex function on M.

Conversely, if f is quasi-(E, F)-convex for any preservative maps E, F of set M, then, take maps $E = F$ as the identity map, one can know that f is quasi-convex on M. The proof is complete. □

Theorem 2.4. *Let functions f, g: $R^n \to R$ be quasi-(E,F)-convex on $M \subseteq R^n$, then for any constants $\alpha, \beta \geq 0$, the function $(\alpha f + \beta g)$ is quasi-(E,F)-convex on M.*

The proof is obvious and omitted.

Theorem 2.5. *Suppose that $M \subseteq R^n$ is an (E, F)-convex set and $\{f_i\}_{i\in I}$ is a family of numerical functions which are quasi-(E, F)-convex and bounded on M. Then the function $f(x) = \sup_{i\in I} f_i(x)$ is quasi-(E, F)-convex on M.*

Proof. For any $x, y \in M$, $\forall\ \lambda \in [0,1]$ and $\forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y)$, by using the quasi-$(E, F)$-convexity of f_i on M, one has

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) = \sup_{i\in I} f_i(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \sup_{i\in I}\{\max\{f_i(\bar{x}),\ f_i(\bar{y})\}\}. \quad (1)$$

In what follows, we will prove the following inequality holds true

$$\sup_{i\in I}\{\max\{f_i(\bar{x}),\ f_i(\bar{y})\}\} \leq \max\{\sup_{i\in I} f_i(\bar{x}),\ \sup_{i\in I} f_i(\bar{y})\}.$$

Noting that

$$f_i(\bar{x}) \leq \sup_{i\in I} f_i(\bar{x}),\ \ f_i(\bar{y}) \leq \sup_{i\in I} f_i(\bar{y}),\ \ \forall\ i \in I,$$

we have

$$\max\{f_i(\bar{x}),\quad f_i(\bar{y})\} \leq \max\{\sup_{i\in I} f_i(\bar{x}),\ \sup_{i\in I} f_i(\bar{y})\},$$

further, we obtain

$$\sup_{i\in I}\{\max\{f_i(\bar{x}),\quad f_i(\bar{y})\}\} \leq \max\{\sup_{i\in I} f_i(\bar{x}),\ \sup_{i\in I} f_i(\bar{y})\},$$

which together with (1) gives

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{\sup_{i\in I} f_i(\bar{x}),\ \sup_{i\in I} f_i(\bar{y})\} = \max\{f(\bar{x}),\ f(\bar{y})\}.$$

It indicates that $f(x) = \sup_{i\in I} f_i(x)$ is quasi-(E, F)-convex on M. □

Theorem 2.6. *Let function $f : R^n \to R$ be quasi-(E, F)-convex on an (E, F)-convex set $M \subseteq R^n$. If E and F are contractive maps of f on M, then, for any $\alpha \in R$, the level set $S_\alpha \overset{\triangle}{=} \{x \in M : f(x) \leq \alpha\}$ is an (E, F)-convex set.*

Proof. For any $x, y \in S_\alpha$, we have

$$x \in M,\ \ y \in M,\ \ f(x) \leq \alpha,\ \ f(y) \leq \alpha. \tag{2}$$

Noting that M is an (E, F)-convex set, we have $\lambda\bar{x} + (1-\lambda)\bar{y} \in M$ for any $\bar{x} \in E(x),\ \ \bar{y} \in E(y)$ and any $\lambda \in [0, 1]$.

Since E, F are contractive maps, we have

$$f(\bar{x}) \leq f(x) \leq \alpha,\ \ f(\bar{y}) \leq f(y) \leq \alpha. \tag{3}$$

By using the quasi-(E, F)-convexity of f and (3), one knows

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{f(\bar{x}), f(\bar{y})\} \leq \alpha, \tag{4}$$

Hence, $\lambda\bar{x} + (1-\lambda)\bar{y} \in S_\alpha$, i.e., S_α is an (E, F)-convex set. □

Theorem 2.7. *Let $M \subseteq R^n$ be an (E, F)-convex set. Suppose that level set $S_\alpha = \{x \in M : f(x) \leq \alpha\}$ for any $\alpha \in R$ is an (E, F)-convex set. If the maps E, F are expanding maps of f on M, then f is quasi-(E, F)-convex on M.*

Proof. For any $x, y \in M$, $\forall \bar{x} \in E(x), \bar{y} \in F(y)$, let $\alpha = \max\{f(\bar{x}), f(\bar{y})\}$. Since E, F are expanding maps, we have

$$f(x) \leq f(\bar{x}) \leq \alpha, \quad f(y) \leq f(\bar{y}) \leq \alpha.$$

So $x, y \in S_\alpha$. Further, for any $\lambda \in (0,1)$, we have $\lambda E(x) + (1-\lambda)F(y) \subseteq S_\alpha$ due to S_α is an (E, F)-convex set. Hence,

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \alpha = \max\{f(\bar{x}), f(\bar{y})\}.$$

It indicates that f is quasi-(E, F)-convex on M. □

3. Quasi-(E, F)-Convex Programming

In this section, some important properties of quasi-(E, F)-convex programmings are discussed. Consider the following nonlinear programming problem

$$\text{(NP)} \quad \min \ \{f(x) : x \in M\}.$$

where $f:\ R^n \to R$ is continuously differentiable and $M \subseteq R^n$.

Definition 3.1. The problem (NP) is said to be a quasi-(E, F)-convex programming if the feasible set M is an (E, F)-convex set and the objective function f is quasi-(E, F)-convex on M.

Furthermore, if f is strictly quasi-(E, F)-convex on M, then problem (NP) is called a strictly quasi-(E, F)-convex programming.

Another programming problem connected with (NP) is as follows

$$\begin{array}{lll} (\mathrm{NP}_{EF}) & \min & f(x) \\ & \text{s.t.} & x \in E(M) \cap F(M). \end{array}$$

Denote

$$\Omega = \{x \in R^n :\ x \text{ is a global optimal solution of (NP)}\},$$
$$\Omega_{EF} = \{x \in R^n :\ x \text{ is a global optimal solution of } (\mathrm{NP}_{\mathrm{EF}})\}.$$

Theorem 3.2. *Assume that problem (NP) is a quasi-(E, F)-convex programming. If E and F are preservative maps of Ω, then Ω is an (E, F)-convex set.*

Proof. $\forall\ x, y \in \Omega$ and $\forall\ \lambda \in [0,1]$, we have $x, y \in M$, further, $\lambda E(x) + (1-\lambda)F(y) \subseteq M$. Denote $f^* \triangleq f(x) = f(y)$.

In what follows, we will prove $\lambda E(x) + (1-\lambda)F(y) \subseteq \Omega$. It is sufficient to prove $\lambda\bar{x} + (1-\lambda)\bar{y} \in \Omega,\ \forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in E(y)$. Noting that E and F are preservative maps of Ω, we have $E(x) \subseteq \Omega,\ F(y) \subseteq \Omega$, so $f(\bar{x}) = f(\bar{y}) = f^*$.

Since f is quasi-(E,F)-convex on M, we have

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{f(\bar{x}), f(\bar{y})\} = f^*,\ \forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y).$$

Therefore, $\lambda\bar{x} + (1-\lambda)\bar{y}$ is a global solution, i.e., $\lambda\bar{x} + (1-\lambda)\bar{y} \in \Omega$. □

Theorem 3.3. *Suppose that problem (NP$_{EF}$) is a quasi-(E,F)-convex programming and $E(M) \cap F(M)$ is a convex set, then*
(1) *The solution set Ω_{EF} is a convex set;*
(2) *If problem (NP$_{EF}$) is a strictly quasi-(E,F)-convex programming and $\bar{x}$ is a local solution of (NP$_{EF}$), then $\bar{x}$ is the unique global solution.*

Proof. (1) For any $\bar{x},\ \bar{y} \in \Omega_{EF}$ and any $\lambda \in (0,1)$, we have $\bar{x} \in E(M) \cap F(M),\ \bar{y} \in E(M) \cap F(M)$. Since $E(M) \cap F(M)$ is a convex set, we can get $\lambda\bar{x} + (1-\lambda)\bar{y} \in E(M) \cap F(M)$, so there exist $x, y \in M$ such that $\bar{x} \in E(x), \bar{y} \in F(y)$. Let f^* be the optimal value of (NP_{EF}). Obviously, $f(\bar{x}) = f(\bar{y}) = f^*$. Since f is quasi-(E,F)-convex on M, we have

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \max\{f(\bar{x}), f(\bar{y})\} = f^*.$$

It means that $\lambda\bar{x}+(1-\lambda)\bar{y}$ is a global solution of (NP$_{EF}$), so $\lambda\bar{x}+(1-\lambda)\bar{y} \in \Omega_{EF}$, i.e., Ω_{EF} is a convex set.

(2) At first, we prove that $\bar{x}$ is a global solution of problem (NP$_{EF}$). Suppose, by contradiction, that $\bar{x}$ is not a global solution of problem (NP_{EF}). So there exists $\bar{y} \in E(M) \cap F(M)$ $(\bar{y} \neq \bar{x})$ such that $f(\bar{y}) < f(\bar{x})$. According to $\bar{x} \in E(M) \cap F(M),\ \bar{y} \in E(M) \cap F(M)$, there exist $x \in M, y \in M$, such that $\bar{x} \in E(x),\ \bar{y} \in F(y)$. Therefore, we have by the strict quasi-(E,F)-convexity of f

$$f(\lambda\bar{y} + (1-\lambda)\bar{x}) < \max\{f(\bar{y}), f(\bar{x})\} = f(\bar{x}),\quad \forall\ \lambda \in [0,1]. \tag{5}$$

Taking into account $\lambda\bar{y}+(1-\lambda)\bar{x} \in E(M) \cap F(M)$ and $\lambda\bar{y}+(1-\lambda)\bar{x} \to \bar{x}$ $(\lambda \to 0^+)$ we know that for λ small enough, the inequality (5) contradicts the fact that $\bar{x}$ is a local solution. Therefore, $\bar{x}$ is a global solution.

Secondly, we prove the uniqueness. If the conclusion is not true, then there exists $\widehat{y} \in E(M) \cap F(M)$ $(\widehat{y} \neq \bar{x})$ such that $f(\widehat{y}) = f(\bar{x})$. Further, there exist

$x, y \in M$ such that $\bar{x} \in E(x), \widehat{y} \in F(y)$. Since f is a strictly quasi-(E,F)-convex, we have $f(\lambda\widehat{y} + (1-\lambda)\bar{x}) < \max\{f(\widehat{y}), f(\bar{x})\} = f(\bar{x})$. Noting that $\lambda\widehat{y} + (1-\lambda)\bar{x} \in E(M) \cap F(M)$. This contradicts the fact that $\bar{x}$ is an optimal solution. Thus, $\bar{x}$ is the unique global solution. □

Theorem 3.4. *Let $M \subseteq R^n$ be an (E,F)-convex set, x^* be a local solution of (NP). If f is strictly quasi-(E,F)-convex and E, F are contractive maps on M, $x^* \in E(x^*)$, then x^* is the unique global optimal solution of problem (NP).*

Proof. We first prove that x^* is a global solution. If it is not true, then there exists $y^* \in M (y^* \neq x^*)$ such that $f(y^*) < f(x^*)$. Since f is a strict quasi-(E,F)-convex function and E, F are contractive maps on M, one has for any $\bar{x} \in E(x^*),\ \bar{y} \in F(y^*)$ and $\lambda \in (0,1)$

$$f(\lambda\bar{y} + (1-\lambda)\bar{x}) < \max\{f(\bar{x}), f(\bar{y})\} \leq \max\{f(x^*), f(y^*)\} \leq f(x^*). \tag{6}$$

Noting that $x^* \in E(x^*)$, we take $\bar{x} = x^*$ and obtain $\lambda\bar{y} + (1-\lambda)x^* \in M,\ f(\lambda\bar{y} + (1-\lambda)x^*) < f(x^*)$. By taking $\lambda \to 0^+$, we can get $\lambda\bar{y} + (1-\lambda)x^* \to x^*$. This contradicts the fact that x^* is a local solution. Hence, x^* is a global solution.

Secondly, we prove the uniqueness. If there exists y^* $(y^* \neq x^*)$ such that $f(y^*) = f(x^*)$. Similar to (6), we have for any $\bar{x} \in E(x^*),\ \bar{y} \in F(y^*)$ and $\lambda \in (0,1)$

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) < \max\{f(\bar{x}), f(\bar{y})\} \leq \max\{f(x^*), f(y^*)\} = f(x^*). \tag{7}$$

Since M is an (E,F)-convex set, $\lambda\bar{x} + (1-\lambda)\bar{y} \in M$. So the inequality (7) contradicts the fact that x^* is a global solution. Thus, x^* is the unique global optimal solution of problem (NP). □

4. Optimality Conditions

In this section, we further extend the concept of quasi-(E,F)-convex function, then discuss the optimality conditions of the corresponding programming. We will consider the following nonlinear programming problems with inequality constraints:

$$(P_g) \qquad \min \quad f(x)$$
$$\text{s.t.} \quad g_i(x) \leq 0,\ i \in I = \{1, 2, ..., m\},$$

where $f,\ g_i\ (i \in I)$ are continuously differentiable on R^n.

Denote the feasible set of (P_g) by $M_g = \{x \in R^n :\ g_i(x) \leq 0,\ i \in I\}$.

Definition 4.1. Let M be a nonempty open set in R^n and $f : M \to R$ be differentiable on M. The function f is said to be pseudo-(E, F)-convex on M if for each $x,\ y \in M,\ \forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y)$ with $\nabla f(\bar{x})^T(\bar{y} - \bar{x}) \geq 0$ we have $f(\bar{y}) \geq f(\bar{x})$; or, equivalently, if $f(\bar{y}) < f(\bar{x})$, then $\nabla f(\bar{x})^T(\bar{y} - \bar{x}) < 0$.

Theorem 4.2. *Let M be a nonempty open and (E, F)-convex set in R^n. If f is (E, F)-convex on M, then f also is pseudo-(E, F)-convex on M.*

Proof. Suppose that $\nabla f(\bar{x})^T(\bar{y} - \bar{x}) \geq 0$, for any $x,\ y \in M,\ \forall\ \bar{x} \in E(x),\ \forall\ \bar{y} \in F(y)$.

Since f is (E, F)-convex on M, we have

$$f(\lambda\bar{y} + (1 - \lambda)\bar{x}) \leq \lambda f(\bar{y}) + (1 - \lambda)f(\bar{x}),\ \forall\ \lambda \in (0, 1). \tag{8}$$

By Taylor expansion, we have

$$f(\lambda\bar{y} + (1 - \lambda)\bar{x}) = f(\bar{x}) + \lambda\nabla f(\bar{x})^T(\bar{y} - \bar{x}) + o(\lambda). \tag{9}$$

We obtain from (8) and (9)

$$f(\bar{y}) - f(\bar{x}) \geq \nabla f(\bar{x})^T(\bar{y} - \bar{x}) + \frac{o(\lambda)}{\lambda}.$$

Taking $\lambda \to 0^+$, we obtain

$$f(\bar{y}) - f(\bar{x}) \geq \nabla f(\bar{x})^T(\bar{y} - \bar{x}) \geq 0.$$

So $f(\bar{y}) \geq f(\bar{x})$, i.e., f is pseudo-(E, F)-convex on M. □

Theorem 4.3. (KKT Sufficient Conditions) Let x^* be a KKT point of problem (P_g). Assume that there exists an ε-neighborhood $N_\varepsilon(x^*)$ about x^*, $\varepsilon > 0$, such that the function f is pseudo-(E, F)-convex and $g_i(x)(i = 1, 2, ..., m)$ are quasi-(E, F)-convex over $N_\varepsilon(x^*)$. If E is a contractive map of function f and g_i $(i \in I)$ on M_g and $x^* \in F(x^*)$, then x^* is a local optimal solution of problem (P_g).

Proof. For any $x \in M_g \cap N_\varepsilon(x^*)$, we have $g_i(x) \leq 0 = g_i(x^*),\ i \in I(x^*) = \{i \in I : g_i(x^*) = 0\}$. Since E is a contractive map of g_i, we have $g_i(\bar{x}) \leq g_i(x),\ \forall \bar{x} \in E(x)$. From the quasi-$(E, F)$-convexity of g_i and $x^* \in F(x^*)$, we obtain

$$g_i(\lambda\bar{x} + (1 - \lambda)x^*) \leq \max\{\ g_i(\bar{x}),\ g_i(x^*)\} = 0, \quad \forall\lambda \in (0, 1), \quad i \in I(x^*).$$

On the other hand, by the differentiability of $g_i(x)$, one has

$$0 \geq g_i(\lambda\bar{x} + (1 - \lambda)x^*) = g_i(x^*) + \lambda\nabla g_i(x^*)^T(\bar{x} - x^*) + o(\lambda).$$

Noting that $g_i(x^*) = 0$ for $i \in I(x^*)$, one gets

$$\lambda \nabla g_i(x^*)^T (\bar{x} - x^*) + o(\lambda) \leq 0, \quad i \in I(x^*).$$

Dividing the above inequality by λ and taking $\lambda \to 0^+$, we have

$$\nabla g_i(x^*)^T (\bar{x} - x^*) \leq 0, \quad i \in I(x^*).$$

Since x^* is a KKT point, there exist multipliers $u_i \geq 0,\ i \in I(x^*)$ such that

$$\nabla f(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0,$$

further, we obtain

$$\nabla f(x^*)^T (\bar{x} - x^*) = - \sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T (\bar{x} - x^*) \geq 0.$$

Since f is pseudo-(E, F)-convex and E is a contractive map of f, we can conclude $f(x) \geq f(\bar{x}) \geq f(x^*)$. Therefore, x^* is a local optimal solution of (P_g). The proof is complete. □

Corollary 4.4. *Let x^* be a KKT point of problem (P_g). Suppose that $f(x)$, $g_i(x)$ $(i \in I)$ are all (E, F)-convex functions on $N_\varepsilon(x^*)$. If $x^* \in E(x^*) \cup F(x^*)$, E, F are contractive maps of function f and g_i $(i \in I)$ on M_g, then x^* is a local optimal solution of (P_g).*

Proof. We have $x^* \in E(x^*)$ or $x^* \in F(x^*)$ from $x^* \in E(x^*) \cup F(x^*)$.

Case A. $x^* \in F(x^*)$. The conclusion holds true from Theorem 4.3.

Case B. $x^* \in E(x^*)$.

For any $x \in M_g \cap N_\varepsilon(x^*)$, from Theorem 4.3, we know $g_i(x)$ $(i \in I)$ are quasi-(E, F)-convex functions, thus,

$$g_i(\lambda \bar{x} + (1 - \lambda) x^*) \leq \max\{g_i(\bar{x}),\ g_i(x^*)\}, \quad x^* \in E(x^*), \quad \forall\, \bar{x} \in F(x).$$

Since E, F are contractive maps of function g_i $(i \in I)$ on M_g, we have

$$\max\{g_i(\bar{x}),\ g_i(x^*)\} \leq \max\{g_i(x),\ g_i(x^*)\} = 0,\ i \in I(x^*).$$

Therefore,

$$0 \geq g_i(\lambda \bar{x} + (1 - \lambda) x^*) = g_i(x^*) + \lambda \nabla g_i(x^*)^T (\bar{x} - x^*) + o(\lambda)$$

$$= \lambda \nabla g_i(x^*)^T(\bar{x} - x^*) + o(\lambda),\ i \in I(x^*).$$

Dividing by λ and taking $\lambda \to 0^+$, one gets

$$\nabla g_i(x^*)^T(\bar{x} - x^*) \leq 0,\ i \in I(x^*).$$

Since x^* is a KKT point of (P_g), we have

$$\nabla f(x^*)^T(\bar{x} - x^*) = -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(\bar{x} - x^*) \geq 0.$$

Noting that the (E, F)-convexity of f, one knows from Theorem 4.2 that f is quasi-(E, F)-convex, so $f(\bar{x}) \geq f(x^*)$. On the other hand, one has $f(\bar{x}) \leq f(x)$, $\forall \bar{x} \in E(x)$ due to the fact that F is a contractive map of function f on M_g. Therefore,

$$f(x^*) \leq f(x).$$

It indicates that x^* is a local optimal solution of (P_g). The proof is complete. □

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