International Journal of Pure and Applied Mathematics
Volume 70 No. 7 2011, 983-991

A PRESENTATION OF $Aut\,(L_{2,3})$

Özge Öztekin
Çukurova University
Balcalı, 01330, Adana, TURKEY

Abstract: We give a set of generators for $Aut\,(L_{2,3})$ and we find a presentation of $Aut\,(L_{2,3})$ by using this generating set.

AMS Subject Classification: 17B01, 17B40
Key Words: free metabelian Lie algebra, automorphism group, group presentation

1. Introduction

For any positive integer $n \geq 2$, let F_n be the free Lie algebra of rank n generated by $X = \{x_1, x_2, ..., x_n\}$ over a field K of characteristic zero. In 1964, Cohn [5] proved that the automorphism group $Aut\,(F_n)$ of F_n is generated by elementary automorphisms. This result is similar to the well-known result in group theory due to Nielsen [9] the corresponding problem for generators of the automorphism groups of free nilpotent groups and free metabelian groups has been studied by Andreadakis [1] and [2], Bachmut [3], Goryaga [7], Gupta [8], Bryant and Gupta [4]. The problem of finding minimal generating sets for the automorphism groups of free nilpotent Lie algebras has been studied by Drensky and Gupta [6]. They gave a minimum set of generators for $Aut\,(L_{n,c})$, the automorphism group of the free nilpotent Lie algebra of class c and rank n, where $n \geq c \geq 2$.

In this work we find a presentation of the automorphism group $Aut\,(L_{2,3})$ of a free nilpotent Lie algebra of rank 2 and class c.

Let F_2 be the free Lie algebra generated by $\{x_1, x_2\}$ over a constructive field K. By GL_2 we denote the general linear group on the subspace of F_2

Received: April 27, 2011

spanned by $\{x_1, x_2\}$. For any positive integer c, let $L_{2,c} = F_2 \diagup \gamma_{c+1}(F_2)$, where $\gamma_{c+1}(F_2)$ is the $(c+1)$th term of the lower central series of F_2. Then $L_{2,c}$ is a free nilpotent Lie algebra of rank 2 and class c. It is clear that $\{x_1 + \gamma_{c+1}(F_n), x_2 + \gamma_{c+1}(F_n)\}$ is a free generating set of $L_{2,c}$. We shall write $x_i + \gamma_{c+1}(F_n)$ as x_i and consider x_i is also an element of $L_{2,c}$.

2. Generators of $Aut(L_{2,3})$

Let F_2 be the free Lie algebra of rank 2 freely generated by $X = \{x_1, x_2\}$ over a constructive field K of characteristic zero. We denote by the following automorphisms:

$$\tau : \begin{array}{l} x_1 \to x_2 \\ x_2 \to x_1 \end{array}, \quad \mu_\lambda : \begin{array}{l} x_1 \to \lambda x_1 \\ x_2 \to x_2 \end{array}, \lambda \in K/\{0\} \quad \upsilon : \begin{array}{l} x_1 \to x_1 + x_2 \\ x_2 \to x_2 \end{array}$$

There is a canonical isomorphism of the groups $Aut(F_2)$ and GL_2, and it is easy to state the presentation of $Aut(F_2)$. We consider the following well known result.

Theorem 2.1. *$GL_2 = \langle \tau, \mu_\lambda, \upsilon \rangle$ with defining relations:*

(1) $\tau^2 = 1$, (2) $(\mu\tau)^4 = 1$, (3) $(\tau\mu\tau\upsilon)^2 = 1$, (4) $(\mu\upsilon)^2 = 1$,

(5) $(\upsilon\tau\mu)^6 = 1$, (6) $(\mu)^2 = 1$, (7) $[[\upsilon, \mu], \upsilon] = 1$, where $\mu = \mu_{-1}$.

If v_1, v_2 are arbitrary elements of $\gamma_2(L_{2,c})$, then there is a unique automorphism ϕ of $(L_{2,c})$ such that $\phi(x_i) = x_i + v_i$, $i = 1, 2$. The group of all automorphisms of this form will be denoted by $IA(L_{2,c})$. The elements of $IA(L_{2,c})$ have often been called IA-automorphism. It is known that $Aut(L_{2,c})$ is the split extension of $IA(L_{2,c})$ by GL_2 [see 6]. It is obvious that $Aut(L_{2,1}) \cong GL_2$.

In 1990, Drensky and Gupta in [6], proved that for all $n \geq c \geq 2$ $Aut(L_{n,c})$ is generated by GL_n and one more automorphism ζ defined by $\zeta(x_1) = x_1 + [x_1 + x_2]$, $\zeta(x_2) = x_2$. Hence as a consequence of their result we have:

Theorem 2.2. *$Aut(L_{2,2}) = \langle GL_2, \zeta \rangle$.*

Definition 2.3. An automorphisms θ of $L_{2,c}$ of the form $\theta : x_i \to x_i + u_i$, where $u_i \in \gamma_c(L_{2,c})$, $i = 1, 2$ is called central automorphism.

The following two propositions can be obtained by straightforward calculations.

Proposition 2.4. *Let $\phi : x_i \to x_i + u_i$, where $u_i \in \gamma_c(L_{2,c})$, $i = 1, 2$, be a central automorphism of $L_{2,c}$ and $a : x_i \to x_i + \omega_i$, $i = 1, 2$, an IA-automorphism of $L_{2,c}$. Then $\alpha\phi = \phi\alpha$.*

Proposition 2.5. *Any commutator in* $Aut\left(L_{2,c}\right)$ *including a repeated central automorphism is trivial.*

Suppose that $x_{i_1}, x_{i_2} \in \{x_1, x_2\}$. We consider the following automorphisms of $L_{2,c}$:

$$\zeta_{i_1 i_2 i_3} : \begin{array}{l} x_1 \to x_1 + [x_{i_1}, x_{i_2}, x_{i_3}] \\ x_2 \to x_2 \end{array} \quad \text{and} \quad \zeta : \begin{array}{l} x_1 \to x_1 + [x_1, x_2] \\ x_2 \to x_2 \end{array}$$

$$\eta_{i_1 i_2 i_3} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + [x_{i_1}, x_{i_2}, x_{i_3}] \end{array} \quad \text{and} \quad \eta : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + [x_1, x_2] \end{array}$$

Since $L_{n,c}$ is naturally isomorphic to $L_{n,c+1} \diagup \gamma_{c+1}\left(L_{n,c+1}\right)$, there is a natural epimorphism $\phi : Aut\left(L_{n,c+1}\right) \to Aut\left(L_{n,c}\right)$.

We have the following lemma.

Lemma 2.6. *The kernel of* ϕ *consists of all central automorphisms of* $L_{n,c+1}$ *and it is generated by all automorphisms of the form* $\varsigma^{-1}\zeta_{i_1 i_2 \ldots i_c}\varsigma$, *where* $\varsigma \in GL_n$.

Proof. If $a \in \ker\phi$ then

$$a : x_i \to x_i + v_i^1 + v_i^2 + ... + v_i^{c+1}, 1 \leq i \leq n.$$

for some homogeneous elements $v_i^k \in L_{n,c+1}$ of length k, where $1 \leq k \leq c+1$. Since $\phi\left(a\right) \in Aut\left(L_{n,c}\right)$ and ϕ is a canonical homomorphism, we have that $v_i^{c+1} = 0$ in $L_{n,c}$. Thus,

$$a : x_i \to x_i + v_i^1 + v_i^2 + ... + v_i^c, 1 \leq i \leq n.$$

Since $\phi\left(a\right)\left(x_i\right) = x_i$ we obtain

$$v_i^1 + v_i^2 + ... + v_i^c = 0, where 1 \leq i \leq n.$$

Therefore we get $v_i^k = 0$ for $1 \leq k \leq c$. Thus α is of the form $a : x_i \to x_i + v_i^{c+1}$, and hence it is central.

Conversely let α be a central automorphism of $L_{n,c+1}$. Then it is of the form

$$a : x_i \to x_i + v_i^{c+1},\ 1 \leq i \leq n,$$

where $v_i^{c+1} \in L_{n,c+1}$ are homogeneous elements of length $c+1$.

By the definition of ϕ

$$\phi\left(a\right) : x_i \to x_i,\ 1 \leq i \leq n.$$

Thus $\phi(a) = 1$ in $Aut(L_{n,c})$. Hence $\alpha \in \ker\phi$. Now we consider the permutation $\delta = (1, i)$ and we calculate $\varsigma^{-1}\zeta_{\delta(1)\delta(2)...\delta(i)...\delta(n)}\varsigma$:

$$\varsigma^{-1}\zeta_{\delta(1)\delta(2)...\delta(i)...\delta(n)}\varsigma(x_1) = x_1 + [[x_1, x_2], ..., x_i, ..., x_n]$$
$$\varsigma^{-1}\zeta_{\delta(1)\delta(2)...\delta(i)...\delta(n)}\varsigma(x_j) = x_j, \;\; j \neq 1$$

Assume that the elements v_i^{c+1} are as the following:

$$v_i^{c+1} = v_{i_1}^{c+1} + v_{i_2}^{c+1} + ... + v_{i_k}^{c+1},$$

where $v_{i_k}^{c+1}$ ($k = 1, 2, ..., s$) are monomials of length $c + 1$.

Let $\varpi : x_i + v_{i_t}^{c+1}$ $(1 \leq i \leq n)$. We use induction on k to show that $a = \varpi_k...\varpi_2\varpi_1$

Let $k = 2$. Then

$$\varpi_2\varpi_1(x_i) = \varpi_2\left(x_i + v_{i_1}^{c+1}\right) = x_i + v_{i_2}^{c+1} + \varpi_2\left(v_{i_1}^{c+1}\right) = x_i + v_{i_2}^{c+1} + v_{i_1}^{c+1}.$$

Now assume that $\varpi_{k-1}...\varpi_2\varpi_1 : x_i \to x_i + v_{i_{k-1}}^{c+1} + ... + v_{i_2}^{c+1} + v_{i_1}^{c+1}$. By definition we have

$$\varpi_k\varpi_{k-1}...\varpi_2\varpi_1 : x_i \to x_i + v_{i_k}^{c+1} + \varpi_{k-1}...\varpi_2\varpi_1\left(v_{i_{k-1}}^{c+1} + ... + v_{i_1}^{c+1}\right),$$
$$\varpi_k\varpi_{k-1}...\varpi_2\varpi_1 : x_i \to x_i + v_{i_k}^{c+1} + v_{i_{k-1}}^{c+1} + ... + v_{i_1}^{c+1}.$$

Hence we get $\varpi_k\varpi_{k-1}...\varpi_2\varpi_1(x_i) = a(x_i)$ for $1 \leq i \leq n$ and from this we obtain $a = \varpi_k\varpi_{k-1}...\varpi_2\varpi_1$.

Now we show that all elements of $\ker\phi$ can be written as a product of automorphisms of the form $\varsigma^{-1}\zeta_{\delta(1)\delta(2)...\delta(i)...\delta(n)}\varsigma$.

If $a \in \ker\phi$, then it can be written as $a : x_i \to x_i + v_{i_1}^{c+1} + ... + v_{i_{k-1}}^{c+1} + v_{i_k}^{c+1}$. Thus $a = \varpi_k\varpi_{k-1}...\varpi_2\varpi_1$. And from $\varsigma^{-1}\varpi_t\varsigma : x_i \to x_i + v_{i_t}^{c+1}$, $(1 \leq t \leq k)$, it is seen that $a = \left(\varsigma^{-1}\varpi_k\varsigma\right)\left(\varsigma^{-1}\varpi_{k-1}\varsigma\right)...\left(\varsigma^{-1}\varpi_1\varsigma\right)$.

Theorem 2.7. *$Aut(L_{2,3}) = \langle GL_2, \zeta, \zeta_{121}\rangle$.*

Proof. Let $\phi : Aut(L_{2,3}) \to Aut(L_{2,2})$ be the natural epimorphism. Note that $\phi(GL_2) = GL_2$ and $\phi(\zeta) = \overline{\zeta}$, where $\overline{\zeta}$ is defined for $L_{2,2}$ in the same way as ζ is defined for $L_{2,3}$. Thus $Aut(L_{2,3})$ is generated by GL_2, ζ and $\ker\phi$. Let $H = \langle GL_2, \zeta, \zeta_{121}\rangle$. We shall prove that $\ker\phi \subseteq H$

Let $\beta \in Aut(L_{2,3})$ Then it is of the form:

$$\beta : \begin{array}{l} x_1 \to ax_1 + bx_2 + c[x_1, x_2] + d[[x_1, x_2], x_1] + e[[x_1, x_2], x_2] \\ x_2 \to a'x_1 + b'x_2 + c'[x_1, x_2] + d'[[x_1, x_2], x_1] + e'[[x_1, x_2], x_2] \end{array}$$

where $a, b, c, d, e, a', b', c', d', e' \in K$ and $ab' - ba' \neq 0$.

If $\beta \in \ker\phi$ then $a' = b = c = c' = 0$. Thus,

$$\beta : \begin{array}{l} x_1 \to ax_1 \;+\; d\,[[x_1, x_2], x_1] + \; e\,[[x_1, x_2], x_2] \\ x_2 \to a'x_1 + d'\,[[x_1, x_2], x_1] + e'\,[[x_1, x_2], x_2] \end{array}$$

From Lemma 6, it is obvious that β can be written by the automorphisms of the form $\delta^{-1}\zeta_{ijk}\delta$, where $i, j, k \in \{1, 2\}$ and $\delta \in GL_2$. It is enough to show that ζ_{ijk} automorphisms can be written by ζ_{121} and automorphisms from GL_2.

For $\upsilon^{-1} : \begin{array}{l} x_1 \to x_1 - x_2 \\ x_2 \to x_2 \end{array}$ we have:

$\zeta_{121}^{-1} = \mu_{-1}\zeta_{121}\mu_{-1}$, $\quad \zeta_{122} = \zeta_{121}^{-1}\upsilon\zeta_{121}\upsilon^{-1}$,

$\zeta_{211} = \zeta_{121}^{-1}$, $\quad \zeta_{212} = \zeta_{122}^{-1} = \upsilon\zeta_{121}\upsilon^{-1}\zeta_{121}^{-1}$

Hence $\beta \in H$. Thus $\ker\phi \subseteq H$ and therefore and $Aut\,(L_{2,3}) = \langle GL_2, \zeta, \ker\phi\rangle \subseteq \langle GL_2, \zeta, \zeta_{121}\rangle$ and

$$Aut\,(L_{2,3}) = \langle GL_2, \zeta, \zeta_{121}\rangle = \langle \tau, \mu_\lambda, \upsilon, \zeta, \zeta_{121}\rangle .$$

3. A Presentation of $Aut(L_{2,3})$

Since $\gamma_2\,(F_2)$ is an ideal of F_2, each automorphism θ of F_2 induces in a natural way an automorphism $\overline{\theta}$ on $L_{2,1}$. Hence there is a natural homomorphism ϕ from $Aut\,(F_2)$ to $Aut\,(L_{2,1})$. The image of this homomorphism is isomorphic to GL_2 and $\ker\phi$ is trivial. Therefore by Theorem 2 we have the following result.

Theorem 3.1. *$Aut\,(L_{2,1}) = \langle \tau, \mu_\lambda, \upsilon\rangle$ with defining relations:*
(1) $\tau^2 = 1$, (2) $(\mu\tau)^4 = 1$, (3) $(\tau\mu\tau\upsilon)^2 = 1$, (4) $(\mu\upsilon)^2 = 1$,
(5) $(\upsilon\tau\mu)^6 = 1$,(6) $(\mu)^2 = 1$,(7) $[[\upsilon, \mu], \upsilon] = 1$, where $\mu = \mu_{-1}$.

Theorem 3.2. *$Aut\,(L_{2,2}) = \langle GL_2, \zeta\rangle$ with defining relations,*
(1) $\tau^2 = 1$, (2) $(\mu\tau)^4 = 1$, (3) $(\tau\mu\tau\upsilon)^2 = 1$, (4) $(\mu\upsilon)^2 = 1$,
(5) $(\upsilon\tau\mu)^6$,(6) $(\mu)^2 = 1$,(7) $[[\upsilon, \mu], \upsilon] = 1$, (8) $[\upsilon, \mu, \zeta] = 1$,
(9) $[\mu, \zeta] = 1$, (10) $[\zeta_{ij}, \zeta_{kl}] = 1$,$i, j, k, l = 1$,2 (11)$(\tau\zeta\tau\mu\tau\zeta)^4 = 1$
(12) $[\upsilon, \zeta] = 1$ where $\mu = \mu_{-1}$.

Proof. By straightforward calculations we can verify that the relations (1)-(12) are relations in $Aut\,(L_{2,2})$.

Now we shall give a presentation of $Aut\,(L_{2,2})$.

We need the following propositions.

Proposition 3.3. *The relations*
(1) $\tau^2 = 1$, *(2)* $(\mu\tau)^4 = 1$, *(3)* $(\tau\mu\tau\upsilon)^2 = 1$, *(4)* $(\mu\upsilon)^2 = 1$,
(5) $(\upsilon\tau\mu)^6 = 1$,*(6)* $(\mu)^2 = 1$,*(7)* $[[\upsilon, \mu], \upsilon] = 1$, *(8)* $[\upsilon, \zeta] = 1$,
(9) $[\mu, \zeta] = 1$, *(10)* $[\zeta_{i_1 i_2 i_3}, \zeta_{i_4 i_5 i_6}] = 1, i_1, i_2, i_3, i_4, i_5, i_6 = 1, 2$
(11)$(\tau\zeta\tau\mu\tau\zeta)^4 = 1$ *(12)* $[\zeta_{121}, \upsilon, \upsilon] = 1$, *(13)* $(\tau\mu\tau\zeta_{121})^2 = 1$,
(14) $[\zeta_{121}, \upsilon, \mu] = 1$, *(15)* $[\zeta_{12i}, \zeta] = 1, i = 1, 2.$
are relations in Aut $(L_{2,3})$.

Proof. By Theorem 3.1 , 1-7 are relations in *Aut* $(L_{2,3})$. By straightforward calculations we can verify that the relations (8)-(15) are relations in *Aut* $(L_{2,3})$.

Let

$$\eta_{21}^{\lambda} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda [x_2, x_1] \end{array} \qquad \eta_{12}^{\lambda} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda [x_1, x_2] \end{array}$$

$$\eta_{212}^{\lambda} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda [x_2, x_1, x_2] \end{array} \qquad \eta_{122}^{\lambda} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda [x_1, x_2, x_2] \end{array}$$

$$\eta_{212}^{\lambda^2} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda^2 [x_2, x_1, x_2] \end{array} \qquad \eta_{121}^{\lambda^2} : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + \lambda^2 [x_1, x_2, x_1] \end{array}$$

$$\zeta_{121}^{\lambda} : \begin{array}{l} x_1 \to x_1 + \lambda [x_1, x_2, x_1] \\ x_2 \to x_2 \end{array} \qquad \zeta_{211}^{\lambda} : \begin{array}{l} x_1 \to x_1 + \lambda [x_2, x_1, x_1] \\ x_2 \to x_2 \end{array}$$

where $\lambda, \lambda^2 \in K$ (K is a constructive field).

Proposition 3.4. *Let* N *be the subgroup generated by*
$\zeta, \zeta_{21}, \eta_{21}, \eta_{12}, \zeta_{121}, \zeta_{122}, \zeta_{211}, \zeta_{212}, \eta_{121}, \eta_{122}, \eta_{211}$, and η_{212}. *Then:*

a) N *is the normal subgroup of Aut* $(L_{2,3})$ *generated by* ζ *and* ζ_{121}.

b) Aut $(L_{2,3}) = GL_2 \cdot N$.

c) The subgroup N *has the presentation*

$$N = \left\langle \begin{array}{l} \zeta, \zeta_{21}, \eta_{21}, \eta_{12}, \\ \zeta_{121}, \zeta_{122}, \zeta_{211}, \\ \zeta_{212}, \eta_{121}, \eta_{122}, \\ \eta_{211}, \eta_{212} \end{array} \middle| \begin{array}{l} [\zeta_{i_1 i_2 i_3}, \zeta_{i_4 i_5}] = 1, [\zeta_{i_1 i_2 i_3}, \eta_{i_4 i_5 i_6}] = 1, \\ [\zeta_{i_1 i_2}, \eta_{i_4 i_5 i_6}] = 1, [\zeta_{i_1 i_2 i_3}, \zeta_{i_4 i_5 i_6}] = 1, \\ [\zeta_{i_1 i_2 i_3}, \eta_{i_4 i_5}] = 1, [\eta_{i_1 i_2 i_3}, \eta_{i_4 i_5 i_6}] = 1, \\ [\eta_{i_1 i_2 i_3}, \eta_{i_4 i_5}] = 1 i_1, i_2, i_3, i_4, i_5, i_6 = 1, 2 \end{array} \right\rangle.$$

Furthermore the defining relations of N *can be derived from the relations (1)-(15).*

Proof. We have that
$\zeta_{121}^{-1} = \mu_{-1}\zeta_{121}\mu_{-1}$, $\zeta_{122} = \zeta_{121}^{-1}\upsilon\zeta_{121}\upsilon^{-1}$, $\zeta_{211} = \zeta_{121}^{-1}, \tau\zeta\tau^{-1} = \eta_{21}$,
$\zeta_{212} = \zeta_{122}^{-1} = \upsilon\zeta_{121}\upsilon^{-1}\zeta_{121}^{-1}, \tau\zeta_{122}\tau^{-1} = \eta_{211}, \tau\zeta_{121}\tau^{-1} = \eta_{212}$,
$\tau\zeta_{212}\tau^{-1} = \eta_{121}, \tau\zeta_{211}\tau^{-1} = \eta_{122}, \tau\mu_{-1}\tau\zeta\tau\mu_{-1}\tau = \zeta_{21}$
$\tau\zeta_{21}\tau^{-1} = \eta_{12}$.

Hence N is contained in the normal subgroup generated by ζ and ζ_{121}.

To prove that N is the normal subgroup generated by ζ and ζ_{121}, it suffices to show that the conjugates of $\zeta, \zeta_{21}, \eta_{21}, \eta_{12}, \zeta_{121}, \zeta_{122}, \zeta_{211}, \zeta_{212}, \eta_{121}, \eta_{122}, \eta_{211}$, and η_{212} under $\tau, \mu_\lambda, \upsilon$ belong to N.

We claim that the following are true:

$$\begin{array}{lll}
\tau\zeta\tau^{-1} = \eta_{21} \in N & \mu_\lambda\zeta\mu_\lambda^{-1} = \zeta \in N & \upsilon\zeta\upsilon^{-1} = \zeta \in N \\
\tau\zeta_{21}\tau^{-1} = \eta_{12} \in N & \mu_\lambda\zeta_{21}\mu_\lambda^{-1} = \zeta_{21} \in N & \upsilon\zeta_{21}\upsilon^{-1} = \zeta_{21} \in N \\
\tau\eta_{21}\tau^{-1} = \zeta \in N & \mu_\lambda\eta_{21}\mu_\lambda^{-1} = \eta_{21}^\lambda \in N & \upsilon\eta_{21}\upsilon^{-1} = \zeta_{211}\eta_{21}\zeta \in N \\
\tau\eta_{12}\tau^{-1} = \zeta_{21} \in N & \mu_\lambda\eta_{12}\mu_\lambda^{-1} = \eta_{12}^\lambda \in N & \upsilon\eta_{12}\upsilon^{-1} = \zeta_{211}\eta_{12}\zeta_{21} \in N \\
\tau\zeta_{121}\tau^{-1} = \eta_{212} \in N & \mu_\lambda\zeta_{121}\mu_\lambda^{-1} = \zeta_{121}^\lambda \in N & \upsilon\zeta_{121}\upsilon^{-1} = \zeta_{121}\zeta_{122} \in N \\
\tau\zeta_{122}\tau^{-1} = \eta_{211} \in N & \mu_\lambda\zeta_{122}\mu_\lambda^{-1} = \zeta_{122} \in N & \upsilon\zeta_{122}\upsilon^{-1} = \zeta_{122} \in N \\
\tau\zeta_{211}\tau^{-1} = \eta_{122} \in N & \mu_\lambda\zeta_{211}\mu_\lambda^{-1} = \zeta_{211}^\lambda \in N & \upsilon\zeta_{211}\upsilon^{-1} = \zeta_{211}\zeta_{212} \in N \\
\tau\zeta_{212}\tau^{-1} = \eta_{121} \in N & \mu_\lambda\zeta_{212}\mu_\lambda^{-1} = \zeta_{212} \in N & \upsilon\zeta_{212}\upsilon^{-1} = \zeta_{212} \in N \\
\tau\eta_{212}\tau^{-1} = \zeta_{121} \in N & \mu_\lambda\eta_{212}\mu_\lambda^{-1} = \eta_{212}^\lambda \in N & \upsilon\eta_{212}\upsilon^{-1} = \zeta_{122}\eta_{212} \in N \\
\tau\eta_{211}\tau^{-1} = \zeta_{122} \in N & \mu_\lambda\eta_{211}\mu_\lambda^{-1} = \eta_{211}^{\lambda^2} \in N & \upsilon\eta_{211}\upsilon^{-1} = \zeta_{122}\zeta_{121}\eta_{212}\eta_{211} \in N \\
\tau\eta_{121}\tau^{-1} = \zeta_{212} \in N & \mu_\lambda\eta_{121}\mu_\lambda^{-1} = \eta_{121}^{\lambda^2} \in N & \upsilon\eta_{121}\upsilon^{-1} = \zeta_{212}\zeta_{211}\eta_{121}\eta_{122} \in N \\
\tau\eta_{122}\tau^{-1} = \zeta_{211} \in N & \mu_\lambda\eta_{122}\mu_\lambda^{-1} = \eta_{122}^{\lambda^2} \in N & \upsilon\eta_{122}\upsilon^{-1} = \eta_{122}\zeta_{212} \in N.
\end{array}$$

By direct calculations we obtain the results. Therefore N is the normal subgroup of $Aut(L_{2,3})$, generated by ζ and ζ_{121}.

b) Since is $Aut(L_{2,3})$ generated by GL_2, ζ and ζ_{121}, we have that $Aut\,(L_{2,3}) = \langle GL_2, \zeta, \zeta_{121}\rangle = \langle GL_2, N\rangle = GL_2 \cdot N$

c) By easy calculations we have,

$[\zeta_{i_1i_2i_3}, \zeta_{i_4i_5}] = 1, [\zeta_{i_1i_2i_3}, \eta_{i_4i_5i_6}] = 1, [\zeta_{i_1i_2}, \eta_{i_4i_5i_6}] = 1, [\zeta_{i_1i_2i_3}, \zeta_{i_4i_5i_6}] = 1,$

$[\zeta_{i_1i_2i_3}, \eta_{i_4i_5}] = 1, [\eta_{i_1i_2i_3}, \eta_{i_4i_5i_6}] = 1, [\eta_{i_1i_2i_3}, \eta_{i_4i_5}] = 1,$

(where $i_1, i_2, i_3, i_4, i_5, i_6 = 1, 2$) in $Aut\,(L_{2,3})$. Hence we have the following presentation of N.

$$N = \left\langle \begin{array}{l} \zeta, \zeta_{21}, \eta_{21}, \eta_{12}, \\ \zeta_{121}, \zeta_{122}, \zeta_{211}, \\ \zeta_{212}, \eta_{121}, \eta_{122}, \\ \eta_{211}, \eta_{212} \end{array} \;\middle|\; \begin{array}{l} [\zeta_{i_1i_2i_3}, \zeta_{i_4i_5}] = 1, [\zeta_{i_1i_2i_3}, \eta_{i_4i_5i_6}] = 1, \\ [\zeta_{i_1i_2}, \eta_{i_4i_5i_6}] = 1, [\zeta_{i_1i_2i_3}, \zeta_{i_4i_5i_6}] = 1, \\ [\zeta_{i_1i_2i_3}, \eta_{i_4i_5}] = 1, [\eta_{i_1i_2i_3}, \eta_{i_4i_5i_6}] = 1, \\ [\eta_{i_1i_2i_3}, \eta_{i_4i_5}] = 1 i_1, i_2, i_3, i_4, i_5, i_6 = 1, 2 \end{array} \right\rangle$$

By easy calculations we can see that the relations of N can be derived from (1)-(14).

Theorem 3.5. *$Aut\,(L_{2,3}) = \langle \tau, \mu_\lambda, \upsilon, \zeta, \zeta_{121}\rangle$ with the defining relations:*
(1) $\tau^2 = 1$, (2) $(\mu\tau)^4 = 1$, (3) $(\tau\mu\tau\upsilon)^2 = 1$, (4) $(\mu\upsilon)^2 = 1$,
(5) $(\upsilon\tau\mu)^6 = 1$,(6) $(\mu)^2 = 1$,(7) $[[\upsilon, \mu], \upsilon] = 1$, (8) $[\upsilon, \zeta] = 1$,
(9) $[\mu, \zeta] = 1$, (10) $[\zeta_{i_1i_2i_3}, \zeta_{i_4i_5i_6}] = 1, i_1, i_2, i_3, i_4, i_5, i_6 = 1, 2$
(11)$(\tau\zeta\tau\mu\tau\zeta)^4 = 1$ (12) $[\zeta_{121}, \upsilon, \upsilon] = 1$, (13) $(\tau\mu\tau\zeta_{121})^2 = 1$,
(14) $[\zeta_{121}, \upsilon, \mu] = 1$, (15) $[\zeta_{12i}, \zeta] = 1, \quad i = 1, 2$.

Proof. Since $Aut\,(L_{2,3}) = \langle GL_2, \zeta, \zeta_{121}\rangle = GL_2 \cdot N$ any automorphisms of

$Aut(L_{2,3})$ are of the form:

$$a_1...a_k\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8},\text{ where } k, m_1, m_2, m_3, m_4, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in \mathbb{Z}$$

and for $i = 1, 2, ..., k$ $a_i \in \{\tau, \mu_\lambda, \upsilon\}$.

Let $a_1...a_k\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} = 1$ relation in $Aut\,(L_{2,3})$.

If $a_1...a_k = 1 \Rightarrow \zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} = 1$

For this station there is no contradiction.

If $a_1...a_k \neq 1 \Rightarrow \zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} \neq 1$ and also $a_1...a_k = (\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8})^{-1}$

If $a_1...a_k = (\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8})^{-1}$ then there is an automorphism $b \in GL_2 \cap N$. $b \in GL_2 \Rightarrow b = a_1...a_k$ and $b \in N \Rightarrow a\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8}a^{-1}, a \in GL_2$. Therefore we obtain

$b = a_1...a_k = a\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8}a^{-1}$ This means

$a^{-1}a_1...a_ka = \zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8}$ where $a_i, a \in GL_2, (i = 1, ..., k)$.

Thus $\zeta, \zeta_{21}, \eta_{21}, \eta_{12,}\zeta_{121}, \zeta_{122}, \zeta_{211}, \zeta_{212}, \eta_{121}, \eta_{122,}\eta_{211}$, and η_{212} can be written by automorphisms of GL_2. This is a contradiction. Hence $GL_2 \cap N = \{1\}$ and $b = \{1\}$. From this, the relation

$$a_1...a_k\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} = 1$$

is satisfy only $a_1...a_k = 1$ and $\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} = 1$. This relations can be derived from (1)-(15), and therefore the relation $a_1...a_k\zeta^{m_1}\zeta_{21}^{m_2}\eta_{21}^{m_3}\eta_{12}^{m_4}\zeta_{121}^{s_1}\zeta_{122}^{s_2}\zeta_{211}^{s_3}\zeta_{212}^{s_4}\eta_{121}^{s_5}\eta_{122}^{s_6}\eta_{211}^{s_7}\eta_{212}^{s_8} = 1$ can be derived from (1)-(15). So the presentation of $Aut(L_{2,3})$ is:

$$Aut(L_{2,3}) = \left\langle \begin{array}{c} \tau, \mu_\lambda, \upsilon, \\ \zeta, \zeta_{121} \end{array} \middle| \begin{array}{l} \tau^2 = 1,\ (\mu\tau)^4 = 1,\ (\tau\mu\tau\upsilon)^2 = 1, \\ (\mu\upsilon)^2 = 1, (\upsilon\tau\mu)^6 = 1, (\mu)^2 = 1, \\ [[\upsilon, \mu], \upsilon] = 1,\ [\upsilon, \zeta] = 1, [\mu, \zeta] = 1, \\ (\tau\zeta\tau\mu\tau\zeta)^4 = 1, [\zeta_{12i}, \zeta] = 1, i = 1, 2 \\ [\zeta_{i_1i_2i_3}, \zeta_{i_4i_5i_6}] = 1, k = \overline{1,6}, i_k = 1, 2 \\ [\zeta_{121}, \upsilon, \upsilon] = 1, (\tau\mu\tau\zeta_{121})^2 = 1, \\ [\zeta_{121}, \upsilon, \mu] = 1, \end{array} \right\rangle$$

References

[1] S. Andreadakis, On the automorphisms of free groups and free nilpotent groups, *Proc. London Math. Soc.*, **3**, No. 15 (1965).

[2] S. Andreadakis, Generators for $Aut(G)$, G free nilpotent, *Arc. Math.*, **42** (1984), 296-300.

[3] S. Bachmuth, Induced automorphisms of free groups and free metabelian groups, *Tarns. Amer. Math. Soc.*, **122** (1966), 1-17.

[4] R.M. Bryant, C.K. Gupta, Automorphism groups of free nilpotent groups, *Arc. Math.*, **52** (1989), 313-320.

[5] P.M. Cohn, Subalgebras of free associative algebras, *Proc. London Math. Soc.*, **3**, No. 14 (1964), 618-632.

[6] V. Drensky, C.K. Gupta, Automorphisms of free nilpotent Lie algebras, *Can. J. Math.*, **XLII**, No. 2 (1990), 259-279.

[7] A.V. Goryaga, Generating elements of the group of automorphisms of a free nilpotent group, *Algebra i Logika*, **15** (1976), 458-463.

[8] C.K. Gupta, IA-automorphisms of two generator metabelian groups, *Arch. Math.*, **37** (1981), 106-112.

[9] J. Nielsen, Die Isomorphismengruppe der freien gruppen, *Math. Ann.*, **91** (1924), 169-209.

992