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# INDEPENDENT AND VERTEX COVERING NUMBER ON KRONECKER PRODUCT OF $C_n$

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**Abstract:** Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph of $G_1$ and $G_2$ has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let $G$ is a simple graph with order m, we prove that, $\alpha(C_n \otimes G) = \max\left\{n\alpha(G), m\lfloor\frac{n}{2}\rfloor\right\}$ and $\beta(C_n \otimes G) = \min\left\{n\beta(G), m\lceil\frac{n}{2}\rceil\right\}$.

**AMS Subject Classification:** 05C69, 05C70, 05C76
**Key Words:** Kronecker product, independent number, vertex covering number

## 1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let $G_1$ and $G_2$ be graphs. The Kronecker product of graph $G_1$ and $G_2$, denote by $G_1 \otimes G_2$, be the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters. A subset $U$ of the vertex set $V(G)$ of $G$ is said to be an independent set of $G$ if the induced subgraph $G[U]$ is a trivial graph. An independent set of $G$ with maximum number of vertices is called a maximum independent set of $G$. The number of

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vertices of a maximum independent set of $G$ is called the independent number of $G$, denoted by $\alpha(G)$.

A vertex of graph $G$ is said to cover the edges incident with it, and a vertex cover of a graph $G$ is a set of vertices covering all the edge of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the vertex covering number of $G$, denoted by $\beta(G)$.

By definitions of independent number and vertex covering number, clearly that $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta(C_n) = \lceil \frac{n}{2} \rceil$.

**Proposition 1.** Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:

(i) $n(V(H)) = n(V(G_1)n(V(G_2);$
(ii) $n(E(H)) = 2n(E(G_1)n(E(G_2);$
(iii) for every $(u,v) \in V(H), d_H((u,v)) = d_{G_1}(u)d_{G_2}(v)$.

Note that for any graph $G$, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$

**Theorem 2.** *Let $G_1$ and $G_2$ be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if $G_1$ or $G_2$ contains an odd cycle.*

**Theorem 3.** *Let $G_1$ and $G_2$ be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.*

Next we get that general form of graph of Kronecker Product of $C_n$ and a simple graph.

**Proposition 4.** Let $G$ be connected graph order $m$, the graph of $C_n \otimes G$ *is*

$$(\bigcup_{i=1}^{n-1} H_i) \cup H_n$$

where $V(H_i)$=$W_i \cup W_{i+1}$ for $i = 1, 2, ..., n-1$; $W_i = \{(i,1),(i,2),...,(i,m)\}$; $E(H_i) = \{(i,u)(i+1,v)/uv \in E(G)\}$ and $V(H_n) = W_n \cup W_{n+1}$; $E(H_n) = \{(n,u)/uv \in E(G)\}$ Moreover, if $G$ has no odd cycle then for each $H_1$ and $H_n$ has exactly two connected components isomorphic to $G$.

Example

## 2. Independent Number of the Graph of $C_n \otimes G$

We now state proposition and prove lemma before stating our main results.

We begin this section by giving the proposition 2.1 which show character of independent set and the Lemma 2.2 which show character of independent set for each $H_i$.

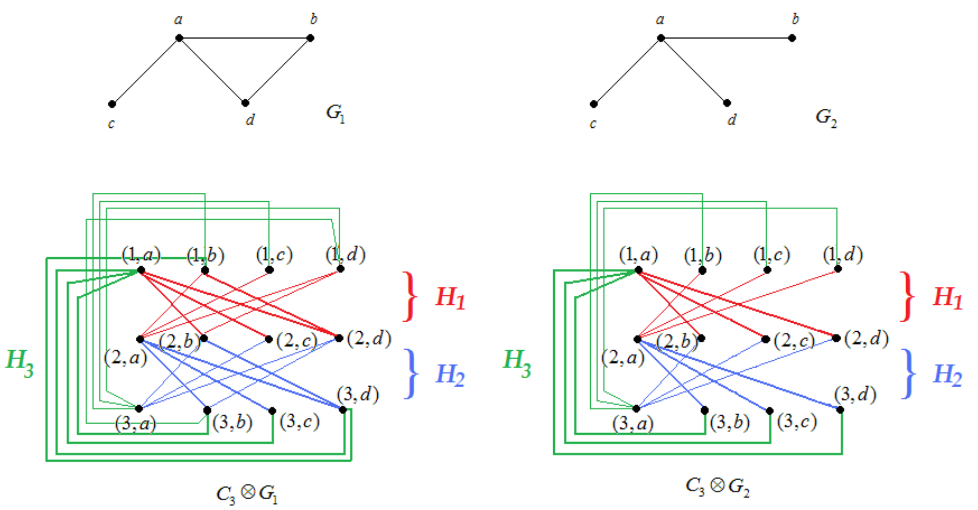


Figure 1: The graph of $C_3 \otimes G_1$ and $C_3 \otimes G_2$

**Proposition 5.** Let $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph $G$ if:

(i) $v_i$ is not adjacent with $v_j$ for all $i \neq j$ and $i, j = 1, 2, ..., k$; and

(ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

**Lemma 6.** Let $C_n \otimes G = (\bigcup_{i=1}^{n-1} H_i) \cup H_n$. For each $H_i$ and $H_n$, then $\alpha(H_i) = \alpha(H_n) = 2\alpha(G)$.

*Proof.* Suppose $G$ has no odd cycle, by proposition 1.4 we get $H_i$=$2G$. So $\alpha(H_i) = 2\alpha(G)$.

if G has odd cycle, for each $H_i$, vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = C_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, $I$ be the maximum independent set of $G$. We get

$\overline{H_i} = 2(G - \overline{e})$ then

$$\alpha(\overline{H_i}) = 2\alpha(G - \overline{e}) = \begin{cases} 2[\alpha(G) + 1], & \text{if } \overline{e} = xy \text{ then } x \in I, \\ & y \notin I \text{ and is not} \\ & \text{adjecent with vertex } z \in I, \\ 2\alpha(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\alpha(G - \overline{e}) = \alpha(G) + 1$ be not impossible because the end vertices of edge $\overline{e}$ are in independent set of $G - \overline{e}$, so $\alpha(H_i) = \alpha(\overline{H_i}) - 1$.

Hence $\alpha(H_i) = 2\alpha(G)$. Similarly, $\alpha(H_n) = 2\alpha(G)$. □

Example

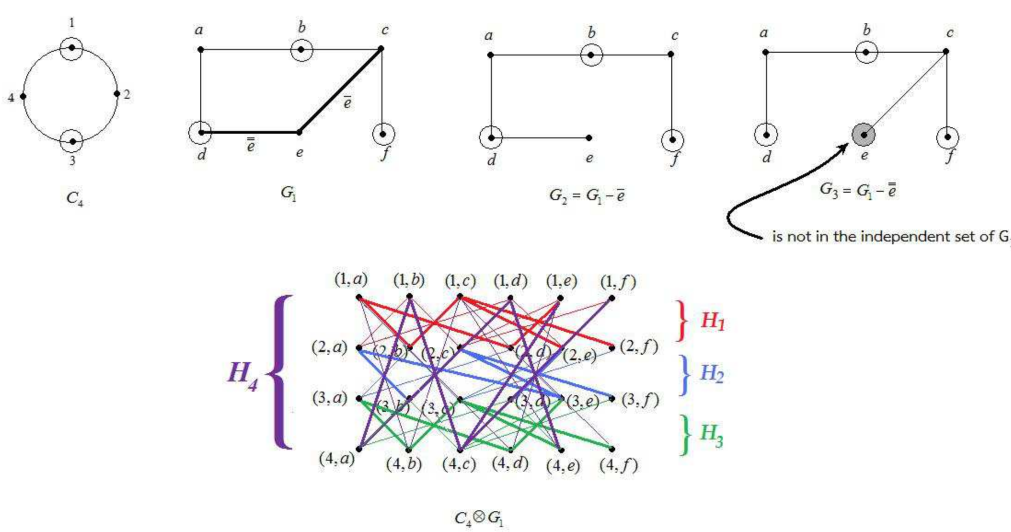


Figure 2: The graph of $C_4 \otimes G_1$

Next, we establish Theorem 2.3 for a maximum independent number of $C_n \otimes G$.

**Theorem 7.** *Let $G$ be connected graph order $m$, then $\alpha(C_n \otimes G) = \max\{n\alpha(G), m\lfloor \frac{n}{2} \rfloor\}$.*

*Proof.* Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_i, i = 1, 2, ..., m\}$, $S_i = \{(v_i, u_j) \in V(C_n \otimes G)/j = 1, 2, ..., m\}, i = 1, 2, ..., n$ and since $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$.

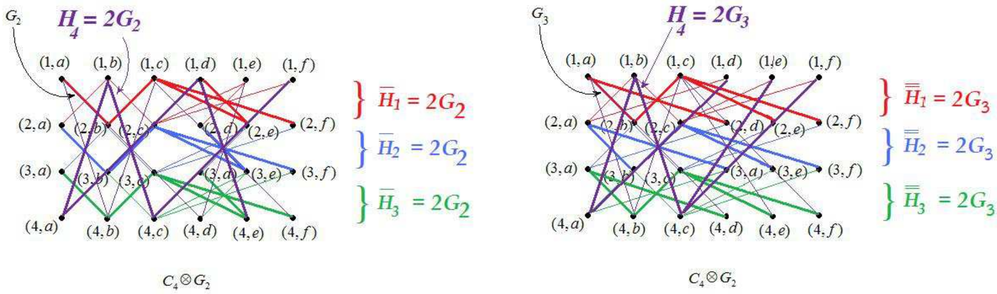


Figure 3: The graph of $C_4 \otimes G_2$ and $C_4 \otimes G_3$

Assume that the maximum independent set of $C_n, G$ be $I_1$=$\{u_1, u_3, ..., u_{2\lfloor \frac{n}{2} \rfloor -1}\}$, $I_2$, respectively.

For each $H_i$, by Lemma 2.1 we have $\alpha(H_i) = 2\alpha(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which is every $H_i$, $H_{i+1}$ and $H_n$, $H_1$ have $\alpha(G)$ common vertrices in their independent set, then $\alpha(C_n \otimes G) \geq n\alpha(G)$.

By remark 2.2, we get a independent set of $C_n \otimes G$ be $S_1 \cup S_3 \cup ... \cup S_{2\lfloor \frac{n}{2} \rfloor -1}$, then $\alpha(C_n \otimes G) \geq m\lfloor \frac{n}{2} \rfloor$.

Hence $\alpha(C_n \otimes G) \geq max\{n\alpha(G), m\lfloor \frac{n}{2} \rfloor\}$.

Suppose that $\alpha(C_n \otimes G) >$max$\{n\alpha(G), m\lfloor \frac{n}{2} \rfloor\}$, then there exists $uv_j(or u_i v) \in V(C_n \otimes G) - W(or S)$;

$$j = k+1, k+2, ..., m; \qquad i = \begin{cases} 2, 4, .., n & \text{where n is even,} \\ 2, 4, ..., 2\lceil \frac{n}{2} \rceil - 2, 2\lceil \frac{n}{2} \rceil - 1 & \text{where n is odd,} \end{cases}$$

which is not adjacent with another vertices in $W$ (or $S$ ), $W = \{uv_k / v_k \in I_2\}$ and $S = \{u_h v / u_h \in I_1\}$. It is not true, because for every $H_i$ and $H_n$ has $V(H_i) - W$=$[\bigcup_{j=1}^{m} N(u_i, v_j)] \cup$ $[\bigcup_{j=1}^{m} N(u_{i+1}, v_j)]$ and $V(H_n) - W$=$[\bigcup_{j=1}^{m} N(u_n, v_j)] \cup$ $[\bigcup_{j=1}^{m} N(u_1, v_j)]$.

Hence $\alpha(C_n \otimes G) = max\{n\alpha(G), m\lfloor \frac{n}{2} \rfloor\}$. □

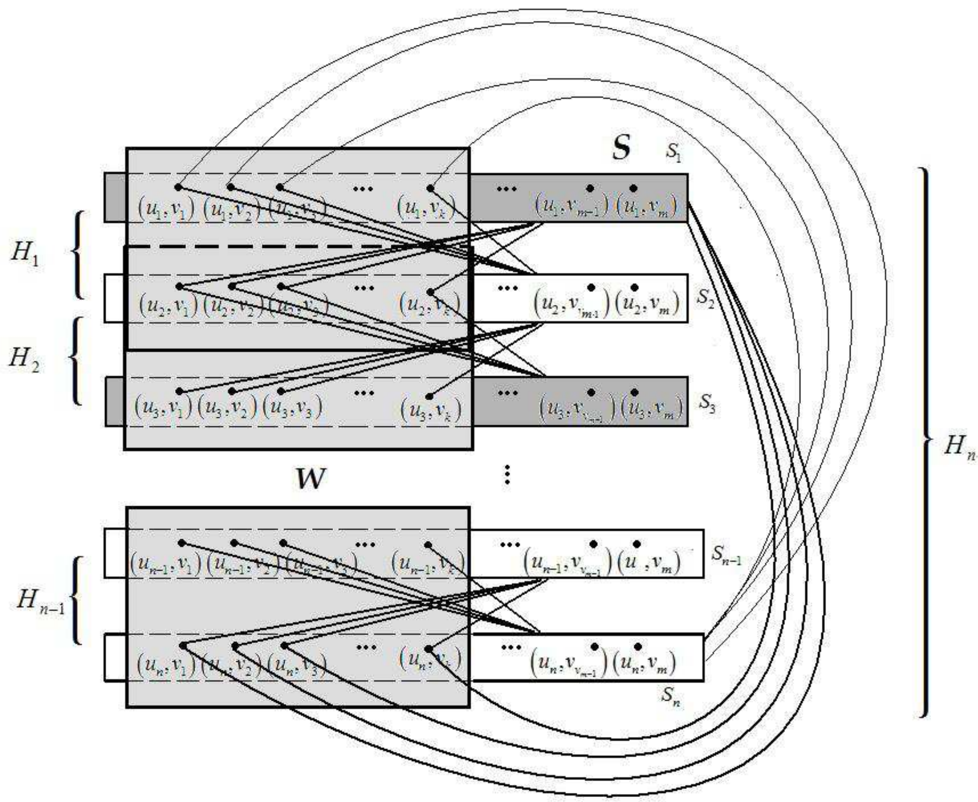


Figure 4: The region of $W, S$ when $n$ is odd

### 3. Vertex Covering Number of the Graph of $C_n \otimes G$

We begin this section by giving the Lemma 3.1 that shows a relation of independent number and vertex covering number and the Lemma 3.2 that show character of independent set for each $H_i$.

**Lemma 8.** (see [2]) Let $G$ be a simple graph with order $n$. Then $\alpha(G) + \beta(G) = n$

**Lemma 9.** Let $C_n \otimes G = (\bigcup_{i=1}^{n-1} H_i) \cup H_n$. For each $H_i$ and $H_n$ then $\beta(H_i) = \beta(H_n) = 2\beta(G)$

*Proof.* Suppose $G$ has no odd cycle, by proposition 1.4, we get $H_i$=$2G$. So $\beta(H_i) = 2\beta(G)$.

if G has odd cycle, for each $(u_{i+1}, v) \in W_i$, $(u_{i+1}, v) \in W_{i+1}$ in $V(H_i)$ and $(u_n, v) \in W_n$ in $V(H_n)$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v) = d_{H_n}((u_n, v)) = d_{H_n}(u_1, v))$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = C_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, $C$ be the minimum vertex covering set of $G$. We get $\overline{H_i} = 2(G - \overline{e})$ then

$$\beta(\overline{H_i}) = 2\beta(G - \overline{e}) = \begin{cases} 2[\beta(G) - 1], & \text{if } \overline{e} = xy \text{ then } x, y \text{ are adjecent,} \\ & \text{with vertices in } C, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\beta(G - \overline{e}) = \beta(G) - 1$ be not impossible because edge $\overline{e}$ is not adjacent with vertices in the vertex covering set of $G - \overline{e}$, so $\beta(H_i) = \beta(\overline{H_i}) + 1$.

Hence $\beta(H_i) = 2\beta(G)$. Similarly, $\beta(H_n) = 2\beta(G)$. □

Next, we establish Theorem 3.3. for a minimum vertex covering number of $C_n \otimes G$.

**Theorem 10.** *Let $G$ be connected graph order $m$, then $\beta(C_n \otimes G) = max\{n\beta(G), m\lceil \frac{n}{2} \rceil\}$*

*Proof.* Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \otimes G)/j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$ and since $\beta(C_n) = \lceil \frac{n}{2} \rceil$. Assume that the maximun vertex covering set of $C_n, G$ be

$$C_1 = \begin{cases} \{u_2, u_4, ..., u_n\} & \text{where } n \text{ is even,} \\ \{u_2, u_4, ..., u_{2\lceil \frac{n}{2} \rceil - 2}, u_{2\lceil \frac{n}{2} \rceil - 1}\} & \text{otherwise.,} \end{cases} \quad C_2 \text{ respectively.}$$

For each $H_i$, by Lemma 3.2 we have $\beta(H_i) = 2\beta(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which is every $H_i$, $H_{i+1}$ and $H_{n-1}$, $H_n$ have $\beta(G)$ common vertrices in their vertex covering set, then $\beta(C_n \otimes G) \leq n\beta(G)$. Since remark 2.2, we get another vertex covering set of $C_n \otimes G$ be

$$\begin{cases} \{S_2, S_4, ..., S_n\} & \text{where } n \text{ is even,} \\ \{S_2, S_4, ..., S_{2\lceil \frac{n}{2} \rceil - 2}, u_{2\lceil \frac{n}{2} \rceil - 1}\} & \text{otherwise.} \end{cases}$$

then $\alpha(C_n \otimes G) \leq m\lceil \frac{n}{2} \rceil$.

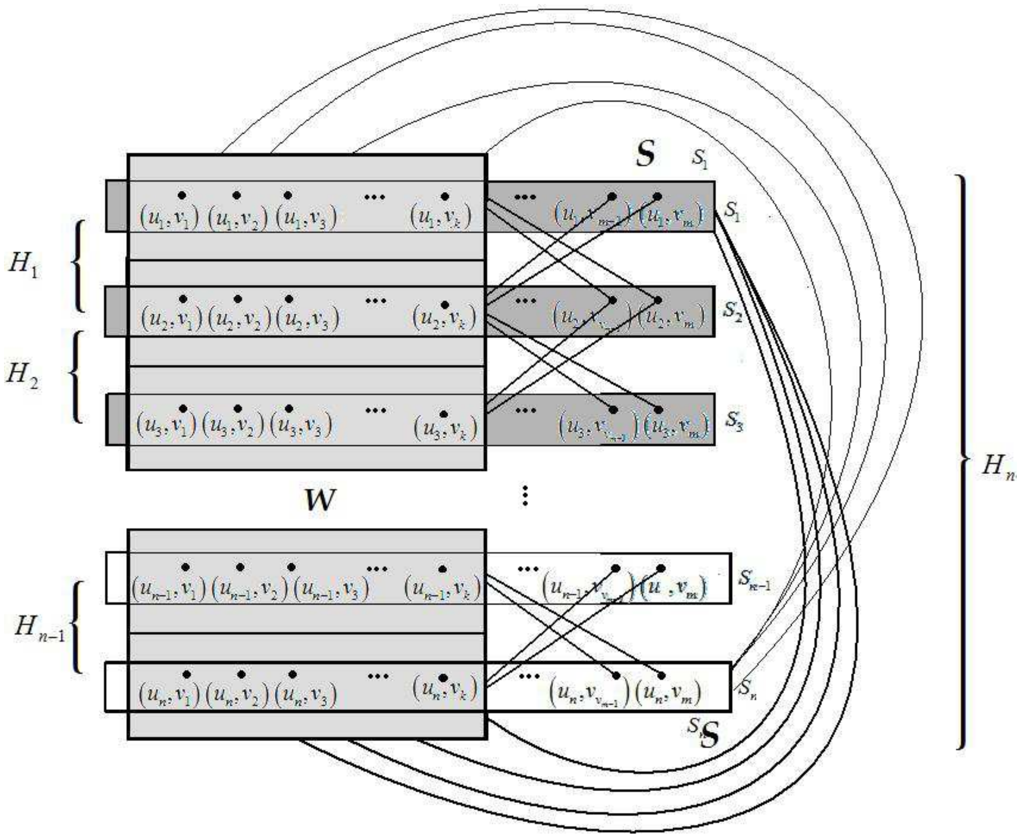


Figure 5: The region of $W, S$ when $n$ is odd

Hence $\alpha(C_n \otimes G) \leq max\{n\alpha(G), m\lceil \frac{n}{2} \rceil\}$.

Suppose that $\beta(C_n \otimes G) < min\{n\beta(G), m\lceil \frac{n}{2} \rceil\}$, then there exists $uv_j(oru_iv) \in V(C_n \otimes G) - W(orS), j = k+1, k+2, ..., m; i = 1, 3, 2\lfloor \frac{n}{2} \rfloor - 1,$ which is not adjacent with another vertices in $W$ (or $S$ ), $W = \{uv_k/v_k \in C_2\}$ and $S = \{u_hv/u_h \in C_1\}$. It is not true, because for every $uv_j(u_iv)$ adjacent with a vertices in $W$ (or $S$).

Hence $\beta(C_n \otimes G) = min\{n\beta(G), m\lfloor \frac{n}{2} \rfloor\}$. □

By Theorem 2.3 and Lemma 3.1, we can also show that:

$$\alpha(C_n \otimes G) + \beta(C_n \otimes G) = mn$$

$$\begin{aligned}\max\{n\alpha(G), m\lceil\tfrac{n}{2}\rceil\} + \beta(C_n \otimes G) &= mn\\ \beta(C_n \otimes G) &= mn - \max\{n\alpha(G), m\lceil\tfrac{n}{2}\rceil\}\\ &= mn + \min\{-n\alpha(G), -m\lceil\tfrac{n}{2}\rceil\}\\ &= \min\{n(m-\alpha(G)), m(n-\lceil\tfrac{n}{2}\rceil\}\\ &= \min\{n\beta(G)), m\lfloor\tfrac{n}{2}\rfloor\}.\end{aligned}$$

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