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EBERLEIN ALMOST PERIODIC FUNCTIONS THAT ARE NOT PSEUDO ALMOST PERIODIC

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Abstract: We construct Eberlein almost periodic functions $f_j : J \to H$ so that $||f_1(\cdot)||$ is not ergodic and thus not Eberlein almost periodic and $||f_2(.)||$ is Eberlein almost periodic, but f_1 and f_2 are not pseudo almost periodic, the Parseval equation for them fails, where $J = \mathbb{R}_+$ or $\mathbb{R}$ and H is a Hilbert space. This answers several questions posed by Zhang and Liu [18].

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1. Introduction and Notation

Recently Zhang and Liu [18] asked, whether for Hilbert space valued Eberlein almost periodic $f : \mathbb{R} \to H$ (see §2) a Parseval equation holds (Fourier coefficients for such f are always defined by [14, Theorem 2.4, for $\mathbb{R}_+$]); this would imply that such f are pseudo almost periodic (see (2.8)). If additionally the range $f(\mathbb{R})$ is relatively norm compact, this is true by results of Goldberg and Irvin [9, Proposition 2.9].

Here we show by examples, that without $f(\mathbb{R})$ relatively compact the f is in general no longer pseudo almost periodic, one has no Parseval equation.

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Throughout this paper, $\mathbb{R}_+ = [0, \infty)$, $J \in \{\mathbb{R}_+, \mathbb{R}\}$, X real or complex Banach space; for $f : J \to X$, $f_s(t) := f(s+t)$, $|f|(t) := ||f(t)||$, $C_b(J, X) = \{f : J \to X : f$ continuous, $||f|| = \sup_{t \in J} ||f(t)|| < \infty\}$, $C_{ub}(J, X) = \{f \in C_b(J, X) : f$ uniformly continuous$\}$, $AP(\mathbb{R}, X) =$ almost periodic functions [1, p. 3], [16, p. 18-19], $AP(\mathbb{R}_+, X) = AP(\mathbb{R}, X)|\mathbb{R}_+$.

2. Eberlein and Pseudo Almost Periodic Functions

A function $f : J \to X$ is called *Eberlein almost periodic* if $f \in C_b(J, X)$ and orbit $O(f) := \{f_s : s \in J\}$ is relatively weakly compact in $C_b(J, X)$ (see [8, Definition 10.1, p. 232], [6, Definition 1.4], [12, p. 467], [5, Definition 2.1, p. 138])

$$EAP(J, X) := \{f : f \text{ Eberlein weakly almost periodic}\}, \tag{2.1}$$

$$EAP_0(J, X) := \{f \in EAP(J, X) : 0 \in \text{ weak closure of } O(f)\}, \tag{2.2}$$

$$EAP_{rc}(J, X) := \{f \in EAP(J, X) : f(J) \text{ relatively norm compact in } X\}. \tag{2.3}$$

By [2], Theorem 2.3.4 and Theorem 2.4.7, [14] one has

$$EAP(J, X) \subset \mathcal{E}(J, X) \cap C_{ub}(J, X), \tag{2.4}$$

where

$$\mathcal{E}(J, X) := \{f \in L^1_{loc}(J, X) : \text{to } f \text{ exists } x \in X,$$

with

$$||\frac{1}{T} \int_s^{s+T} f(t)\, dt - x|| \to 0 \text{ as } T \to \infty, \quad \text{uniformly in } s \in J\}, \tag{2.5}$$

then $m_B(f) := x$ is called the Bohr-mean.

For J and X as in Section 1 one has a decomposition theorem [13], p. 18, (in $f = g + h$ the $g \in AP(J, X)$, $h \in EAP_0(J, X)$ are unique)

$$EAP(J, X) = AP(J, X) \oplus EAP_0(J, X). \tag{2.6}$$

The class of *pseudo almost periodic* functions introduced by Zhang [16], [17], Definition 5.1, p. 57, [3], (1.1) is given by

$$PAP_0(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X),$$

$$m_B(|f|) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f|(t)dt \text{ exists } = 0\}, \quad (2.7)$$

similarly for $\mathbb{R}_+$, with $m_B(|f|) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |f|(t)dt$,

$$PAP(J,X) := AP(J,X) \oplus PAP_0(J,X). \quad (2.8)$$

Now by [9], Proposition 2.9, one has

$$f \in EAP_{rc}(J,X) \text{ implies } |f| \in EAP(J,\mathbb{C}), \text{ and so } |f|^2 \in EAP(J,\mathbb{C}), \quad (2.9)$$

by [8], Theorem 12.1, p. 234.

So, if X = complex Hilbert space H, the polarisation formula ([11, p. 24, (2)]) yields $(f(\cdot), g(\cdot))_H \in EAP(J,\mathbb{C})$ if $f, g \in EAP_{rc}(J,H)$, (2.4) shows that $(f,g) := m_B(f(.), g(.))_H$ is well defined. With this (semi-definite) scalar product one gets [9, Theorems 5.2 and 5.7] a Parseval equation for $f \in EAP_{rc}(J,H)$.

So, (2.6), (2.4) and [9], Corollary 4.19, give

$$EAP_{rc}(\mathbb{R},H) = AP(\mathbb{R},H) + \{f \in EAP_{rc}(\mathbb{R},H) : (f,f) = 0\},$$
$$\text{with } (f,f) = 0 \text{ if and only if } m_B(|f|) = 0,\ f \in EAP_{rc}(\mathbb{R},H). \quad (2.10)$$

So, with (2.8) one gets for any complex Hilbert space H

$$EAP_{rc}(\mathbb{R},H) \subset PAP(\mathbb{R},H). \quad (2.11)$$

Without the "range relatively compact" however all this is no longer true.

3. Examples

For the following we need a converse of Mazur's theorem (see [15], p. 120, Theorem 2), namely

Proposition 3.1. *A sequence $(x_n) \subset X$ weakly converges to $x \in X$ if and only if, for each subsequence (x'_n) of (x_n), there exists a sequence (y_n) of finite convex combinations of the elements of (x'_n) with $||y_n - x|| \to 0$ as $n \to \infty$.*

For a proof see [4], Proposition 1.8, p. 17.

Example 3.2. For $J = \mathbb{R}_+$ or $\mathbb{R}$ and H infinite dimensional Hilbert space there exists $f \in EAP_0(J,H)$ so that $m_B(|f|)$ and $m_B(|f|^2)$ do not exist, $m_B(|f|) = \lim_{T\to\infty} \frac{1}{T} \int_0^T ||f(t)||\, dt$ respectively $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T ||f(t)||\, dt$ if $J = \mathbb{R}_+$ respectively $\mathbb{R}$. So $|f|$ and $|f|^2 = (f(\cdot), f(\cdot))_H$ are not in $EAP(J,\mathbb{C})$.

Proof. Choose an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ from H. Define $h:\mathbb{R}\to H$ by $h:=0$ on $(-\infty,\frac{1}{2}]$, $h(n):=e_n$, $n\in\mathbb{N}$, h linear on $[n-\frac{1}{2},n]$ and on $[n,n+\frac{1}{2}]$, with $h(n-\frac{1}{2})=0$, $n\in\mathbb{N}$. Then h is well defined and $\in C_{ub}(\mathbb{R},H)$.

Define further $\phi:\mathbb{R}\to[0,1]$ for given $I_n=[\alpha_n,\beta_n]$, $\beta_n=\alpha_{n+1}\in\mathbb{N}$, $\alpha_n<\beta_n$, $n\in\mathbb{N}$, $I_1=[0,1]$, as follows :

$\phi:=0$ on $(-\infty,0]$ and all I_n with odd n, $\phi=1$ on $[\alpha_{2n}+\frac{1}{10},\beta_{2n}-\frac{1}{10}]$ and ϕ linear on $[\alpha_{2k},\alpha_{2k}+\frac{1}{10}]$ and $[\beta_{2k}-\frac{1}{10},\beta_{2k}]$, $k\in\mathbb{N}$. Then also ϕ is well defined and $\in C_{ub}(\mathbb{R},\mathbb{R})$.

To get a non-ergodic ϕ, choose the I_n recursively with $I_1=[0,1]$ as follows (Zorn's Lemma):

If $I_1,\cdots,I_{2k}$ are defined, take $\alpha_{2k+1}:=\beta_{2k}$ and β_{2k+1} such that $\frac{\alpha_{2k}}{\beta_{2k+1}}<\frac{1}{5}$;

If $I_1,\cdots,I_{2k-1}$ are defined, take $\alpha_{2k}:=\beta_{2k-1}$ and β_{2k} such that

$$\frac{\beta_{2k}-\alpha_{2k}-\frac{1}{5}-2}{\beta_{2k}}>\frac{3}{4}.$$

Finally, define $f:=\phi h|J,\in C_{ub}(J,H)$. Then

$$\lim\inf_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f|(t)\,dt=lim\inf_{T\to\infty}\frac{1}{2T}\int_0^T|f|(t)\,dt\le$$

$$\lim\inf_{T\to\infty}\frac{1}{2T}\int_0^T\phi(t)\,dt\le\lim\inf_{k\to\infty}\frac{1}{\beta_{2k+1}}\int_0^{\beta_{2k+1}}\phi(t)\,dt\le\frac{1}{5},$$

$$\lim\sup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f|^2(t)dt=limsup_{T\to\infty}\frac{1}{2T}\int_0^T||\phi(t)h(t)||^2\,dt\ge$$

$$\lim\sup_{k\to\infty}\frac{1}{\beta_{2k}}\int_0^{\beta_{2k}}|\phi|^2(t)|h|^2(t)\,dt\ge\lim\sup_{k\to\infty}\frac{1}{\beta_{2k}}[\beta_{2k}-\alpha_{2k}-\frac{1}{5}-2]1^2\frac{1}{3}\ge\frac{1}{4}.$$

Since $|f|^2\le|f|$, the above shows that $m_B(|f|)$ and and $m_B(|f|^2)$ do not exist, $J=\mathbb{R}$ or $\mathbb{R}_+$.

$f\in EAP(J,H)$: With the Eberlein-Smulian theorem [7], p. 430, Theorem 1, one has to show: To each sequence (b_n) from J there exists a subsequence (a_m) and $g\in V:=C_b(J,H)$ with $f_{a_m}\to g$ weakly in V. Now if (b_n) is bounded, there exists a subsequence (c_k) and $c\in J$ with $c_k\to c$, so $f_{c_k}\to f_c$ uniformly on J and so even in the norm of V, since $f\in C_{ub}(J,H)$.

Now assume $a_m\to\infty$; by taking a further subsequence, one can assume $a_{m+1}-a_m>1$ and $a_1\ge1$, $m\in\mathbb{N}$. To apply Proposition 3.1, let (c_k) be any subsequence of (a_m) and $\varepsilon > 0$, then there exist $q,\ k_1, .., k_q \in \mathbb{N}$ with $\frac{1}{q} < \varepsilon^2$, $c_{k_{j+1}} - c_{k_j} > 1$ and $c_{k_1} \geq 1$, $1 \leq j \leq q$. Then the c_{k_j} are in different $[n - \frac{1}{2}, n + \frac{1}{2}]$ intervals for different j, so for any $t \in \mathbb{R}$, $f_{c_{k_j}}(t) = f(c_{k_j} + t) = r_{j,t} e_{p(j,t)}$ with $0 \leq r_{j,t} \leq 1$ and $p(i,t) < p(j,t)$ if $i < j$ and $c_{k_i} + t > \frac{1}{2}$. With i_0 minimal if such i exist and $\theta_{k_j} = \frac{1}{q}$, $1 \leq j \leq q$, else $= 0$, one gets

$$||\sum_{j=1}^{k_q} \theta_{k_j} f(c_{k_j} + t)||_H^2 = ||\sum_{j=1}^{q} \frac{1}{q} f(c_{k_j} + t)||_H^2 =$$

$$||\sum_{j=i_0}^{q} \frac{1}{q} r_{j,t} e_{p(j,t)}||_H^2 = \frac{1}{q^2} \sum_{j=i_0}^{q} (r_{j,t})^2 \leq \frac{q}{q^2} = \frac{1}{q};$$

if no such i_0 exists, the above sum is even $0 \leq \frac{1}{q}$.

This holds for any $t \in \mathbb{R}$, so $|| \sum_{j=1}^{k_q} \theta_{k_j} f_{c_{k_j}} ||_V < \varepsilon$. Therefore by Proposition 3.1 indeed $f_{c_k} \to 0$ weakly in V, $J = \mathbb{R}$ or $\mathbb{R}_+$.

The case $a_m \to -\infty$ ($J = \mathbb{R}$) follows similarly.

$f \in EAP_0(J, H)$: For $(b_n) = (n)$ the above shows $0 \in$ weak closure of orbit $O(f)$.

$|f|$ and $(f(.), f(.))$ not Eberlein almost periodic follows with (2.4). □

Since for the f of Example 3.2 the Bohr mean $m_B(|f|^2)$ does not exist, one has no Parseval equation.

$EAP(J, X) \subset PAP(J, X)$ is also false, already for X = Hilbert space:

Assume $f \in EAP(J, X) \subset PAP(J, X)$. Then $f = g + h$, $g \in AP(J, X)$, $h \in PAP_0(J, X)$; now for $f \in EAP_0(J, X)$ one can show that all Fourier coefficients vanish (for J $= \mathbb{R}_+$ see [14, Theorem 2.4]), for h the same holds , implying $g = 0$, then $f = h \in PAP_0(J, X)$ and so the existence of $m_B(|f|)$, a contradiction for f of Example 3.2.

The proof of Example 3.2 works also for $X = l^p(N, \mathbb{C})$, $1 < p \leq \infty$ and c_0, so $EAP(J, X) \subset PAP(J, X)$ is also false for these X.

Since for any $f \in EAP(J, X)$ the range $f(J)$ is relatively weakly compact, and if $X = l^1 = l^1(M, \mathbb{C})$, any M, this implies $f(J)$ relatively norm compact [10, p. 281 (2)], one has

$$EAP(J, l^1) = EAP_{rc}(J, l^1) \subset PAP(J, l^1).$$

Example 3.3. For $J = \mathbb{R}$ or $\mathbb{R}_+$ and H separable infinite dimensional Hilbert space there exist $f \in EAP_0(J, H)$ with $|f|, |f|^2 \in AP(J, \mathbb{R}) \subset EAP(J, \mathbb{R})$, but $f(J)$ is not relatively compact, $m_B(|f|)$ and $m_B(|f|^2)$ exist and are > 0.

So a converse of (2.9) is not true, even with $|f|, |f|^2 \in EAP(J, \mathbb{R})$ the Parseval equation can fail, such f need not be pseudo almost periodic.

Proof. Choose an orthonormal sequence $(e_n)_{n\in\mathbb{Z}}$ from H. Define $f : \mathbb{R} \to H$ by $f(n) := e_n$, $n \in \mathbb{Z}$, f linear on $[n-\frac{1}{2}, n]$ and on $[n, n+\frac{1}{2}]$, with $f(n-\frac{1}{2}) = 0$, $n \in \mathbb{Z}$. Then f is well defined and $\in C_{ub}(\mathbb{R}, H)$. One can prove that $f \in EAP_0(\mathbb{R}, H)$ as in the proof of Example 3.2. Obviously, $|f| \in C_{ub}(\mathbb{R}, \mathbb{R})$ has period 1 and so $|f| \in AP(\mathbb{R}, \mathbb{R}) \subset EAP(\mathbb{R}, \mathbb{R})$. $f|\mathbb{R}_+$ has the same properties. □

Added in proof: By communication from Kreulich and Ruess, they can construct to each bounded uniformly continuous $g : \mathbb{R} \to [0, \infty)$ an $f \in EAP(\mathbb{R}, H)$ with $|f| = g$ on $\mathbb{R}$.

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