International Journal of Pure and Applied Mathematics
Volume 71 No. 1 2011, 67-82

ON THE CONVOLUTION EQUATION RELATED TO THE KLEIN-GORDON OPERATOR

Amphon Liangprom[1], Kamsing Nonlaopon[2 §]

Department of Mathematics
Khon Kaen University
Khon Kaen, 40002, THAILAND

Abstract: In this paper, we study the distribution $e^{\alpha x}(\Box + m^2)^k\delta$, where $(\Box + m^2)^k$ is the Klein-Gordon operator iterated k times defined by (1.14), k is a non-negative integer, δ is the Dirac-delta distribution, m is a non-negative real number, $x = (x_1, x_2, \ldots, x_n)$ is a variable and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a constant and both are the points in the n-dimensional Euclidean spaces $\mathbb{R}^n$.

At first, the properties of $e^{\alpha x}(\Box+m^2)^k\delta$ are studied and after that we study the application of $e^{\alpha x}(\Box + m^2)^k\delta$ for solving the solution of the convolution equation

$$e^{\alpha x}(\Box + m^2)^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{M} C_r(\Box + m^2)^r\delta,$$

where $u(x)$ is the generalized function and C_r is a constant. It found that the type of solutions of this convolution equation, such as the ordinary function and the singular distribution depend on the relationship between the values of k and M.

AMS Subject Classification: 46F10, 46F12
Key Words: convolution equation, tempered distribution, Klein-Gordon operator, Dirac-delta distribution

1. Introduction

The n-dimensional ultra-hyperbolic operator $\Box^k$ iterated k times defined by

Received: May 5, 2011

[§]Correspondence author

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \quad (1.1)$$

where $p+q=n, n$ is the dimension of space $\mathbb{R}^n$ and k is a non-negative integer.

Consider the linear differential equation of the form

$$\Box^k u(x) = f(x), \quad (1.2)$$

where $u(x)$ and $f(x)$ are generalized functions and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$.

I. M. Gelfand and G. E. Shilov [1] have first introduced the fundamental solution of (1.2), which is a complicated form. Later, S. E. Trione [14] has shown that the generalized function $R_{2k}(x)$ defined by (2.2) is the unique fundamental solution of (1.2) and M. A. Tellez [13] has also proved that $R_{2k}(x)$ exists only for case when p is odd with n odd or even and $p+q=n$.

Next, A. Kananthai [6] has first introduced the operator $\diamondsuit^k$ and was named the diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \quad p+q=n, \quad (1.3)$$

where n is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)$ and k is a non-negative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k\Box^k = \Box^k\triangle^k, \quad (1.4)$$

where $\triangle^k$ is the Laplace operator iterated k times and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \quad (1.5)$$

and $\Box^k$ is the ultra-hyperbolic operator iterated k times defined by (1.1). He has shown that the convolution $(-1)^k S_{2k}(x) * R_{2k}(x)$ is the fundamental solution of the operator $\diamondsuit^k$. That is,

$$\diamondsuit^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta, \quad (1.6)$$

where $S_{2k}(x)$ is defined by

$$S_\gamma(x) = \frac{|x|^{\gamma-n}}{H_n(\gamma)}, \quad (1.7)$$

and

$$H_n(\gamma) = \frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}, \tag{1.8}$$

where α is a complex parameter, n is the dimension of $\mathbb{R}^n$, $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ and the generalized function $R_{2k}(x)$ is defined by (2.2).

In 1997, A. Kananthai [4] studied the properties of the distribution $e^{\alpha x}\Box^k\delta$ and after that he studied the application of the distribution $e^{\alpha x}\Box^k\delta$ for solving the fundamental solution of the equation of the ultra-hyperbolic type by using the convolution method.

In 1998, A. Kananthai [2] studied the properties of the distribution $e^{\alpha x}\Diamond^k\delta$ and its application for solving the solution of the convolution equation

$$e^{\alpha x}\Diamond^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Diamond^r \delta, \tag{1.9}$$

where $\Diamond^k$ is the diamond operator iterated k times defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{1.10}$$

with $p + q = n$, dimension of the Euclidean space $\mathbb{R}^n$. Recently, K. Nonlaopon gave some generalizations of this paper; for more details, see [8].

In 2000, A. Kananthai [3] studied the application of the distribution $e^{\alpha x}\Box^k\delta$ for solving the solutions of the convolution equation

$$e^{\alpha x}\Box^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Box^r \delta, \tag{1.11}$$

which is related to the ultra-hyperbolic equation.

In 2009, P. Sasopa and K. Nonlaopon [10] studied the properties of the distribution $e^{\alpha x}\Box_c^k\delta$ and its application to solve the solution of the convolution equation

$$e^{\alpha x}\Box_c^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Box_c^r \delta, \tag{1.12}$$

where $\Box_c^k$ is the operator which related to the ultra-hyperbolic type operator iterated k times defined by

$$\Box_c^k = \left(\frac{1}{c^2}\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^k, \tag{1.13}$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$.

In 1988, S. E. Trione [16] studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated k times defined by

$$(\Box+m^2)^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} + m^2\right)^k. \tag{1.14}$$

The fundamental solution of the operator $(\Box+m^2)^k$ is $W_{2k}(x,m)$, and is defined by (2.8) with $\gamma = 2k$. Next, M. A. Tellez [12] has studied the convolution product of $W_\alpha(x,m) * W_\beta(x,m)$, where α and β are any complex parameter. Later, S. E. Trione [15] has studied the fundamental $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator iterated k times and she has also studied the convolution of such fundamental solution.

In this paper, we study the properties of the distribution $e^{\alpha x}(\Box+m^2)^k\delta$ and the application of $e^{\alpha x}(\Box+m^2)^k\delta$ for solving the solutions of the convolution equation

$$e^{\alpha x}(\Box+m^2)^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{M} C_r(\Box+m^2)^r\delta, \tag{1.15}$$

where $(\Box+m^2)^k$ is the Klein-Gordon operator iterated k times defined by (1.14), $u(x)$ is the generalized function and C_r is a constant. In finding the type of solution $u(x)$ of (1.15), we use the method of convolution of the generalized functions.

Before going to that point, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimension of the Euclidean space $\mathbb{R}^n$. Denote by

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

the nondegenerated quadratic form and $p+q=n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$ and $\overline{\Gamma}_+$ denote its closure. For any complex number γ, define the function

$$R_\gamma(x) = \begin{cases} \frac{v^{(\gamma-n)/2}}{K_n(\gamma)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\gamma-n}{2}\right)\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma(\gamma)}{\Gamma\left(\frac{\gamma-p+2}{2}\right)\Gamma\left(\frac{p-\gamma}{2}\right)}. \tag{2.3}$$

The function $R_\gamma(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [9]. It is well known that such function is an ordinary function if $Re(\gamma) \geq n$ and is a distribution of γ if $Re(\gamma) < n$. Let supp $R_\gamma(x)$ denote the support of $R_\gamma(x)$ and suppose that supp $R_\gamma(x) \subset \overline{\Gamma}_+$, that is, supp $R_\gamma(x)$ is compact.

By putting $p = 1$ in (2.1) and (2.2), and taking into account Legendre's duplication formula for $\Gamma(z)$, that is

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.4}$$

we obtain

$$I_\gamma(x) = \frac{w^{(\gamma-n)/2}}{H_n(\gamma)}, \tag{2.5}$$

and $w = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\gamma) = \pi^{(n-2)/2}2^{\gamma-1}\Gamma\left(\frac{\gamma+2-n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.6}$$

$I_\gamma(x)$ is called the hyperbolic kernel of Marcel Riesz.

Lemma 2.1. *Given the equation $\Box^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\Box^k$ is defined by (1.1), then*

$$u(x) = R_{2k}(x),$$

where $R_{2k}(x)$ is defined by (2.2), with $\gamma = 2k$.

We obtain $R_{2k}(x)$ is the fundamental solution of the operator $\Box^k$, that is,

$$\Box^k R_{2k}(x) = \delta. \tag{2.7}$$

The proof of this Lemma is given in [14].

It can be shown that $R_{-2k}(x) = \Box^k\delta$, for k is non-negative integer.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and the function $W_\gamma(x,m)$ is defined by

$$W_\gamma(x,m) = \sum_{\nu=0}^{\infty}\binom{-\gamma/2}{\nu}m^{2\nu}R_{\gamma+2\nu}(x), \tag{2.8}$$

where γ is a complex parameter, m is a non-negative real number and $R_{\gamma+2\nu}(x)$ is defined by (2.2) .

From the definition of $W_\gamma(x,m)$ and by putting $\gamma = -2k$, for k is non-negative integer, we have

$$W_{-2k}(x,m) = \sum_{\nu=0}^{\infty} \binom{k}{\nu} m^{2\nu} R_{2(-k+\nu)}(x). \tag{2.9}$$

Since the operator $(\Box + m^2)^k$ defined by (1.14) is linearly continuous and has $1-1$ mapping of this possess its own inverses. From Lemma 2.1, we obtain

$$W_{-2k}(x,m) = \sum_{\nu=0}^{\infty} \binom{k}{\nu} m^{2\nu} \Box^{k-\nu}\delta = (\Box + m^2)^k \delta. \tag{2.10}$$

By putting $k = 0$ in (2.10), we have $W_0(x,m) = \delta$. And by putting $\gamma = 2k$ into (2.8), we have

$$W_{2k}(x,m) = \binom{-k}{0} m^{2(0)} R_{2k+0}(x) + \sum_{\nu=1}^{\infty} \binom{-k}{\nu} m^{2\nu} R_{2k+2\nu}(x). \tag{2.11}$$

The second summand of the right-hand member of (2.11) vanishes for $m^2 = 0$ and then, we have

$$W_{2k}(x, m = 0) = R_{2k}(x),$$

is the fundamental solution of the ultra-hyperbolic operator $\Box^k$.

Lemma 2.2. *Let $W_\gamma(x,m)$ is defined by (2.8), then*

$$W_\gamma(x,m) * W_{2k}(x,m) = W_{\gamma+2k}(x,m)$$

for k is a non-negative integer.

The proof of this Lemma is given in [12].

Lemma 2.3. *Given the equation*

$$(\Box + m^2)^k u(x) = \delta, \tag{2.12}$$

where $(\Box + m^2)^k$ is the Klein-Gordon operator iterated k times defined by

$$(\Box + m^2)^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \right)^k, \tag{2.13}$$

k is a nonnegative integer and δ is the Dirac-delta distribution. Then $u(x) = W_{2k}(x,m)$ is the fundamental solution of the Klein-Gordon operator iterated k times $(\Box + m^2)^k$, where $W_{2k}(x,m)$ is defined by (2.8) with $\gamma = 2k$.

The proof of this Lemma is given in [5].

3. Properties of the Distribution $e^{\alpha x}(\square + m^2)^k\delta$

First, we shall consider the distribution $e^{\alpha x}(\square + m^2)\delta$ with $k = 1$.

Lemma 3.1. *The distribution $e^{\alpha x}(\square + m^2)\delta$ has the following properties :*

Proposition 3.1. *For the Klein-Gordon operator $\square+m^2$ defined by (2.13) with $k = 1$, then*

$$e^{\alpha x}(\square + m^2)\delta = (\square + m^2)\delta - 2\left(\sum_{i=1}^{p} \alpha_i \frac{\partial \delta}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \delta}{\partial x_j}\right) + \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right)\delta \quad (3.1)$$

and $e^{\alpha x}(\square + m^2)\delta$ is a tempered distribution of second order with support $\{0\}$.

Proof. Let $\varphi \in \mathscr{D}$ be the space of testing functions, infinitely differentiable with compact supports and $\mathscr{D}'$ be the space of distributions. Now

$$\left\langle e^{\alpha x}(\square + m^2)\delta, \varphi(x)\right\rangle = \left\langle \delta, (\square + m^2)e^{\alpha x}\varphi(x)\right\rangle,$$

for $e^{\alpha x}(\square + m^2)\delta \in \mathscr{D}'$. By computing directly, we obtain

$$\begin{aligned}(\square + m^2)e^{\alpha x}\varphi(x) &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2\right) e^{\alpha x}\varphi(x) \\ &= e^{\alpha x}(\square + m^2)\varphi(x) + 2e^{\alpha x}\left(\sum_{i=1}^{p} \alpha_i \frac{\partial \varphi(x)}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \varphi(x)}{\partial x_j}\right) \\ &\quad + e^{\alpha x}\left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right)\varphi(x). \end{aligned} \quad (3.2)$$

Then

$$\begin{aligned}\left\langle \delta, (\square + m^2)e^{\alpha x}\varphi(x)\right\rangle &= (\square + m^2)\varphi(0) + 2\left(\sum_{i=1}^{p} \alpha_i \frac{\partial \varphi(0)}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \varphi(0)}{\partial x_j}\right) \\ &\quad + \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right)\varphi(0)\end{aligned}$$

$$= \left\langle (\Box + m^2)\delta - 2\left(\sum_{i=1}^{p} \alpha_i \frac{\partial \delta}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \delta}{\partial x_j}\right) + \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2\right)\delta, \varphi(x)\right\rangle. \quad (3.3)$$

By equality of distributions, we obtain (3.1) as required. To show that $e^{\alpha x}(\Box + m^2)\delta$ is a tempered distribution, from (3.1) $\delta, \partial\delta/\partial x_i, \partial\delta/\partial x_j$ and $(\Box + m^2)\delta$ have support $\{0\}$ which is compact, hence, by L. Schwartz [11], they are tempered distributions. From (3.1), it follows that $e^{\alpha x}(\Box + m^2)\delta$ is also tempered distributions and by A. H. Zemanian [17, Theorem 3.5-2, p.98] $e^{\alpha x}(\Box + m^2)\delta$ is of second order with point support $\{0\}$. □

Proposition 3.2. (Boundedness property). *For every testing function $\varphi \in S$, the Schwartz space and $e^{\alpha x}(\Box + m^2)\delta \in S'$, the space of tempered distributions then $\left|\left\langle e^{\alpha x}(\Box + m^2)\delta, \varphi\right\rangle\right| \leq CM$ where C and M are constant with*

$$\begin{aligned} M &= \max\left\{|\varphi(0)|, \left|\frac{\partial \varphi(0)}{\partial x_i}\right|, \left|\frac{\partial \varphi(0)}{\partial x_j}\right|, \left|(\Box + m^2)\varphi(0)\right|\right\} \\ C &= 1 + 2\sum_{i=1}^{p}|\alpha_i| + 2\sum_{j=p+1}^{p+q}|\alpha_j| + \sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2 \end{aligned} \quad (3.4)$$

Proof. Since $\left\langle e^{\alpha x}(\Box + m^2)\delta, \varphi(x)\right\rangle = \left\langle \delta, (\Box + m^2)e^{\alpha x}\varphi(x)\right\rangle$, hence by (3.3) we have

$$\left|\left\langle e^{\alpha x}(\Box + m^2)\delta, \varphi\right\rangle\right| \leq \left|(\Box + m^2)\varphi(0)\right| + 2\sum_{i=1}^{p}|\alpha_i|\left|\frac{\partial \varphi(0)}{\partial x_i}\right| + 2\sum_{j=p+1}^{p+q}|\alpha_j|\left|\frac{\partial \varphi(0)}{\partial x_j}\right| + \left(\sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2\right)|\varphi(0)|.$$

Let $M = \max\left\{|\varphi(0)|, |\partial\varphi(0)/\partial x_i|, |\partial\varphi(0)/\partial x_j|, \left|(\Box + m^2)\varphi(0)\right|\right\}$, then

$$\left|\left\langle e^{\alpha x}(\Box + m^2)\delta, \varphi\right\rangle\right| \leq \left(1 + 2\sum_{i=1}^{p}|\alpha_i| + 2\sum_{j=p+1}^{p+q}|\alpha_j| + \sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2\right) M.$$

It follows that $\left|\left\langle e^{\alpha x}(\Box + m^2)\delta, \varphi\right\rangle\right| \leq CM$, where C is defined by (3.4). □

Lemma 3.2. *Given $u(x)$ any distribution in the space S', then*

$$e^{\alpha x}(\Box+m^2)\delta*u(x)=(\Box+m^2)u(x)-2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial u(x)}{\partial x_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial u(x)}{\partial x_j}\right)+\left(\sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{p+q}\alpha_j^2\right)u(x). \tag{3.5}$$

Proof. Convolving both sides of (3.1) by $u(x)$, we obtain (3.5) as required. If L is the operator defined by

$$L\equiv(\Box+m^2)-2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial x_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial x_j}\right)+\left(\sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{p+q}\alpha_j^2\right). \tag{3.6}$$

Then (3.5) can be written as $e^{\alpha x}(\Box+m^2)\delta*u(x)=Lu(x)$. □

Lemma 3.3. *(*The generalization of Lemma 3.2*).*

$$e^{\alpha x}(\Box+m^2)^k\delta*u(x)=L^ku(x), \tag{3.7}$$

where L^k is the operator defined by (3.6) iterated k times with $L^0u(x)=u(x)$.

Proof. We have $\langle e^{\alpha x}(\Box+m^2)^k\delta,\varphi(x)\rangle=\langle(\Box+m^2)^k\delta,e^{\alpha x}\varphi(x)\rangle$ for every $\varphi(x)\in\mathscr{D}$ and $e^{\alpha x}(\Box+m^2)^k\delta\in\mathscr{D}'$. So

$$\begin{aligned}\left\langle(\Box+m^2)^k\delta,e^{\alpha x}\varphi(x)\right\rangle&=\left\langle(\Box+m^2)^{k-1}\delta,(\Box+m^2)e^{\alpha x}\varphi(x)\right\rangle\\&=\left\langle(\Box+m^2)^{k-1}\delta,e^{\alpha x}T\varphi(x)\right\rangle,\end{aligned}$$

where T is the partial differential operator from (3.2) defined by

$$T\equiv(\Box+m^2)+2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial x_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial x_j}\right)+\left(\sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{p+q}\alpha_j^2\right). \tag{3.8}$$

Thus,

$$\begin{aligned}\left\langle(\Box+m^2)^{k-1}\delta,e^{\alpha x}T\varphi(x)\right\rangle&=\left\langle(\Box+m^2)^{k-2}\delta,(\Box+m^2)e^{\alpha x}T\varphi(x)\right\rangle\\&=\left\langle(\Box+m^2)^{k-2}\delta,e^{\alpha x}T(T\varphi(x))\right\rangle\\&=\left\langle(\Box+m^2)^{k-2}\delta,e^{\alpha x}T^2\varphi(x)\right\rangle.\end{aligned}$$

By keeping on operating $(\Box + m^2)$ with $k-2$ times, we obtain

$$\left\langle (\Box + m^2)^{k-2}\delta, e^{\alpha x}T^2\varphi(x)\right\rangle = \left\langle \delta, e^{\alpha x}T^k\varphi(x)\right\rangle = T^k\varphi(0),$$

where T^k is the operator of (3.8) iterated k times. Now

$$T^k\varphi(0) = \left\langle \delta, T^k\varphi(x)\right\rangle = \left\langle L\delta, T^{k-1}\varphi(x)\right\rangle,$$

by the operator L in (3.6) and the derivative of distribution. Continuing this process, we obtain $T^k\varphi(0) = \left\langle L^k\delta, \varphi(x)\right\rangle$ or $\left\langle e^{\alpha x}(\Box + m^2)^k\delta, \varphi(x)\right\rangle = \left\langle L^k\delta, \varphi(x)\right\rangle$. It follows that

$$e^{\alpha x}(\Box + m^2)^k\delta = L^k\delta. \tag{3.9}$$

Convolving both sides of (3.9) by distribution $u(x)$, then we obtain (3.7) as required. $\Box$

4. Main Results

Theorem 4.1. *Let L be the partial differential operator defined by*

$$L \equiv (\Box + m^2) - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial x_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right),$$

where this operator appears in (3.1). Consider the equation

$$Lu(x) = \delta, \tag{4.1}$$

where $u(x)$ is any distribution in S', then $u(x) = e^{\alpha x}W_2(x,m)$ is the fundamental solution of (4.1), where $W_2(x,m)$ is defined by (2.8) with $\gamma = 2$.

Proof. From (3.1) and (4.1) we can write $e^{\alpha x}(\Box+m^2)\delta * u(x) = Lu(x) = \delta$. Convolving both sides by $e^{\alpha x}W_2(x,m)$, we have

$$e^{\alpha x}W_2(x,m) * e^{\alpha x}(\Box + m^2)\delta * u(x) = e^{\alpha x}W_2(x,m) * \delta.$$

Then

$$e^{\alpha x}(W_2(x,m) * (\Box + m^2)\delta) * u(x) = e^{\alpha x}W_2(x,m),$$

or equivalently,

$$e^{\alpha x}\left((\Box + m^2)W_2(x,m)\right) * u(x) = e^{\alpha x}W_2(x,m).$$

Because $(\Box + m^2)W_2(x,m) = \delta$ by Lemma 2.3 with $k = 1$, we obtain $(e^{\alpha x}\delta) * u(x) = e^{\alpha x}W_2(x,m)$. Since $e^{\alpha x}\delta = \delta$, then $\delta * u(x) = e^{\alpha x}W_2(x,m)$. It follows that $u(x) = e^{\alpha x}W_2(x,m)$ is the fundamental solution of the operator L. $\Box$

Theorem 4.2. *(The generalization of Theorem 4.1). From Lemma 3.3, consider*

$$e^{\alpha x}(\Box + m^2)^k\delta * u(x) = \delta \tag{4.2}$$

or

$$L^k u(x) = \delta, \tag{4.3}$$

then $u(x) = e^{\alpha x}W_{2k}(x, m)$ *is the fundamental solution of the operator* L^k.

Proof. We can prove by using equation (4.2) or (4.3) as well. If we start with equation (4.2), by convolving both sides of (4.2) by $e^{\alpha x}W_{2k}(x, m)$, we obtain

$$e^{\alpha x}W_{2k}(x, m) * \Big(e^{\alpha x}(\Box + m^2)^k\delta * u(x)\Big) = e^{\alpha x}W_{2k}(x, m) * \delta,$$

or $e^{\alpha x}((\Box+m^2)^k W_{2k}(x, m))*u(x) = e^{\alpha x}W_{2k}(x, m)$. Since $(\Box+m^2)^k W_{2k}(x, m) = \delta$ by Lemma 2.3, we have $(e^{\alpha x}\delta)*u(x) = e^{\alpha x}W_{2k}(x, m)$ or $u(x) = e^{\alpha x}W_{2k}(x, m)$ as required. Or if we use equation (4.3), by convolving both sides of (4.3) by $e^{\alpha x}W_2(x, m)$, then we obtain

$$e^{\alpha x}W_2(x, m) * L^k u(x) = e^{\alpha x}W_2(x, m) * \delta,$$

or $L(e^{\alpha x}W_2(x, m)) * L^{k-1}u(x) = e^{\alpha x}W_2(x, m)$. By Theorem 4.1, we obtain $L^{k-1}u(x) = e^{\alpha x}W_2(x, m)$. By keeping on convolving $e^{\alpha x}W_2(x, m)$ with $k-1$ times, we obtain

$$u(x) = e^{\alpha x}(W_2(x, m) * W_2(x, m) * \cdots * W_2(x, m)) = e^{\alpha x}W_{2k}(x, m),$$

by Lemma 2.2 and [7, p.196]. □

In particular, if we put $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ in (4.2), then (4.2) reduces to (2.12) and we obtain $u(x) = W_{2k}(x, m)$ is the fundamental solution of the Klein-Gordon operator iterated k times.

Theorem 4.3. *Given the convolution equation*

$$e^{\alpha x}(\Box + m^2)^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{M} C_r(\Box + m^2)^r\delta, \tag{4.4}$$

where $(\Box + m^2)^k$ *is the Klein-Gordon operator iterated* k *times defined by*

$$(\Box + m^2)^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} + m^2\right)^k,$$

$p+q=n, n$ is odd with p odd and q even, or n even with p odd and q odd, the variable $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, the constant $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in\mathbb{R}^n$, m is a non-negative real number, δ is the Dirac-delta distribution with $(\Box+m^2)^0\delta=\delta, (\Box+m^2)^1\delta=(\Box+m^2)\delta$ and C_r is a constant. Then, the type of solutions $u(x)$ of (4.4) that depend on the relationship between the values of k and M are as the following cases:

(1) If $M<k$ and $M=0$, then the solution of (4.4) is

$$u(x)=C_0e^{\alpha x}W_{2k}(x,m).$$

Now $W_{2k}(x,m)$ is defined by (2.8) with $\gamma=2k$. If $2k\geq n$ and for any α, then $e^{\alpha x}W_{2k}(x,m)$ is the ordinary function.

(2) If $0<M<k$, then the solution of (4.4) is

$$u(x)=e^{\alpha x}\sum_{r=1}^{M}C_rW_{2k-2r}(x,m)$$

which is the ordinary function for $2k-2r\geq n$ with any arbitrary constant α.

(3) If $M\geq k$ and for any α, suppose $k\leq M\leq N$, then (4.4) has

$$u(x)=e^{\alpha x}\sum_{r=k}^{N}C_r(\Box+m^2)^{r-k}\delta$$

as a solution which is the singular distribution.

Proof. (1) For $M<k$ and $M=0$, then (4.4) becomes

$$e^{\alpha x}(\Box+m^2)^k\delta*u(x)=C_0e^{\alpha x}\delta=C_0\delta$$

and by Theorem 4.2, we obtain $u(x)=C_0e^{\alpha x}W_{2k}(x,m)$. Now $W_{2k}(x,m)$, is defined by (2.8) with $\gamma=2k$, is the ordinary function for $2k\geq n$, since $R_{2k}(x)$ is the ordinary function for $2k\geq n$, then $R_{2k+2\nu}(x)$ is also the ordinary function for $2k\geq n$. It follows that $C_0e^{\alpha x}W_{2k}(x,m)$ is the ordinary function for $2k\geq n$ with any α.

(2) For $0<M<k$, we have

$$\begin{aligned}&e^{\alpha x}(\Box+m^2)^k\delta*u(x)\\&\qquad=e^{\alpha x}\left[C_1(\Box+m^2)\delta+C_2(\Box+m^2)^2\delta+\cdots+C_M(\Box+m^2)^M\delta\right].\end{aligned}$$

Convolving both sides by $e^{\alpha x}W_{2k}(x,m)$ and by Lemma 2.3, we obtain

$$u(x) = e^{\alpha x}\left[C_1(\Box + m^2)W_{2k}(x,m) + \cdots + C_M(\Box + m^2)^M W_{2k}(x,m)\right].$$

Now $(\Box + m^2)^k W_{2k}(x,m) = \delta$, then $(\Box + m^2)^{k-r}(\Box + m^2)^r W_{2k}(x,m) = \delta$ for $r < k$. Convolving both sides by $W_{2k-2r}(x,m)$, we obtain

$$W_{2k-2r}(x,m) * (\Box + m^2)^{k-r}(\Box + m^2)^r W_{2k}(x,m) = W_{2k-2r}(x,m),$$

or

$$(\Box + m^2)^{k-r}W_{2k-2r}(x,m) * (\Box + m^2)^r W_{2k}(x,m) = W_{2k-2r}(x,m),$$

or

$$(\Box + m^2)^r W_{2k}(x,m) = W_{2k-2r}(x,m)$$

for $r < k$. It follows that

$$u(x) = e^{\alpha x}\left[C_1 W_{2k-2}(x,m) + C_2 W_{2k-4}(x,m) + \cdots + C_M W_{2k-2M}(x,m)\right],$$

or

$$u(x) = e^{\alpha x}\sum_{r=1}^{M} C_r W_{2k-2r}(x,m).$$

Similarly, as in the case (1), $e^{\alpha x}W_{2k-2r}(x,m)$ is the ordinary function for $2k - 2r \geq n$ with any α. It follows that

$$u(x) = e^{\alpha x}\sum_{r=1}^{M} C_r W_{2k-2r}(x,m)$$

is also the ordinary function with any α.

(3) For $M \geq k$ and for any α, suppose $k \leq M \leq N$, we have

$$\begin{aligned} &e^{\alpha x}(\Box + m^2)^k \delta * u(x) \\ &\quad = e^{\alpha x}\left[C_k(\Box + m^2)^k\delta + C_{k+1}(\Box + m^2)^{k+1}\delta + \cdots + C_N(\Box + m^2)^N\delta\right]. \end{aligned}$$

Convolving both sides by $e^{\alpha x}W_{2k}(x,m)$ and by Lemma 2.3 again, we have

$$u(x) = e^{\alpha x}\left[C_k(\Box + m^2)^k W_{2k}(x,m) + \cdots + C_N(\Box + m^2)^N W_{2k}(x,m)\right].$$

Now

$$(\Box + m^2)^M W_{2k}(x,m) = (\Box + m^2)^{M-k}(\Box + m^2)^k W_{2k}(x,m) = (\Box + m^2)^{M-k}\delta$$

for $k \leq M \leq N$. So

$$\begin{aligned} u(x) &= e^{\alpha x}\Big[C_k\delta + C_{k+1}(\square + m^2)\delta + C_{k+2}(\square + m^2)^2\delta + \cdots \\ &\quad + C_N(\square + m^2)^{N-k}\delta\Big] \\ &= e^{\alpha x}\sum_{r=k}^{N} C_r(\square + m^2)^{r-k}\delta. \end{aligned}$$

Now, by (3.6) and (3.9) we have

$$\begin{aligned} &e^{\alpha x}(\square + m^2)^{r-k}\delta \\ &\quad = (\square + m^2)^{r-k}\delta + (\text{the terms of lower order of partial derivative of } \delta), \end{aligned}$$

for $k \leq r \leq N$. Since all terms of the right-hand side of above equation are singular distribution, it follows that

$$u(x) = e^{\alpha x}\sum_{r=k}^{N} C_r(\square + m^2)^{r-k}\delta$$

is the singular distribution. That completes the proof. $\square$

Acknowledgments

This work was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission, through the Cluster of Research to Enhance the Quality of Basic Education.

References

[1] I.M. Gelfand, G.E. Shilov, *Generalized Functions, Volume I: Properties and Operations*, Academic Press, New York (1964).

[2] A. Kananthai, On the convolution equation related to the diamond kernel of Marcel Riesz, *Journal of Computational and Applied Mathematics*, **100** (1998), 33-39.

[3] A. Kananthai, On the convolution equation related to the N-dimentional ultra-hyperbolic operator, *Journal of Computational and Applied Mathematics*, **115** (2000), 301-308.

[4] A. Kananthai, On the distribution related to the ultra-hyperbolic equations, *Journal of Computational and Applied Mathematics*, **84** (1997), 101-106.

[5] A. Kananthai, On the green function of the diamond operator related to the Klein-Gordon operator, *Bulletin of Calcutta Mathematical Society*, **93**, No. 5 (2001), 353-360.

[6] A. Kananthai, On the solution of the n-dimensional diamond operator, *Applied Mathematics and Computation*, **88** (1997), 27-37.

[7] R.P. Kanwal, *Generalized Functions: Theory and Technique*, Second Edition, Birkhäuser, Boston, Hamilton Printing (1998).

[8] K. Nonlaopon, The convolution equation related to the diamond kernel of Marcel Riesz, *Far East Journal of Mathematical Sciences*, **45**, No. 1 (2010), 121-134.

[9] Y. Nozaki, On Riemann-Liouville integral of ultra-hyperbolic type, *Kodai Mathematical Seminar Reports*, **6**, No. 2 (1964), 69-87.

[10] P. Sasopa, K. Nonlaopon, The convolution equation related to the ultra-hyperbolic equation, *Far East Journal of Mathematical Sciences*, **35**, No. 3 (2009), 295-308

[11] L. Schwartz, *Theories des Distribution*, Volumes 1 and 2, Actualites Scientifiques et Industriel, Hermann and Cie, Paris (1957, 1959).

[12] M.A. Tellez, The convolution product of $W_\alpha(u,m) * W_\beta(u,m)$, *Mathematicae Notae*, **38** (1995-96), 105-111.

[13] M.A. Tellez, The ditribution Hankel transform of Marcel Riesz's ultra-hyperbolic kernel, *Studied in Applied Mathematics*, **93** (1994), 133-162.

[14] S.E. Trione, On Marcel Riesz's ultra hyperbolic kernel, *Trabajos de Mathematica*, **116** (1987), 1-12.

[15] S.E. Trione, On the elementary $(P \pm i0)^\lambda$ ultrahyperbolic solution of the Klein-Gordon operator iterated k-times, *Integral Transforms and Special Functions*, **9**, No. 2 (2000), 149-162.

[16] S.E. Trione, On the elementary retarded ultra-hyperbolic solution of the Klein-Gordon operator iterated k-times, *Studies in Applied Mathematics*, **79**, No. 2 (1988), 121-141.

[17] A.H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York (1964).