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ORBIT DIRICHLET SERIES FOR POWERS OF MAPS

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Abstract: The orbit Dirichlet series for maps are studied. The main results find the orbit Dirichlet series of prime powers of maps with polynomial orbit growth.

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1. Introduction

A *Dirichlet series* is any series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where s and a_n are complex numbers, $n = 1, 2, 3, \cdots$. The most famous Dirichlet series is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the Riemann zeta function (see details in [1]).

In 2009, Pakapongpun and Ward [2] studied the orbit Dirichlet series

$$d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s}$$

where T is a continuous mapping $T : X \to X$, X is a compact metric space and $O_n(T)$ is the number of closed orbit of length n under T.

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Recently, Pakapongpun studied $d_{T\times T\times\cdots\times T}(s)$ where there are m terms in the cartesian product (the detailed calculation is in [3]).

The orbit Dirichlet series $d_{T^p}(s)$ is studied in this paper, where $O_n(T) = n^a$ and p is a prime number. This is a simple example of a set of problems studied in [2] and [3].

2. Preliminary Notes

Let T be a map. A *closed orbit* τ of length $|\tau|$ is a set of the form

$$\{x, Tx, T^2x, \cdots, T^{|\tau|}x = x\}$$

with cardinality $|\tau|$. Write $O_n(T)$ for the number of closed orbits of length n under T. We always assume that $O_n(T) < \infty$ for all $n \geq 1$.

The number of points of period n is

$$F_n(T) = \sum_{d|n} dO_d(T),$$

and the relation with the number of closed orbit is given by

$$O_n(T) = \frac{1}{n}\sum_{d|n} \mu(\frac{n}{d})F_d(T)$$

where $\mu : \mathbb{N} \to \mathbb{R}$ is the Möbius function (see Definition 2.3 below).

The orbit Dirichlet series associated to T is

$$d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s}.$$

Notice that if $O_n(T) = n^a$, then $d_T(s) = \zeta(s - a)$.

The asymptotic behaviour of expression like

$$\pi_T(N) = \#\{\tau : |\tau| \leq N\} = \sum_{n\leq N} O_n(T),$$

where τ is a closed orbit of length $|\tau| = n$ is studied in dynamical systems (see details in [1]).

The number of orbits of length n under an iterate T^p for some prime p will be shown in the next theorem.

Theorem 2.1. *Let p be a prime. Then*

$$O_n(T^p) = \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T)) & \text{if } p \mid n. \end{cases}$$

Proof. We know that

$$O_n(T^p) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) F_{pd}(T), \quad \text{since} \quad F_n(T^p) = F_{pn}(T).$$

It $p \nmid n$ then

$$\begin{aligned} \sum_{d|pn} \mu\left(\frac{pn}{d}\right) F_d(T) &= \sum_{\substack{d|pn \\ p|d}} \mu\left(\frac{pn}{d}\right) F_d(T) + \sum_{\substack{d|pn \\ p\nmid d}} \mu\left(\frac{pn}{d}\right) F_d(T) \\ &= \sum_{\substack{d|pn \\ p|d}} \mu\left(\frac{pn}{d}\right) F_d(T) - \sum_{d|n} \mu\left(\frac{n}{d}\right) F_d(T) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) F_{pd}(T) - \sum_{d|n} \mu\left(\frac{n}{d}\right) F_d(T). \end{aligned}$$

Thus,

$$\begin{aligned} O_n(T^p) &= \frac{1}{n} \sum_{d|pn} \mu\left(\frac{pn}{d}\right) F_d(T) + \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) F_d(T) \\ &= pO_{pn}(T) + O_n(T). \end{aligned}$$

If $p \mid n$ then

$$O_n(T^p) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) F_{pd}(T).$$

Let $pd = k$, $\frac{k}{p} = d$, so $k \mid pn$. Then

$$\begin{aligned} O_n(T^p) &= \frac{1}{n} \sum_{d|pn} \mu\left(\frac{pn}{d}\right) F_d(T) \\ &= pO_{pn}(T). \end{aligned}$$

So,

$$O_n(T^p) = \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T)) & \text{if } p \mid n. \end{cases}$$

$\square$

Definition 2.2. For real or complex k and integer $n \geq 1$ we define

$$\sigma_k(n) = \sum_{d|n} d^k$$

the sum of the k^{th} power of the divisor of n. When $k = 1$, $\sigma_1(n)$ is the sum of divisors of n; this is often denoted by $\sigma(n)$.

Definition 2.3. The Möbius function $\mu : \mathbb{N} \to \mathbb{R}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1; \\ 0 & \text{if any } a_i > 1, \end{cases}$$

where for $n > 1$, we write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $a_i \geq 1$ for the factorization of n into distinct primes.

Theorem 2.4. *Suppose that the series $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ does not converge for all s or diverge for all s. Then there exists a real number x_a, called the abscissa of absolute convergence, such that the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely if $\Re > x_a$ but does not converge absolutely if $\Re < x_a$.*

Theorem 2.5. *(Perron's formula). Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be absolutely convergent for $\Re(s) > x_a$. Then for $c > x_a$,*

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} F(z) \frac{n^z}{z} dz = \sideset{}{^*}\sum_{n \leq N} f(n)$$

for any $N > 0$. Here $\sum^*$ means that the last term in the sum must be multiplied by $\frac{1}{2}$ when N is an integer.

Definition 2.2, 2.3, and Theorem 2.4, 2.5 are taken from [1].

3. Main Results

Theorem 3.1. *Let T be map, let p be a prime number and let $k \in \mathbb{N}_0$. If $d_T(s) = \zeta(s-k)$, then*

1. $d_{T^p}(s) = \left(p^{k+1} + 1 - p^{k-s}\right) \zeta(s-k)$,
2. *the abscissa of convergence of d_{T^p} is $k+1$, and*
3. $\pi_{T^p}(N) \sim \left(p^{k+1} + 1 - p^{-1}\right) \frac{N^{k+1}}{k+1}$.

Proof. 1. Clearly, $d_T(s) = \zeta(s-k)$ if and only if $O_n(T) = n^k$, and we know that

$$\begin{aligned}
O_n(T^p) &= \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) F_{pd}(T) \\
&= \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma(pd) \\
&= \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T)) & \text{if } p \mid n; \end{cases} \quad \text{by Theorem 2.1} \\
&= \begin{cases} p^{k+1}n^k + n^k & \text{if } p \nmid n; \\ p^{k+1}n^k & \text{if } p \mid n. \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
d_{T^p}(s) &= \sum_{n=1}^{\infty} \frac{p^{k+1}n^k}{n^s} + \sum_{p\nmid n} \frac{n^k}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{p^{k+1}n^k}{n^s} + \sum_{n=1}^{\infty} \frac{n^k}{n^s} - \sum_{n=1}^{\infty} \frac{(pn)^k}{(pn)^s} \\
&= \left(p^{k+1} + 1\right) \zeta(s-k) - p^{k-s}\zeta(s-k) \\
&= \left(p^{k+1} + 1 - p^{k-s}\right) \zeta(s-k).
\end{aligned}$$

2. As we know the abscissa of convergence of the Riemann zeta function is 1, so the abscissa of convergence of $d_T(s) = \zeta(s-k)$ is $k+1$ (and $\left(p^{k+1} + 1 - p^{k-s}\right)$ convergences for any $s \in \mathbb{C}$). Hence, the abscissa of convergence d_{T^p} is $k+1$.

3. Notice that $F(s) = \sum_{n=1}^{\infty} \frac{O_n(T^p)}{n^s} = d_{T^p}(s)$ is absolutely convergent for $\Re(s) > k+1$. If $\Re(s) > k+1-\epsilon$ for some small $\epsilon > 0$, then

$$\begin{aligned}
\sum_{n\le N}^{*} O_n(T^p) &= \frac{1}{2\pi i}\int_{\epsilon-\infty i}^{\epsilon+\infty i} F(z)\frac{N^z}{z}dz \qquad \text{(by Perron's formula)} \\
&= Res\left(d_{T^p}(s)\frac{N^z}{z}\right)_{z=k+1} \\
&= \left(p^{k+1} + 1 - p^{-1}\right)\frac{N^{k+1}}{k+1}.
\end{aligned}$$

Hence

$$\pi_{T^p}(N) \sim \left(p^{k+1} + 1 - p^{-1}\right)\frac{N^{k+1}}{k+1}.$$

□

Example 3.2. Let $d_T(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Then

$$d_{T^3}(s) = \left(\sigma(3) - \frac{(\sigma(3) - 3)}{3^s}\right)\zeta(s) = \left(4 - \frac{1}{3^s}\right)\zeta(s)$$

and the abscissa of convergence d_{T^3} is 1 Moreover,

$$\pi_{T^p}(N) = \sum_{n \leq N} O_n(T^p) \sim Res\left(d_{T^p}(s)\frac{N^s}{s}\right)_{s=1} = (4 - \frac{1}{3})N = \frac{11}{3}N.$$

Theorem 3.1 is a simple example of more general results that properties of the orbit Dirichlet series for products and iterates of maps. These results allow analytic methods to be used to find orbit growth asymptotics.

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