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A UNIVERSAL METHOD OF SOLVING QUARTIC EQUATIONS

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Abstract: All the existing methods of solving quartic equations (Descartes-Euler-Cardano's, Ferrari-Lagrange's, Neumark's, Christianson-Brown's, and Yacoub-Fraidenraich-Brown's ones) are particular versions of some universal method. It enables producing an arbitrary number of other particular algorithms with some desired properties.

**Dedicated to Vera P. Filinova
my mathematics teacher.**

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Introduction

Quartic equations (often referred to as just "quartics") in one unknown

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{1}$$

are the highest degree polynomials which can be solved analytically, by radicals, with no iterative techniques. There are several algorithms for this purpose, each involving a subsidiary cubic equation, the so-called "resolvent". In his paper [1] Herbison-Evans examines them in application to computer graphics tasks. Let us list these five algorithms with their resolvents.

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1. Descartes-Euler-Cardano's one [2]:

$$y^3 + (2b - 3a^2/4)y^2 + (3a^4/16 - a^2b + ac + b^2 - 4d)y - \left(\frac{a^3}{8} - \frac{ab}{2} + c\right)^2 = 0; \quad (2)$$

2. Ferrari-Lagrange's [3]:

$$y^3 + by^2 + (ac - 4d)y + (a^2d + c^2 - 4bd) = 0; \quad (3)$$

3. Neumark's [4]:

$$y^3 - 2by + (ac + b^2 - 4d)y + (a^2d - abc + c^2) = 0; \quad (4)$$

4. Christianson-Brown's: in 1967 Brown [5] published an original method of solving palindromic quartics, and in 1991 Christianson [6] extended it to a general quartic equation:

$$y^3 + \frac{4a^2b - 4b^2 - 4ac + 16d - 3a^4/4}{a^3 - 4ab + 8c}y^2 + (3a^2/16 - b/2)y + ab/16 - a^3/64 - c/8) = 0; \quad (5)$$

5. Yacoub-Fraidenraich-Brown's: Yacoub and Fraidenraich [7] have shown a different way of applying Brown's technique to a general quartic:

$$(a^3 - 4ab + 8c)y^3 + (a^2b - 4b^2 + 2ac + 16d)y^2 + (a^2c - 4bc + 8ad)y + (a^2d - c^2) = 0. \quad (6)$$

By Herbison-Evans' analysis [1], none of these five would completely suit the needs of stable computations. He writes: "*It would be good to find more algorithms or variations on the five listed here which allowed other combinations of coefficient signs to be handled in a stable fashion. It is far from clear why there are five and only five algorithms so far discovered. Are there more? Are there an infinite number of algorithms?*"

Let us try to answer these questions.

The basic algorithm is, as we will see, Ferrari-Lagrange's one (with a minor change).

1. Basic Algorithm

Represent Eq. (1) as the product of two quadratic trinomials

$$
\begin{aligned}
&(x^2+g_1x+h_1)(x^2+g_2x+h_2)= \\
&\qquad = x^4+(g_1+g_2)x^3+(g_1g_2+h_1+h_2)x^2+(g_1h_2+g_2h_1)x+h_1h_2,
\end{aligned}
$$

i.e.

$$
\begin{cases}
g_1+g_2=a, \\
g_1g_2+h_1+h_2=b, \\
g_1h_2+g_2h_1=c, \\
h_1h_2=d.
\end{cases}
\tag{7}
$$

These pair sums and products of g and h allow Vièta's theorem [8] to be applied to express these coefficients as the roots of the subsidiary quadratic equations

$$
\begin{cases}
g^2-ag+b-y=0, \\
h^2-yh+d=0,
\end{cases}
\tag{8}
$$

where y is derived from the only equation with the cross products g_1h_2 and g_2h_1. Having explicitly written the roots, substituted them to the equation for c, and eliminated the radical, one obtains the cubic equation for y (Appendix, lines 1–9):

$$y^3-by^2+(ac-4d)y-a^2d-c^2+4bd=0. \tag{9}$$

This is the cubic resolvent of Eq. (1). In our case, it is identical to that of Ferrari-Lagrange's method, except for the sign at the even powers of y.

The algorithm is as follows:

1. Calculate the coefficients of Eq. (9) and find its real root (any cubic has at least one real root).

2. Calculate the coefficients of Eqs (8) and solve them to get g_1, g_2, h_1, and h_2. Eq. (1) is therefore factored into two quadratics.

3. Solve these quadratics.

This algorithm allows variations to be considered below.

2. A Shift of y

Introduce a new variable $y_s \equiv y - \delta$ (i.e. $y = y_s + \delta$), where δ is an arbitrary shift. For dimension reasons, it must look as $\delta = \alpha a^2 + \beta b$, where α and β are dimensionless coefficients; more sophisticated terms like $a^k b^l c^m d^n$, where $k + 2l + 3m + 4n = 2$, should not be involved. The resolvent takes the form:

$$(y_s + \delta)^3 - b(y_s + \delta)^2 + (ac - 4d)(y_s + \delta) - a^2 d - c^2 + 4bd = \\ = y_s^3 + Ay_s^2 + By_s + C = 0, \quad (10)$$

where

$$\begin{cases} A = 3\delta - b, \\ B = 3\delta^2 - 2b\delta + ac - 4d, \\ C = \delta^3 - b\delta^2 + (ac - 4d)\delta - a^2 d - c^2 + 4bd \end{cases} \quad (11)$$

(Appendix, lines 10–11). On finding a real root y_s, a backward shift is due with return to the rest of the master algorithm.

In **Neumark**'s method $\delta = b$, and

$$\begin{cases} A = 3b - b = 2b, \\ B = 3b^2 - 2b^2 + ac - 4d = ac + b^2 - 4d, \\ C = b^3 - b^3 + (ac - 4d)b - a^2 d - c^2 + 4bd = abc + c^2 - a^2 d \end{cases} \quad (12)$$

(Appendix, line 12), and in **Ferrari-Lagrange**'s one, $\delta = 0$, but y_s in both cases is negated, so the resolvent's coefficients at the even powers of y_s reverse their signs, along with the sign at y_s in Eqs (8).

3. A Shift of x

Introduce a new variable $x_s \equiv x - \Delta$ (i.e. $x = x_s + \Delta$), where Δ is an arbitrary shift. For dimension reasons, it must look as $\Delta = \gamma a$, where γ is a dimensionless coefficient; more sophisticated terms like $a^k b^l c^m d^n$, where $k+2l+3m+4n = 1$, should not be involved. Derive the coefficients of the initial quartic with respect to x_s:

$$(x_s + \Delta)^4 + a(x_s + \Delta)^3 + b(x_s + \Delta)^2 + c(x_s + \Delta) + d = \\ = x_s^4 + (a + 4\Delta)x_s^3 + (b + 3a\Delta + 6\Delta^2)x_s^2 + (c + 2b\Delta + 3a\Delta^2 + 4\Delta^3)x_s + \\ + d + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4 = x_s^4 + a_s x_s^3 + b_s x_s^2 + c_s x_s + d_s = 0,$$

where $a_s = a + 4\Delta$, $b_s = b + 3a\Delta + 6\Delta^2$, $c_s = c + 2b\Delta + 3a\Delta^2 + 4\Delta^3$, $d_s = d + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4$ (Appendix, line 16). Substituting these expressions to Eqs (11) for simultaneous application of both shifts, we have:

$$\begin{cases} A = 3\delta - b_s = -b - 3a\Delta + 3\delta - 6\Delta^2, \\ B = 3\delta^2 - 2b_s\delta + a_s c_s - 4d_s = \\ \quad = 3\delta^2 - 2(b + 3a\Delta + 6\Delta^2)\delta + (a + 4\Delta)(c + 2b\Delta + 3a\Delta^2 + \\ \quad + 4\Delta^3) - 4(d + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4), \\ C = \delta^3 - b_s\delta^2 + (a_s c_s - 4d_s)\delta - a_s^2 d_s - c_s^2 + 4b_s d_s = \\ \quad = \delta^3 - (b + 3a\Delta + 6\Delta^2)\delta^2 + ((a + 4\Delta)(c + 2b\Delta + 3a\Delta^2 + \\ \quad + 4\Delta^3) - 4(d + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4))\delta - (a + 4\Delta)^2(d + \\ \quad + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4) - (c + 2b\Delta + 3a\Delta^2 + 4\Delta^3)^2 + \\ \quad + 4(b + 3a\Delta + 6\Delta^2)(d + c\Delta + b\Delta^2 + a\Delta^3 + \Delta^4). \end{cases} \tag{13}$$

Calculations show that when $\delta = \alpha a_s^2 + \beta b_s$, where $8\alpha + 3\beta = 1$, the coefficients of the resolvent do not change upon any shift of x. In **Descartes-Euler-Cardano**'s method, the shift is chosen just in this way: $\delta = b_s - a_s^2/4$ ($\alpha = -1/4$, $\beta = 1$). Additionally, $\Delta = -a/4$, so $\delta = b - 3a^2/8$ and

$$\begin{cases} A = 2b - \dfrac{3a^2}{4}, \\ B = \dfrac{3a^4}{16} - a^2 b + ac + b^2 - 4d, \\ C = -\left(\dfrac{a^3}{8} - \dfrac{ab}{2} + c\right)^2 \end{cases} \tag{14}$$

(Appendix, lines 18–20).

4. Hyperbolic Transformation of *y*

Let's assign the subscripts CB and YFB to the roots of Christianson-Brown's and Yacoub-Fraidenraich-Brown's resolvents, respectively, that of the DEC resolvent remaining unsubscripted. Then

$$y = \frac{a^3 + 8c - 4ab}{16 y_{CB}} = \frac{a^3 + 8c - 4ab}{16\left(y_{YFB} + \frac{a}{4}\right)}$$

(note that the numerator $(a^3 + 8c - 4ab)$ is invariant with respect to the shift Δ of x [9]). Substitute these expressions into Descartes-Euler-Cardano's resolvent (14), eliminate the denominator (with possible loss of roots), and come to the known coefficients (Appendix, lines 21–25).

Instead of immediate recalculation of their y_{CB} and y_{YFB} into y with reducing the task to that already solved, the authors of the palindromic methods propose their own sets of formulae, which hides the relation between several forms of representation of the same, in essence, resolvent.

5. A New Particular Method

Having the described general scheme, one can easily produce an arbitrary number of particular algorithms. For example, let us require that the cubic resolvent should contain no y^2 term ($A = 0$), then it follows from Eqs (11) that $\delta = b/3$ (note that the condition $8\alpha + 3\beta = 1$ holds true and the coefficients of the future resolvent will be invariant to Δ) provided $\Delta = 0$, and

$$\begin{cases} B = 3\left(\frac{b}{3}\right)^2 - 2b\frac{b}{3} + ac - 4d = ac - \frac{b^2}{3} - 4d, \\ C = \left(\frac{b}{3}\right)^3 - b\left(\frac{b}{3}\right)^2 + (ac - 4d)\frac{b}{3} - a^2d - c^2 + 4bd = \\ \quad = \frac{abc}{3} - a^2d - \frac{2}{27}b^3 - c^2 + \frac{8}{3}bd \end{cases} \tag{15}$$

(Appendix, line 13). The subsidiary quadratics will be

$$\begin{cases} g^2 - ag + \frac{2}{3}b - y_s = 0, \\ h^2 - \left(y_s + \frac{b}{3}\right)h + d = 0, \end{cases} \tag{16}$$

their roots are

$$g_{1,2} = \frac{a \pm \sqrt{a^2 - 8b/3 + 4y_s}}{2}, \tag{17}$$

$$h_{1,2} = \frac{y_s + b/2 \pm \sqrt{(y_s + b/3)^2 - 4d}}{2}, \tag{18}$$

$$\tag{19}$$

and, finally, the roots of the initial quartic are

$$x_{1,2} = \frac{-g_1 \pm \sqrt{g_1^2 - 4h_1}}{2}, \tag{20}$$

$$x_{3,4} = \frac{-g_2 \pm \sqrt{g_2^2 - 4h_2}}{2}. \tag{21}$$

A "new" algorithm for solving quartics is thus "invented", it provides the cubic resolvent just in its reduced form.

6. Conclusion

Of course, nobody could guarantee computational success to any of an arbitrary number of particular algorithms. We may guess that the five previously published methods have been elaborated so as to attach some algebraic quality to the resolvent's coefficients, but their robustness in actual computations should be a subject of further analysis.

Excessive complications may be adverse. E.g., if in the palindromic algorithms the expression $(a^3 + 8c - 4ab)$ takes on zero or a near-zero value, this will cause a fault or an error. The case of y_{CB} or $(y_{YFB} + a/4)$ close to zero is also unfavorable. Simple shifts of the roots seem more safe, provided that no close values are subtracted.

A similar technique of shifts and hyperbolic transformations can be applied to cubic equations as well.

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Appendix: MAXIMA's script

(%i1) `g^2-a*g+b-y$ solve(%,g)$`

(%i3) `h^2-y*h+d$ solve(%,h)$`

(%i5) `rhs(%th(3)[1])*rhs(%th(1)[2])+rhs(%th(3)[2])* rhs(%th(1)[1])-c,expand;`

(%o5) $-\frac{\sqrt{4\,y-4\,b+a^2}\,\sqrt{y^2-4\,d}}{2}+\frac{a\,y}{2}-c$

(%i6) `pickapart(%,1);`

(%t6) $-\frac{\sqrt{4\,y-4\,b+a^2}\,\sqrt{y^2-4\,d}}{2}$

(%t7) $\frac{a\,y}{2}$

(%o7) $-c+\%t7+\%t6$

(%i8) `expandwrt(expand(%t6^2-(c-%t7)^2),y);`

(%o8) $y^3-b\,y^2-4\,d\,y+a\,c\,y+4\,b\,d-a^2\,d-c^2$

(%i9) `create_list(expand(coeff(%,y,n)),n,[3,2,1,0]);`

(%o9) $[1,-b,a\,c-4\,d,4\,b\,d-a^2\,d-c^2]$

(%i10) `subst([y=ys+D],%th(2)),expand;`

(%o10) $D^3+3\,ys\,D^2-b\,D^2+3\,ys^2\,D-2\,b\,ys\,D-4\,d\,D+a\,c\,D+ys^3-b\,ys^2-4\,d\,ys+a\,c\,ys+4\,b\,d-a^2\,d-c^2$

(%i11) `[A,B,C]:create_list(coeff(%,ys,n),n,[2,1,0]);`

(%o11) $[3\,D-b,3\,D^2-2\,b\,D-4\,d+a\,c,D^3-b\,D^2-4\,d\,D+a\,c\,D+4\,b\,d-a^2\,d-c^2]$

(%i12) `subst([D=b],%),eval,expand;`

(%o12) $[2\,b,-4\,d+a\,c+b^2,-a^2\,d-c^2+a\,b\,c]$

(%i13) `subst([D=b/3],%th(2)),eval,expand;`

(%o13) $[0,-4\,d+a\,c-\frac{b^2}{3},\frac{8\,b\,d}{3}-a^2\,d-c^2+\frac{a\,b\,c}{3}-\frac{2\,b^3}{27}]$

(%i14) `x^4+a*x^3+b*x^2+c*x+d$ subst([x=xs+D],%),expand$`

(%i16) `[as,bs,cs,ds]:create_list(coeff(%,xs,n),n,[3,2,1,0]);`

(%o16) $[4\,D+a,6\,D^2+3\,a\,D+b,4\,D^3+3\,a\,D^2+2\,b\,D+c,D^4+a\,D^3+b\,D^2+c\,D+d]$

(%i17) `subst([D=-a/4],%),expand;`

(%o17) $[0,b-\frac{3\,a^2}{8},c-\frac{a\,b}{2}+\frac{a^3}{8},d-\frac{a\,c}{4}+\frac{a^2\,b}{16}-\frac{3\,a^4}{256}]$

```
(%i18)   subst([D=%[2],a=%[1],b=%[2],c=%[3],d=%[4]],%o12),expand;
```

(%o18) $\left[2\,b-\frac{3\,a^2}{4},-4\,d+a\,c+b^2-a^2\,b+\frac{3\,a^4}{16},-c^2+a\,b\,c-\frac{a^3\,c}{4}-\frac{a^2\,b^2}{4}+\frac{a^4\,b}{8}-\frac{a^6}{64}\right]$

```
(%i19)   factor(%[3]);
```

(%o19) $-\frac{\left(8\,c-4\,a\,b+a^3\right)^2}{64}$

```
(%i20)   y^3+%th(2)[1]*y^2+%th(2)[2]*y+%th(2)[3];
```

(%o20) $y^3+\left(2\,b-\frac{3\,a^2}{4}\right)\,y^2+\left(-4\,d+a\,c+b^2-a^2\,b+\frac{3\,a^4}{16}\right)\,y-c^2+a\,b\,c-\frac{a^3\,c}{4}-\frac{a^2\,b^2}{4}+\frac{a^4\,b}{8}-\frac{a^6}{64}$

```
(%i21)   subst([y=(a^3+8*c-4*a*b)/(4*(a+4*yYFB))],%),ratsimp$
(%i22)   -num(%)/(a^3+8*c-4*a*b),ratsimp;
```

(%o22) $\left(8c-4ab+a^3\right)yYFB^3+\left(16d+2ac-4b^2+a^2b\right)yYFB^2+$
$+\left(8ad+\left(a^2-4b\right)c\right)yYFB+a^2d-c^2$

```
(%i23)   subst([y=(a^3+8*c-4*a*b)/(16*yCB)],%th(3)),ratsimp$
(%i24)   expandwrt(ratsimp(num(%)/coeff(expand(num(%)),yCB,3)),
         yCB)$
(%i25)   create_list(coeff(%,yCB,n),n,[2,1,0]),ratsimp;
```

(%o25) $\left[\frac{64\,d-16\,a\,c-16\,b^2+16\,a^2\,b-3\,a^4}{32\,c-16\,a\,b+4\,a^3},-\frac{8\,b-3\,a^2}{16},-\frac{8\,c-4\,a\,b+a^3}{64}\right]$

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