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WEAKLY ULTRA SEPARATION AXIOMS VIA $\alpha\psi$-OPEN SETS

Y.K. Kim[1 §], R. Devi[2], A. Selvakumar[3]

[1]Department of Mathematics
Myongji University
Kyunggi, 449-728, SOUTH KOREA

[2]Department of Mathematics
Kongunadu Arts and Science College
Coimabtore, 641029, Tamil Nadu, INDIA

Abstract: In this paper, we introduce the concept of *weakly ultra-$\alpha\psi$-separation* of two sets in a topological space using *$\alpha\psi$-open* sets. The *$\alpha\psi$-closure* and the *$\alpha\psi$-kernel* are defined in terms of this *weakly ultra-$\alpha\psi$*-separation. We also investigate some of the properties of weak separation axioms like $\alpha\psi$-T_0 and $\alpha\psi$-T_1 spaces.

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1. Introduction

The notion of *$\alpha\psi$-closed* set was introduced and studied by R. Devi et al (see [2]). In this paper, we define that a set A is *weakly ultra-$\alpha\psi$-separated* from B if there exists a *$\alpha\psi$-open* set G containing A such that $G \cap B = \phi$. Using this concept, we define the *$\alpha\psi$-closure* and the *$\alpha\psi$-kernel*. Also we define the *$\alpha\psi$-derived* set and the *$\alpha\psi$-shell* of a set A of a topological space (X, τ).

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let A be a subset of a space X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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§Correspondence author

2. Preliminaries

Before entering to our work, we recall the following definitions, which are useful in the sequel.

Definition 1. A subset A of a space (X, τ) is called

(i) a *semi-open* set (see [4]) if $A \subseteq cl(int(A))$ and a *semi-closed* set if $int(cl(A)) \subseteq A$ and

(ii) an α-*open* set (see [5]) if $A \subseteq int(cl(int(A)))$ and an α-*closed* set if $cl(int(cl(A))) \subseteq A$.

The *semi*-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all *semi-closed* (resp. α-*closed*) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 2. A subset A of a topological space (X, τ) is called

(i) a *semi-generalized closed* (briefly *sg-closed*) set (see [1]) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is *semi-open* in (X, τ). The complement of *sg-closed* set is called *sg-open* set,

(ii) a ψ-*closed* set (see [6]) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is *sg-open* in (X, τ). The complement of ψ-*closed* set is called ψ-*open* set and

(iii) a $\alpha\psi$-*closed* set (see [2]) if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-*open* in (X, τ). The complement of $\alpha\psi$-*closed* set is called $\alpha\psi$-*open* set.

The $\alpha\psi$-closure of a subset A of a space (X, τ) is the intersection of all $\alpha\psi$-*closed* sets that contain A and is denoted by $\alpha\psi cl(A)$ or $\alpha\psi$-$cl(A)$. The $\alpha\psi$-interior of a subset A of a space (X, τ) is the union of all $\alpha\psi$-*open* sets that are contained in A and is denoted by $\alpha\psi int(A)$. By $\alpha\psi O(X, \tau)$ or $\alpha\psi O(X)$, we denote the family of all $\alpha\psi$-*open* sets of (X, τ).

Definition 3. The intersection of all $\alpha\psi$-*open* subsets of (X, τ) containing A is called the $\alpha\psi$-*kernel* of A (briefly, $\alpha\psi$-$ker(A)$), i.e.

$$\alpha\psi\text{-}ker(A) = \cap\{G \in \alpha\psi O(X) : A \subseteq G\}.$$

Definition 4. Let $x \in X$. Then $\alpha\psi$-*kernel* of x is denoted by $\alpha\psi$-$ker(\{x\}) = \cap\{G \in \alpha\psi O(X) : x \in G\}$.

Definition 5. Let X be a topological space and $x \in X$, then a subset N_x of X is called an $\alpha\psi$-neighborhood (briefly, $\alpha\psi$-*nbd*) of X if there exists an $\alpha\psi$-*open* set G such that $x \in G \subseteq N_x$.

Definition 6. In a space X, a set A is said to be *weakly ultra-$\alpha\psi$-separated* from a set B if there exists an *$\alpha\psi$-open* set G such that $A \subseteq G$ and $G \cap B = \phi$ or $A \cap \alpha\psi cl(B) = \phi$.

By the definition 6, we have the following for $x, y \in X$ of a topological space,

(i) $\alpha\psi\text{-}cl(\{x\}) = \{y : \{y\}\ is\ not\ weakly\ ultra\text{-}\alpha\psi\text{-}separated\ from\{x\}\}$

(ii) $\alpha\psi\text{-}ker(\{x\}) = \{y : \{y\}is\ not\ weakly\ ultra\text{-}\alpha\psi\text{-}separated\ from\{y\}\}$.

Definition 7. For any point x of a space X:

(i) The *$\alpha\psi$-derived* (briefly, $\alpha\psi\text{-}d(\{x\})$) *set* of x is defined to be the set $\alpha\psi\text{-}d(\{x\}) = \alpha\psi\text{-}cl(\{x\}) - \{x\} = \{y : y \neq x$ and $\{y\}$ is not *weakly -ultra-$\alpha\psi$*-separated from$\{x\}\}$,

(ii) the $\alpha\psi$-shell (briefly, $\alpha\psi\text{-}shl(\{x\})$) of a singleton set $\{x\}$ is defined to be the set $\alpha\psi\text{-}shl(\{x\}) = \alpha\psi\text{-}ker(\{x\}) - \{x\} = \{y : y \neq x\ and\ \{x\}is\ not\ weakly\text{-}ultra\text{-}\alpha\psi\text{-}separated\ from\{y\}\}$.

Definition 8. Let X be a topological space. Then we define

(i) $\alpha\psi\text{-}N\text{-}D = \{x : x \in X\ and\ \alpha\psi\text{-}d(\{x\}) = \phi\}$.

(ii) $\alpha\psi\text{-}N\text{-}shl = \{x : x \in X\ and\ \alpha\psi\text{-}shl(\{x\}) = \phi\}$.

(iii) $\alpha\psi\text{-}\langle x\rangle = \alpha\psi\text{-}cl(\{x\}) \cap \alpha\psi\text{-}ker(\{x\})$.

3. $\alpha\psi$-T_i Spaces, i=0, 1

Definition 9. A topological space X is said to be $\alpha\psi$-T_0 if for $x, y \in X$, $x \neq y$, there exists $U \in \alpha\psi O(X)$ such that U contains only one of x or y but not the other.

Definition 10. (see [3]) A topological space X is said to be $\alpha\psi$-T_1 if for $x, y \in X$, $x \neq y$, there exist $U, V \in \alpha\psi O(X)$ such that $x \in U$ and $y \in V$ but $y \notin U$ and $x \notin V$.

Remark 11. Every $\alpha\psi$-T_1 space is $\alpha\psi$-T_0.

Theorem 12. *A topological space X is an $\alpha\psi$-T_0 space if and only if the $\alpha\psi$-closure of distinct points are distinct.*

Proof. Let $x \neq y$ implies $\alpha\psi$-$cl(\{x\}) \neq \alpha\psi$-$cl(\{y\})$. Then there exists at least one $z \in \alpha\psi$-$cl(\{x\})$ but $z \notin \alpha\psi$-$cl(\{y\})$. Let $x \in \alpha\psi$-$cl(\{y\})$. Then $\alpha\psi$-$cl(\{x\}) \subseteq \alpha\psi$-$cl(\{y\})$, which is a contradiction that $z \notin \alpha\psi$-$cl(\{y\})$. Hence $x \in X - \alpha\psi$-$cl(\{y\})$. Conversely, let X be an $\alpha\psi$-T_0 space. Take $x, y \in X$ and $x \neq y$. Then there exists an $\alpha\psi$-*open* set G such that $x \in G$ and $y \notin G$, which implies $y \in X - G = F(say)$. Now $\alpha\psi$-$cl(\{y\}) = \cap\{F : cl(\{y\}) \subseteq F$ *and* F *is an* $\alpha\psi$ -*closed set*$\}$. This implies that $y \in \alpha\psi$-$cl(\{y\})$and $x \notin \alpha\psi$-$cl(\{y\})$. □

Theorem 13. *A space X is $\alpha\psi$-T_0 if and only if any of the following conditions holds.*

(i) For arbitrary $x, y \in X$, $x \neq y$, either $\{x\}$ is weakly ultra-$\alpha\psi$-separated from $\{y\}$ or $\{y\}$ is weakly ultra-$\alpha\psi$-separated from $\{x\}$.

(ii) $y \in \alpha\psi$-$cl(\{x\})$ implies $x \notin \alpha\psi$-$cl(\{y\})$.

(iii) For all $x, y \in X$ if $x \neq y$, then $\alpha\psi$-$cl(\{x\}) \neq \alpha\psi$-$cl(\{y\})$.

Proof. (i) Obvious from the definitions of $\alpha\psi$-T_0 and *weakly ultra-$\alpha\psi$-separation*.

(ii) By assumption, $y \in \alpha\psi$-$cl(\{x\})$ and so $\{y\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{x\}$. As X is $\alpha\psi$-T_0, $\{x\}$ should be *weakly ultra-$\alpha\psi$-separated* from $\{y\}$, that is $x \notin \alpha\psi$-$cl(\{y\})$.

(iii) If X is $\alpha\psi$-T_0, then for all $x, y \in X$ and $x \neq y$, $\alpha\psi$-$cl(\{x\}) \neq \alpha\psi$-$cl(\{y\})$ as evidenced by (ii). Now let us prove the converse. Let $\alpha\psi$-$cl(x) \neq \alpha\psi$-$cl(\{y\})$. Then there exists $z \in X$, such that $z \in \alpha\psi$-$cl(\{x\})$ and $z \notin \alpha\psi$-$cl(\{y\})$. If $\{x\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{y\}$, then $x \in \alpha\psi$-$cl(\{y\})$. So $\alpha\psi$-$cl(\{x\}) \subseteq \alpha\psi$-$cl(\{y\})$. Then $z \in \alpha\psi$-$cl(\{y\})$, which is a contradiction. □

Theorem 14. *A space is $\alpha\psi$-T_0 if and only if $\alpha\psi$-$d(\{x\}) \cap \alpha\psi$-$shl(\{x\}) = \phi$.*

Proof. Let X be $\alpha\psi$-T_0. Suppose we have $\alpha\psi$-$d(\{x\}) \cap \alpha\psi$-$shl(\{x\}) \neq \phi$. Let $z \in \alpha\psi$-$d(\{x\})$ and $z \in \alpha\psi$-$shl(\{x\})$. Then $z \neq x$ and $z \in \alpha\psi$-$cl(\{x\})$ and $z \in \alpha\psi$-$ker(\{x\})$. Then $\{z\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{x\}$ and also $\{x\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{z\}$, which is a contradiction.

Conversely, let $\alpha\psi$-$d(\{x\}) \cap \alpha\psi$-$shl(\{x\}) = \phi$. Then there exists $z \neq x$, $z \in \alpha\psi$-$cl(\{x\})$ and $z \notin \alpha\psi$-$ker(\{x\})$. Hence if we have $\{z\}$, which is not *weakly ultra-$\alpha\psi$-separated* from $\{x\}$, then $\{x\}$ is *weakly ultra-$\alpha\psi$-separated* from $\{z\}$. Hence, X is $\alpha\psi$-T_0. □

Theorem 15. *A topological space X is $\alpha\psi$-T_1 if and only if $\{x\}$ is $\alpha\psi$-closed in X for every $x \in X$.*

Proof. If $\{x\}$ is *$\alpha\psi$-closed* in X, for $x \neq y$, $X - \{x\}$, $X - \{y\}$ are *$\alpha\psi$-open* sets such that $y \in X - \{x\}$ and $x \in X - \{y\}$. Therefore, X is $\alpha\psi$-T_1.

Conversely, if X is $\alpha\psi$-T_1 and if $y \in X - \{x\}$ then $x \neq y$. Therefore, there exists an *$\alpha\psi$-open* sets U_x, V_y in X such that $x \in U_x$ but $y \notin U_x$ and $y \in V_y$ but $x \notin V_y$. Let G be the union of all such V_y. Then G is an *$\alpha\psi$-open* set and $G \subseteq X - \{x\} \subseteq X$. Therefore, $X - \{x\}$ is a *$\alpha\psi$-open* set in X. □

Theorem 16. *A topological space is an $\alpha\psi$-T_1 space if and only if every subset of X is an $\alpha\psi$-closed set.*

Proof. Assume that every singleton subset $\{x\}$ of X is a *$\alpha\psi$-closed* set. A finite subset of X is the union of finite number of singleton sets. Hence *$\alpha\psi$-closed*. Conversely, every singleton set $\{x\}$ is a finite subset of X. □

Theorem 17. *A space X is $\alpha\psi$-T_1 if and only if one of the following conditions holds.*

(i) For arbitrary $x, y \in X$, $x \neq y$, $\{x\}$ is weakly ultra-$\alpha\psi$-separated from $\{y\}$.

(ii) For every $x \in X$, $\alpha\psi$-$cl(\{x\}) = \{x\}$.

(iii) For every $x \in X$, $\alpha\psi$-$d(\{x\}) = \phi$ or $\alpha\psi$-N-$D = X$.

(iv) For every $x \in X$, $\alpha\psi$-$ker(\{x\}) = \{x\}$.

(v) For every $x \in X$, $\alpha\psi$-$shl(\{x\}) = \phi$ or $\alpha\psi$-N-$shl = X$.

(vi) For every $x, y \in X$, $x \neq y$, $\alpha\psi$-$cl(\{x\}) \cap \alpha\psi$-$cl(\{y\}) = \phi$.

(vii) For every arbitrary $x, y \in X$, $x \neq y$, we have $\alpha\psi$-$ker(\{x\}) \cap \alpha\psi$-$ker(\{y\}) = \phi$.

Proof. (i) This is just a reformulation of the definition of $\alpha\psi$-T_1.

(ii) If $\{x\}$ is *weakly ultra-$\alpha\psi$-separated* from $\{y\}$, then for $y \neq x$, we have $y \notin \alpha\psi$-$cl(\{x\})$, and hence $x \notin \alpha\psi$-$ker(\{y\})$. Therefore we get that $\alpha\psi$-$ker(\{y\}) = \{y\}$. Its converse is just a reformulation of the above proof.

(iii), (iv) and (v) are obvious.

(vi) As X is $\alpha\psi$-T_1, $\alpha\psi$-$cl(\{x\}) = \{x\}$ and $\alpha\psi$-$cl(\{y\}) = \{y\}$. So, when $x \neq y$, $\alpha\psi$-$cl(\{x\}) \cap \alpha\psi$-$cl(\{y\}) = \phi$.

(vii) Obvious from (vi). □

Theorem 18. *If X is an $\alpha\psi$-T_1 space, then the intersection of $\alpha\psi$-nbd of an arbitrary point of X is a singleton set.*

Proof. Let X be an $\alpha\psi$-T_1 space. Also let $x \in X$ and N_x be the $\alpha\psi$-*nbd* of x. If y is a point of X and $y \neq x$, then there exists an $\alpha\psi$-*open* set containing x but not y. Since y is arbitrary, N_x has no point other than x. Conversely, the intersection of $\alpha\psi$-*nbd* of x is the singleton set $\{x\}$, which does not contain any other point y. Hence X is an $\alpha\psi$-T_1 space. □

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