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MULTISOLITON SOLUTIONS OF THE CLASSICAL BOUSSINESQ EQUATION BASED ON WRONSKIAN TECHNIQUE

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Abstract: In this paper, we change the classical Boussinesq equation into a member of AKNS hierarchy by equivalent transformation, and get multisoliton solutions of AKNS equation by means of constructing the double Wronskian deteminant matrix, and then we can get the corresponding multisoliton solutions of the Boussinesq equation.

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1. Introduction

So far, there are many methods which are used to search for exact solutions of the soliton equations, and the Wronskian technique is a kind of method which is efficient and widely used. Soliton solutions can be written in Wronskian form, which was first introduced by Satsuma (see[1]) in 1979. But Satsuma did not connect this representation with the bilinear forms of soliton equations. Until 1983, Freeman and Nimmo (see[2-6]) had developed the Wronskian technique which admits direct verification of solitons in Wronskian form of the bilinear equations. The main process as follows: First we should get the bilinear forms of soliton equations. Then choose the proper function ϕ_j, and contribute the Wronskian determinant solutions. Finally substitute the Wronskian determinant solutions into the bilinear equations, and verify the solutions are correct. The whole process of certification and calculation is simple, the solutions can be

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verified directly. This is just the advantage of Wronskian technique. Many soliton equations, such as the nonlinear Schrödinger equation (see[7]), the AKNS hierarchy (see[8]), the 2-dimenxional Toda lattice (see[9]) and some equations which constrained from the KP hierarchy (see[10]) admit solutions in double Wronskian form.

In this paper, let us start from the Boussinesq equation

$$\begin{cases} \zeta_t + [(1+\zeta)\,u]_x = -\frac{1}{4}u_{xxx} \\ u_t + uu_x + \zeta_x = 0 \end{cases} \tag{1.1}$$

where ζ is the elevation of a water wave, u is the surface velocity of water along x direction.

The Lax pair reads

$$\begin{cases} \varphi_{xx} = \left(\lambda^2 + \lambda u + \frac{u^2}{4} - \zeta - 1\right)\varphi \\ \varphi_t = \frac{1}{4}u_x\varphi + \left(\lambda - \frac{u}{2}\right)\varphi_x \end{cases} \tag{1.2}$$

Introducing following transformation

$$q = e^{\int u dx},\ r = -\left(1 + \zeta - \tfrac{1}{2}u_x\right)e^{-\int u dx} \tag{1.3}$$

or

$$u = \tfrac{q_x}{q},\ \zeta = -1 - qr + \tfrac{1}{2}u_x \tag{1.4}$$

By a direct calculation, we get an equation as followed

$$\begin{cases} q_t + \frac{1}{2}q_{xx} - q^2 r - q = 0 \\ r_t - \frac{1}{2}r_{xx} + qr^2 + r = 0 \end{cases} \tag{1.5}$$

The equation (1.5) is a member of AKNS hierarchy, and its Lax pair reads

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{1.6}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr + \frac{1}{2} & \lambda q - \frac{q_x}{2} \\ \lambda r + \frac{r_x}{2} & \lambda^2 - \frac{1}{2}qr + \frac{1}{2} \end{pmatrix} \tag{1.7}$$

Introducing the dependent variable transformation to the equation (1.5)

$$q = \tfrac{g}{f},\ r = \tfrac{h}{f} \tag{1.8}$$

If g, h and f satisfy the bilinear equations

$$ff_{xx} - f_x^2 + gh = 0 \tag{1.9}$$
$$2\left(g_t f - g f_t\right) + g_{xx} f - 2 g_x f_x + g f_{xx} - 2gf = 0 \tag{1.10}$$
$$2\left(h_t f - h f_t\right) - h_{xx} f + 2 h_x f_x - h f_{xx} + 2hf = 0 \tag{1.11}$$

then q and r are the solutions of AKNS equation (1.5)

Consider the following matrix equation

$$\phi_x = -A\phi, \quad \psi_x = A\psi, \quad \phi_t = -\phi_{xx} + \tfrac{1}{2}\phi, \quad \psi_t = \psi_{xx} - \tfrac{1}{2}\psi \tag{1.12}$$

where $A = (a_{ij})$ is an $(n+m+2) \times (n+m+2)$ arbitrary real matrix independent of (x, t). ϕ and ψ are vectors

$$\phi = \left(\phi_1, \phi_2, ..., \phi_{n+m+2}\right)^T, \quad \psi = \left(\psi_1, \psi_2, ..., \psi_{n+m+2}\right)^T \tag{1.13}$$

First, we will prove the double Wronskian determinants

$$f = W^{n+1,m+1}\left(\phi;\psi\right) = \left|\widehat{n};\widehat{m}\right| \tag{1.14}$$
$$g = 2W^{n+2,m}\left(\phi;\psi\right) = 2\left|\widehat{n+1};\widehat{m-1}\right| \tag{1.15}$$
$$h = -2W^{n,m+2}\left(\phi;\psi\right) = -2\left|\widehat{n-1};\widehat{m+1}\right| \tag{1.16}$$

are solutions of the bilinear equations (1.9), (1.10), (1.11), where the vectors ϕ and ψ satisfy the condition (1.12), and

$$W^{j,l}\left(\phi;\psi\right) = \left|\phi, \partial_x\phi, ..., \partial_x^{j-1}\phi; \psi, \partial_x\psi, ..., \partial_x^{l-1}\psi\right| = \left|\widehat{j-1};\widehat{l-1}\right| \tag{1.17}$$

$\left|\widehat{j-1};\widehat{l-1}\right|$ is the abbreviation of this determinant.

The paper is organized as follows. In Section 2, solutions in the form of double Wronskian determinants are proven and given. In Section 3, multisoliton solutions are given. Finally, we give the conclusion in Section 4.

2. Solutions in Wronskian Form

In this section, we start from the following two lemmas (see [3]).

Lemma 1.

$$|Q, a, b|\,|Q, c, d| - |Q, a, c|\,|Q, b, d| + |Q, a, d|\,|Q, b, c| = 0 \tag{2.1}$$

where Q is an $n \times (n-2)$ matrix, a, b, c and d represent n column vectors.

Lemma 2.

$$\sum_{j=1}^{n} \gamma_j \left|\alpha_1, \alpha_2, ..., \alpha_n\right| = \sum_{j=1}^{n} \left|\alpha_1, \alpha_2, ..., \gamma\alpha_j, ..., \alpha_n\right| \tag{2.2}$$

where α_j $(j = 1, 2, ..., n)$ *are* n *column vectors, and*

$$\gamma\alpha_j = (\gamma_1\alpha_{1j}, \gamma_2\alpha_{2j}, ..., \gamma_n\alpha_{nj})^T \tag{2.3}$$

Theorem 1. *The double Wronskian determinants (1.14), (1.15), (1.16) with the conditions*

$$\phi_{j,x} = -k_j\phi_j, \quad \psi_{j,x} = k_j\psi_j \tag{2.4}$$

$$\phi_{j,t} = -\phi_{j,xx} + \tfrac{1}{2}\phi_j, \quad \psi_{j,t} = \psi_{j,xx} - \tfrac{1}{2}\psi_j \quad (j = 1, 2, ..., m + n + 2) \tag{2.5}$$

solve the equation (1.9), (1.10), (1.11).

Prof. First, we will prove that the double Wronskian determinant (1.14), (1.15), (1.16) satisfy the equation (1.9), (1.10), (1.11).

The derivatives of f with respect to x are

$$f_x = \left|\widehat{n-1}, n+1; \widehat{m}\right| + \left|\widehat{n}; \widehat{m-1}, m+1\right| \tag{2.6}$$

$$\begin{aligned} f_{xx} = &\left|\widehat{n-2}, n, n+1; \widehat{m}\right| + \left|\widehat{n-1}, n+2; \widehat{m}\right| + 2\left|\widehat{n-1}, n+1; \widehat{m-1}, m+1\right| \\ &+ \left|\widehat{n}; \widehat{m-2}, m, m+1\right| + \left|\widehat{n}; \widehat{m-1}, m+2\right| \end{aligned} \tag{2.7}$$

Note that the identity

$$|\widehat{n}; \widehat{m}| \left(\sum_{j=1}^{n+m+2} k_j\right)^2 |\widehat{n}; \widehat{m}| = \left(\sum_{j=1}^{n+m+2} k_j \left|\widehat{n}; \widehat{m}\right|\right)^2 \tag{2.8}$$

which yields identity

$$\begin{aligned} &|\widehat{n}; \widehat{m}| \left(\left|\widehat{n-2}, n, n+1; \widehat{m}\right| + \left|\widehat{n-1}; n+2; \widehat{m}\right| - 2\left|n - \widehat{1, n} + 1; \widehat{m-1}, m+1\right| \right. \\ &\left. + \left|\widehat{n}; \widehat{m-2}, m, m+1\right| + \left|\widehat{n}; \widehat{m-1}, m+2\right|\right) \\ &= \left(\left|\widehat{n-1}, n+1; \widehat{m}\right| - \left|\widehat{n}; \widehat{m-1}, m+1\right|\right)^2 \end{aligned} \tag{2.9}$$

then we have

$$
\begin{aligned}
ff_{xx} &= |\widehat{n};\widehat{m}|\left(\left|\widehat{n-2},n,n+1;\widehat{m}\right| + \left|\widehat{n-1},n+2;\widehat{m}\right| + 2\right.\\
&\quad \left.\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| + \left|\widehat{n};\widehat{m-2},m,m+1\right| + \left|\widehat{n};\widehat{m-1},m+2\right|\right)\\
&= |\widehat{n};\widehat{m}|\left(\left|\widehat{n-2},n,n+1;\widehat{m}\right| + \left|\widehat{n-1},n+2;\widehat{m}\right| - 2\right.\\
&\quad \left.\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| + \left|\widehat{n};\widehat{m-2},m,m+1\right| + \left|\widehat{n};\widehat{m-1},m+2\right|\right)\\
&\quad +4\,|\widehat{n};\widehat{m}|\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right|\\
&= \left(\left|\widehat{n-1},n+1;\widehat{m}\right| - \left|\widehat{n};\widehat{m-1},m+1\right|\right)^2 +\\
&\quad 4\,|\widehat{n};\widehat{m}|\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right|
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
& ff_{xx} - f_x^2 + gh\\
&= \left(\left|\widehat{n-1},n+1;\widehat{m}\right| - \left|\widehat{n};\widehat{m-1},m+1\right|\right)^2 + 4\,|\widehat{n};\widehat{m}|\\
&\quad \left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| - \left(\left|\widehat{n-1},n+1;\widehat{m}\right| - \left|\widehat{n};\widehat{m-1},m+1\right|\right)^2\\
&\quad -4\left|\widehat{n+1};\widehat{m-1}\right|\left|\widehat{n-1};\widehat{m+1}\right|\\
&= -4\left(\left|\widehat{n-1},n+1;\widehat{m}\right|\left|\widehat{n};\widehat{m-1},m+1\right| - |\widehat{n};\widehat{m}|\right.\\
&\quad \left.\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| + \left|\widehat{n+1};m-1\right|\left|\widehat{n-1};\widehat{m+1}\right|\right)\\
&= -4\left(\left|\widehat{n-1};\widehat{m-1},n+1,m\right|\left|\widehat{n-1};\widehat{m-1},n,m+1\right| -\right.\\
&\quad \left|\widehat{n-1};\widehat{m-1},n,m\right|\left|\widehat{n-1};\widehat{m-1},n+1,m+1\right| +\\
&\quad \left.\left|\widehat{n-1};\widehat{m-1},n,n+1\right|\left|\widehat{n-1};\widehat{m-1},m,m+1\right|\right)
\end{aligned}
\tag{2.11}
$$

Take matrix $Q = \left(\widehat{n-1};\widehat{m-1}\right)$, utilizing lemma 1 we can know that equation (2.11) is equal to zero. Thus, we have proven (1.9).

The derivatives of g with respect to x are

$$
g_x = 2\left|\widehat{n},n+2;\widehat{m-1}\right| + 2\left|\widehat{n+1};\widehat{m-2},m\right| \tag{2.12}
$$

$$
\begin{aligned}
g_{xx} = 2&\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| + 2\left|\widehat{n},n+3;\widehat{m-1}\right| +\\
4&\left|\widehat{n},n+2;\widehat{m-2},m\right| + 2\left|\widehat{n+1};\widehat{m-3},m-1,m\right| +\\
2&\left|\widehat{n+1};\widehat{m-2},m+1\right|
\end{aligned}
\tag{2.13}
$$

From (2.5), we have

$$g_t = (2+n-m)\left|\widehat{n+1};\widehat{m-1}\right| + 2\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| - 2\left|\widehat{n},n+3;\widehat{m-1}\right| - 2\left|\widehat{n+1};\widehat{m-3},m-1,m\right| + 2\left|\widehat{n+1};\widehat{m-2},m+1\right| \tag{2.14}$$

$$f_t = \frac{1}{2}(n-m)\left|\widehat{n};\widehat{m}\right| + \left|\widehat{n-2},n,n+1;\widehat{m}\right| - \left|\widehat{n-1},n+2;\widehat{m}\right| - \left|\widehat{n};\widehat{m-2},m,m+1\right| + \left|\widehat{n};\widehat{m-1},m+2\right| \tag{2.15}$$

So

$$\begin{aligned}&(2g_t+g_{xx})f\\ &= 2(2+n-m)\left|\widehat{n+1};\widehat{m-1}\right|\left|\widehat{n};\widehat{m}\right| +\\ &\quad\Big(6\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| - 2\left|\widehat{n},n+3;\widehat{m-1}\right| +\\ &\quad 4\left|\widehat{n},n+2;\widehat{m-2},m\right| - 2\left|\widehat{n+1};\widehat{m-3},m-1,m\right| +\\ &\quad 6\left|\widehat{n+1};\widehat{m-2},m+1\right|\Big)\left|\widehat{n};\widehat{m}\right|\end{aligned} \tag{2.16}$$

$$\begin{aligned}&g(-2f_t+f_{xx})\\ &= -2(n-m)\left|\widehat{n+1};\widehat{m-1}\right|\left|\widehat{n};\widehat{m}\right| + 2\left|\widehat{n+1};\widehat{m-1}\right|\\ &\quad\Big(-\left|\widehat{n-2},n,n+1;\widehat{m}\right| + 3\left|\widehat{n-1},n+2;\widehat{m}\right|\\ &\quad +2\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right|\\ &\quad +3\left|\widehat{n};\widehat{m-2},m,m+1\right| - \left|\widehat{n};\widehat{m-1},m+2\right|\Big)\end{aligned} \tag{2.17}$$

$$\begin{aligned}-2g_xf_x = -4&\left(\left|\widehat{n},n+2;\widehat{m-1}\right| + \left|\widehat{n+1};\widehat{m-2},m\right|\right)\\ &\left(\left|\widehat{n-1},n+1;\widehat{m}\right| + \left|\widehat{n};\widehat{m-1},m+1\right|\right)\end{aligned} \tag{2.18}$$

And utilize the following identities which are similar to (2.9)

$$\begin{aligned}&\left|\widehat{n};\widehat{m}\right|\Big(\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| + \left|\widehat{n},n+3;\widehat{m-1}\right| -\\ &2\left|\widehat{n},n+2;\widehat{m-2},m\right| + \left|\widehat{n+1};\widehat{m-3},m-1,m\right| + \left|\widehat{n+1};\widehat{m-2},m+1\right|\Big)\\ &= \left(\left|\widehat{n-1},n+1;\widehat{m}\right| - \left|\widehat{n};\widehat{m-1},m+1\right|\right)\\ &\quad\left(\left|\widehat{n},n+2;\widehat{m-1}\right| - \left|\widehat{n+1};\widehat{m-2},m\right|\right)\end{aligned} \tag{2.19}$$

$$
\begin{aligned}
&\left|\widehat{n+1};\widehat{m-1}\right|\left(\left|\widehat{n-2},n,n+1;\widehat{m}\right|+\left|\widehat{n-1},n+2;\widehat{m}\right|-\right.\\
&\left.2\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right|+\left|\widehat{n};\widehat{m-2},m,m+1\right|+\left|\widehat{n};\widehat{m-1},m+2\right|\right)\\
&=\left(\left|\widehat{n},n+2;\widehat{m-1}\right|-\left|\widehat{n+1};\widehat{m-2},m\right|\right)\\
&\quad\left(\left|\widehat{n-1},n+1;\widehat{m}\right|-\left|\widehat{n};\widehat{m-1},m+1\right|\right)
\end{aligned}
\tag{2.20}
$$

Then (1.10) becomes

$$
\begin{aligned}
&|\widehat{n};\widehat{m}|\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right|+|\widehat{n};\widehat{m}|\left|\widehat{n+1};\widehat{m-2},m+1\right|+\\
&\left|\widehat{n+1};\widehat{m-1}\right|\left|\widehat{n-1},n+2;\widehat{m}\right|+\left|\widehat{n+1};\widehat{m-1}\right|\left|\widehat{n};\widehat{m-2},m,m+1\right|-\\
&\left|\widehat{n},n+2;\widehat{m-1}\right|\left|\widehat{n-1},n+1;\widehat{m}\right|-\left|\widehat{n+1};\widehat{m-2},m\right|\left|\widehat{n};\widehat{m-1},m+1\right|
\end{aligned}
\tag{2.21}
$$

take matrix $Q=\left(\widehat{n-1};\widehat{m-1}\right)$ or $\left(\widehat{n};\widehat{m-2}\right)$, utilizing lemma 1 we can know that equation (2.21) is equal to zero. Thus, we have proven (1.10).

Similarly, we can prove the double Wronskian determinant(1.14), (1.15), (1.16) satisfy (1.11).

The general solutions of equation (2.4), (2.5) are

$$
\phi_j=e^{-\xi_j}c_j,\quad \psi_j=e^{\xi_j}d_j,\quad \xi_j=\left(k_j^2-\tfrac{1}{2}\right)t+k_jx \tag{2.22}
$$

where c_j and $d_j\,(j=1,2,...,n+m+2)$ are arbitrary constants. Taking $c_j=d_j=1$, the double Wronskian solutions of the bilinear equations (1.9), (1.10), (1.11) are

$$
f=\left|e^{-\xi_j},\partial x e^{-\xi_j},...,\partial_x^n e^{-\xi_j};e^{\xi_j},\partial_x e^{\xi_j},...,\partial_x^m e^{\xi_j}\right| \tag{2.23}
$$

$$
g=2\left|e^{-\xi_j},\partial x e^{-\xi_j},...,\partial_x^{n+1} e^{-\xi_j};e^{\xi_j},\partial_x e^{\xi_j},...,\partial_x^{m-1} e^{\xi_j}\right| \tag{2.24}
$$

$$
h=-2\left|e^{-\xi_j},\partial x e^{-\xi_j},...,\partial_x^{n-1} e^{-\xi_j};e^{\xi_j},\partial_x e^{\xi_j},...,\partial_x^{m+1} e^{\xi_j}\right| \tag{2.25}
$$

Take $n=m=0$

$$
f=e^{\xi_2-\xi_1}-e^{\xi_1-\xi_2},\ g=2\left(k_1-k_2\right)e^{-\xi_1-\xi_2},\ h=2\left(k_1-k_2\right)e^{\xi_1+\xi_2} \tag{2.26}
$$

and the solutions of AKNS equations (1.5) are

$$
q=\left(k_1-k_2\right)\tfrac{e^{-\xi_1-\xi_2}}{\sinh(\xi_2-\xi_1)},\quad r=\left(k_1-k_2\right)\tfrac{e^{\xi_1+\xi_2}}{\sinh(\xi_2-\xi_1)} \tag{2.27}
$$

and the solutions of Boussinesq equation (1.1) are

$$
u=-\left(k_1+k_2\right)-\left(k_1-k_2\right)\coth\left(\xi_2-\xi_1\right) \tag{2.28}
$$

$$
\zeta=-1-\tfrac{1}{2}\left(k_2-k_1\right)^2\csc h^2\left(\xi_2-\xi_1\right) \tag{2.29}
$$

Take $n = 1, m = 0$

$$f = (k_1 - k_2)\, e^{-\xi_1 - \xi_2 + \xi_3} + (k_3 - k_1)\, e^{-\xi_1 + \xi_2 - \xi_3} + (k_2 - k_3)\, e^{\xi_1 - \xi_2 - \xi_3} \tag{2.30}$$

$$g = 2\,(k_1 - k_2)\,(k_2 - k_3)\,(k_1 - k_3)\, e^{-\xi_1 - \xi_2 - \xi_3} \tag{2.31}$$

$$h = -2\left[(k_3 - k_2)\, e^{-\xi_1 + \xi_2 + \xi_3} + (k_2 - k_1)\, e^{\xi_1 + \xi_2 - \xi_3} + (k_1 - k_3)\, e^{\xi_1 - \xi_2 + \xi_3}\right] \tag{2.32}$$

then

$$q = 2\frac{(k_1-k_2)(k_2-k_3)(k_1-k_3)e^{-\xi_1-\xi_2-\xi_3}}{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3}+(k_3-k_1)e^{-\xi_1+\xi_2-\xi_3}+(k_2-k_3)e^{\xi_1-\xi_2-\xi_3}} \tag{2.33}$$

$$r = 2\frac{(k_2-k_3)e^{-\xi_1+\xi_2+\xi_3}+(k_1-k_2)e^{\xi_1+\xi_2-\xi_3}+(k_3-k_1)e^{\xi_1-\xi_2+\xi_3}}{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3}+(k_3-k_1)e^{-\xi_1+\xi_2-\xi_3}+(k_2-k_3)e^{\xi_1-\xi_2-\xi_3}} \tag{2.34}$$

$$u = -\,(k_1 + k_2 + k_3) - \frac{(-k_1-k_2+k_3)D_1+(-k_1+k_2-k_3)D_2+(k_1-k_2-k_3)D_3}{D_1+D_2+D_3} \tag{2.35}$$

$$\zeta = -1 + 2D_4\frac{\left[(k_3-k_2)e^{-2\xi_1}+(k_2-k_1)e^{-2\xi_3}+(k_1-k_3)e^{-2\xi_2}\right]}{D_1+D_2+D_3} \tag{2.36}$$

where

$$D_1 = (k_1 - k_2)\, e^{-\xi_1 - \xi_2 + \xi_3},\ D_2 = (k_3 - k_1)\, e^{-\xi_1 + \xi_2 - \xi_3},$$

$$D_3 = (k_2 - k_3)\, e^{\xi_1 - \xi_2 - \xi_3},\ D_4 = (k_1 - k_2)\,(k_2 - k_3)\,(k_1 - k_3)\,.$$

Take $n = 0, m = 1$, similarly, we have

$$q = -2\frac{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3}+(k_3-k_1)e^{-\xi_1+\xi_2-\xi_3}+(k_2-k_3)e^{\xi_1-\xi_2-\xi_3}}{(k_1-k_2)e^{\xi_1+\xi_2-\xi_3}+(k_3-k_1)e^{\xi_1-\xi_2+\xi_3}+(k_2-k_3)e^{-\xi_1+\xi_2+\xi_3}} \tag{2.37}$$

$$r = 2\frac{(k_1-k_2)(k_2-k_3)(k_1-k_3)e^{\xi_1+\xi_2+\xi_3}}{(k_1-k_2)e^{\xi_1+\xi_2-\xi_3}+(k_3-k_1)e^{\xi_1-\xi_2+\xi_3}+(k_2-k_3)e^{-\xi_1+\xi_2+\xi_3}} \tag{2.38}$$

$$u = \frac{(k_1-k_2)(-k_1-k_2+k_3)F_1+(k_3-k_1)(-k_1+k_2-k_3)F_2+(k_2-k_3)(k_1-k_2-k_3)F_3}{(k_1-k_2)F_1+(k_3-k_1)F_2+(k_2-k_3)F_3}$$
$$-\frac{(k_2-k_1)(k_1+k_2-k_3)\frac{1}{F_1}+(k_1-k_3)(k_1-k_2+k_3)\frac{1}{F_2}+(k_3-k_2)(-k_1+k_2+k_3)\frac{1}{F_3}}{(k_2-k_1)\frac{1}{F_1}+(k_1-k_3)e\frac{1}{F_2}+(k_3-k_2)\frac{1}{F_3}}$$

(2.39)

$$\zeta=-1-\frac{1}{2}\left\{\frac{-4(k_1-k_2)(k_2-k_3)(k_1-k_3)\left[(k_1-k_2)e^{2\xi_3}+(k_3-k_1)e^{2\xi_2}+(k_2-k_3)e^{2\xi_1}\right]}{\left[(k_2-k_1)\frac{1}{F_1}+(k_1-k_3)\frac{1}{F_2}+(k_3-k_2)\frac{1}{F_3}\right]^2}\right.$$
$$-\frac{(k_1-k_2)(-k_1-k_2+k_3)^2F_1+(k_3-k_1)(-k_1+k_2-k_3)^2F_2+(k_2-k_3)(k_1-k_2-k_3)^2F_3}{(k_1-k_2)F_1+(k_3-k_1)F_2+(k_2-k_3)F_3}$$
$$\left.+\left[\frac{(k_1-k_2)(-k_1-k_2+k_3)F_1+(k_3-k_1)(-k_1+k_2-k_3)F_2+(k_2-k_3)(k_1-k_2-k_3)F_3}{(k_1-k_2)F_1+(k_3-k_1)F_2+(k_2-k_3)F_3}\right]^2\right\} \tag{2.40}$$

where

$$F_1=e^{-\xi_1-\xi_2+\xi_3},\ F_2=e^{-\xi_1+\xi_2-\xi_3},\ F_3=e^{\xi_1-\xi_2-\xi_3}.$$

Next we will further prove the double Wronskian determinants (1.14), (1.15), (1.16) constituted by ϕ and ψ which are determined by general matrix equation (1.12) also meet the bilinear equations (1.9), (1.10), (1.11).

Lemma 3. (see [11])

$$\sum_{i=1}^{n}|b_1,...,p_ib_i,...,b_n|=\sum_{j=1}^{n}\begin{vmatrix} b'_1\\ \vdots\\ p'_jb'_j\\ \vdots\\ b'_n\end{vmatrix} \tag{2.41}$$

where $P=(p_{ij})$ is $n\times n$ operator matrix, p_{ij} is differential operator, $B=(b_{ij})$ is $n\times n$ function matrix, b_j and b'_j represent column vector and transversal vector, and $p_ib_i=(p_{1i}b_{1i},p_{2i}b_{2i},...,p_{ni}b_{ni})^T$, $p'_jb'_j=(p_{j1}b_{j1},p_{j2}b_{j2},...,p_{jn}b_{jn})$.

Utilizing this lemma, we can get the conclusion we want. In fact, we only need to prove identities (2.9), (2.19) and (2.20) still hold in this condition. If the trace of $A=(a_{ij})$ is not zero, that is $trA\neq 0$, and take the elements of operator matrix P as $p_{ij}=-\partial_x\,(1\le i\le n+m+2; 1\le j\le n+1)$ and $p_{ij}=\partial_x\,(1\le i\le n+m+2; n+2\le j\le n+m+2)$. Then make the operator matrix P work into the matrix $A=(a_{ij})$, utilize lemma 3 and the matrix equation (1.12), we can directly calculate that

$$trA\,|\widehat{n};\widehat{m}|=-\left|\widehat{n-1},n+1;\widehat{m}\right|+\left|\widehat{n};\widehat{m-1},m+1\right| \tag{2.42}$$

Then we can have

$$(trA)^2 \left|\widehat{n};\widehat{m}\right| = \left|\widehat{n-2},n,n+1;\widehat{m}\right| + \left|\widehat{n-1},n+2;\widehat{m}\right| - 2\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| + \left|\widehat{n};\widehat{m-2},m,m+1\right| + \left|\widehat{n};\widehat{m-1},m+2\right| \tag{2.43}$$

$$(trA)^2 \left|\widehat{n+1};\widehat{m-1}\right| = \left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| + \left|\widehat{n},n+3;\widehat{m-1}\right| - 2\left|\widehat{n},n+2;\widehat{m-2},m\right| + \left|\widehat{n+1};\widehat{m-3},m-1,m\right| + \left|\widehat{n+1};\widehat{m-2},m+1\right| \tag{2.44}$$

Utilize the identity (2.8)

$$\left|\widehat{n};\widehat{m}\right| \left(\sum_{j=1}^{n+m+2} k_j\right)^2 \left|\widehat{n};\widehat{m}\right| = \left(\sum_{j=1}^{n+m+2} k_j \left|\widehat{n};\widehat{m}\right|\right)^2$$

We can know that (2.9) hold. And we can also get identities (2.19), (2.20).

If $trA = 0$, we have

$$\left|\widehat{n-1},n+1;\widehat{m}\right| = \left|\widehat{n};\widehat{m-1},m+1\right| \tag{2.45}$$

$$\left|\widehat{n-1},n+2;\widehat{m}\right| - 2\left|\widehat{n-1},n+1;\widehat{m-1},m+1\right| + \left|\widehat{n};\widehat{m-2},m,m+1\right| = -\left|\widehat{n-2},n,n+1;\widehat{m}\right| - \left|\widehat{n+1};\widehat{m-1},m+2\right| \tag{2.46}$$

$$\left|\widehat{n},n+3;\widehat{m-1}\right| - 2\left|\widehat{n},n+2;\widehat{m-2},m\right| + \left|\widehat{n+1};\widehat{m-3},m-1,m\right| = -\left|\widehat{n-1},n+1,n+2;\widehat{m-1}\right| - \left|\widehat{n+1};\widehat{m-2},m+1\right| \tag{2.47}$$

Utilizing the identities above, we can easily prove that when $trA = 0$ the double Wronskian determinants (1.14), (1.15), (1.16) satisfy the bilinear equations (1.9), (1.10), (1.11).

General solutions of matrix equation (1.12) are

$$\phi = e^{-\left(A^2-\frac{1}{2}I\right)t-Ax}C, \quad \psi = e^{\left(A^2-\frac{1}{2}I\right)t+Ax}D \tag{2.48}$$

where $C = (c_1, c_2, ..., c_{n+m+2})^T$ and $D = (d_1, d_2, ..., d_{n+m+2})^T$ are arbitrary constant vectors.

We have the following theorem.

Theorem 2 *$A = (a_{ij})$ is arbitrary $(n+m+2) \times (n+m+2)$ real matrix, then the double Wronskian determinants (1.14), (1.15), (1.16) constituted by column vector (2.48) are the solutions of the bilinear equations (1.9), (1.10), (1.11) and the solutions of the corresponding second-order AKNS (1.5) equation are*

$$q = 2\frac{W^{n+2,m}(\phi,\psi)}{W^{n+1,m+1}(\phi,\psi)}, \quad r = -2\frac{W^{n,m+2}(\phi,\psi)}{W^{n+1,m+1}(\phi,\psi)} \tag{2.49}$$

3. Multisoliton Solutions

Develop the general solutions into series

$$\phi = e^{-\left(A^2-\frac{1}{2}I\right)t}e^{-Ax}C = \sum_{s=0}^{\infty}\left[\sum_{l=0}^{\frac{s}{2}}\frac{(-1)^{s-l}}{l!(s-2l)!}t^l x^{s-2l}\left(A^2-\frac{1}{2}I\right)^l A^{s-2l}\right]C \tag{3.1}$$

$$\psi = e^{\left(A^2-\frac{1}{2}I\right)t}e^{Ax}D = \sum_{s=0}^{\infty}\left[\sum_{l=0}^{\frac{s}{2}}\frac{1}{l!(s-2l)!}t^l x^{s-2l}\left(A^2-\frac{1}{2}I\right)^l A^{s-2l}\right]D \tag{3.2}$$

Choose some special matrix A, we can get multisoliton solutions of AKNS equation (1.5) or the relevant bilinear equations (1.9),(1.10),(1.11).

Take A as diagonal matrix

$$A = \begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_{n+m+2} \end{bmatrix}, \; k_i \neq k_j \; (i \neq j) \tag{3.3}$$

We have

$$\phi_j = c_j e^{-\left(k_j^2-\frac{1}{2}\right)t-k_j x}, \; \psi_j = d_j e^{\left(k_j^2-\frac{1}{2}\right)t+k_j x}, \; (j = 1, 2, ..., n+m+2) \tag{3.4}$$

so this kind of double Wronskian determinants are the multisoliton solutions of Boussinesq equation (1.1) and AKNS equation (1.5) in common sense.

4. Conclusion

Using the method of constructing the double Wronskian determinant matrix, we get the multisoliton solutions of a second order AKNS equation, correspondingly, we can get the multisoliton solutions of Boussinesq equation. For each column of Wronskian determinant is derivative of former, so its higher order derivatives are constituted by the summary of several determinants which have the same order, and solution can be substituted into the equation directly for verification. From the above we can see that the Wronskian tecnique not only accurate out the soliton equation effectively, and the representation of soliton solution obtained is also varied.

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