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LIOUVILLE'S THEOREM AND POWER SERIES FOR QUATERNIONIC FUNCTIONS

J.A. Marão[1], M.F. Borges[2 §]

[1]Department of Mathematics
UFMA - Federal University of Maranhão
65085-580, Maranhão, BRAZIL

[2]UNESP - São Paulo State University
S.J. Rio Preto Campus
15054-000, São José do Rio Preto, BRAZIL

Abstract: In recent years quaternionic functions have been an intense and prosperous object of research, and important results were determined [1]-[6]. Some of these results are similar to well known cases in one complex variable, op. cit. [5], [6]. In this paper the hypercomplex expansion of a function in a power series as well as determination of a Liouville's type theorem have been investigated to the quaternionic functions. In this case, it is observed that the Liouville's type theorem is true for second order derivatives, which differs from its classical version.

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1. Introduction and Motivation

The development in series of powers is a natural resource to determine how a function might be expanded as a sum of powers. It helps the determination of integration in some cases. To fix ideas, it is necessary to show what happens to the case of functions of one real variable, as seen in [7]. The Taylor series for a analytic function $f(z)$, is such that:

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§Correspondence author

$$f(z) = f(a) + \frac{z-a}{1!}f'(z) + \frac{(z-a)^2}{2!}f''(z) + \ldots + \frac{(z-a)^n}{n!}f^n(z) + R_n(z), \quad (1)$$

$$f(z) = \sum_{m=0}^{\infty} a_n (z-a)^n, \quad (2)$$

where $a_n = \frac{f^n(a)}{n!}$ e $\lim_{n\to\infty} R_n(z) = 0$.

The series given in (1) are called the Taylor series centered at a. For the case where $a = 0$, the series is called Maclaurin Series.

The determination of the Taylor series for quaternionic functions, you need some steps on their rigorous proof. Thus, an issue raised in determining the convergence of the rest to zero as n tends to infinity requires the area of 4-dimensional hyper sphere. To this end, the formula generally used is [8]:

$$\frac{2\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(\frac{n}{2})}, \quad (3)$$

where $\Gamma(*)$ is Gamma Function. This formula is necessary determine the area of hyper surface in the case $n = 4$. Thus, it follows that:

$$A_{HE} = 2\pi^2 r^3. \quad (4)$$

2. Liouville's Theorem and Liouville's Like Theorems

Liouville's like Theorems have all been treated as main source of research in few number of recent mathematical works [9], [10], [11], [12]. They have, for instance been used for radial solutions for the bi-harmonic operator and for studying both quasi-harmonic functions and foliations with complex leaves. Under a quaternionic point of view, a long-time ago quaternion holomorphs were presented and a Liouville's Theorem helped to approach a solution of some regular integral functions.

The Liouville's Theorem, was first developed by French mathematician Joseph Liouville (1809-1882) for functions of a complex variable. Your statement is given below:

"If $f(z)$ is analytic and limited in absolute terms for a complex z finite number, then $f(z)$ is a constant", see [7].

To fix ideas, it will be shown now a relationship that will assist in the statement of Liouville's theorem to the quaternionic case. Following [9], the derivation by Cauchy's integral formula, is such that:

$$f^n(q) = \frac{1}{(i+j+2k)\pi}\int_\Lambda \frac{f(q')}{(q'-q)}dq' \quad (5)$$

So, considering Λ a hyper sphere of radius r centered at q and M is still a maximum of $|f(q)|$ on Λ. It follows that:

$$|f^n(q)| = \frac{n!}{6\pi}|\int_\Lambda \frac{f(q')}{(q'-q)}dq'| \leq \frac{n!}{6\pi}\frac{K}{r^{n+1}}2\pi^2 r^3 \quad (6)$$

$$|f^n(q)| \leq \frac{n!\pi K}{r^{n-2}}. \quad (7)$$

Using the statement above, will now be made a statement to the Liouville theorem for quaternions.

Theorem 1. *If $f''(q)$ is analytic and limited in absolute terms for a complex q finite number, then $f''(q)$ is a constant.*

Demonstration. By assumption, $|f''(q)| < M$ for any q Thus, $f'''(q) = \frac{M}{r}$. Since this is valid for all r, can be considered r as large as you want, and conclude that $f'''(q) = 0$.

As it is easyly observed in this theorem there is a fundamental difference compared to classical Liouville theorem, where now the theorem is initially true for second derivatives.

3. Power Series for the Quaternionic Case

Take a quaternionic function that has derivatives in a neighborhood of the point $q = a$. Considering now a hypersphere Λ which exists in that neighborhood and has a center at a. The Cauchy Formula, seen in [5] can be applied as follows:

$$f(q) = \frac{1}{(i+j+2k)\pi}\int_\Lambda \frac{f(q')}{q'-q}dq' \quad (8)$$

where the point q is an arbitrary point and fixed the interior of Λ and q' is the variable of integration. Now follows that:

$$\frac{1}{q'-q} = \frac{1}{q'-a(q-a)} = \frac{1}{(q'-a)(1-\frac{q-a}{q'-a})}. \quad (9)$$

With data on q' and q it follows that:

$$|\frac{q-a}{q^*-a}| < 1.$$

Moreover, it follows that:

$$\frac{1}{1-\frac{q-a}{q'-a}} = 1 + \frac{q-a}{q'-a} + [\frac{q-a}{q'-a}]^2 + \ldots + [\frac{q-a}{q'-a}]^n + \frac{[\frac{q-a}{q'-a}]^{n+1}}{\frac{q'-q}{q'-a}}$$

which was determined by geometric progression. Here, now, by replacing in (8) that $f(q)$ is given by:

$$\begin{aligned} f(q) = {} & \frac{1}{(i+j+2k)\pi} \int_\Lambda \frac{f(q')}{q'-a} dq' + \frac{q-a}{(i+j+2k)\pi} \int_\Lambda \frac{f(q')}{q'-a} dq' \\ & + \ldots + \frac{(q-a)^n}{(i+j+2k)\pi} \int_\Lambda \frac{f(q')}{q'-q} dq' + R_n(q) \end{aligned} \tag{10}$$

where

$$R_n(q) = \frac{(q-a)^n}{(i+j+2k)\pi} \int_\Lambda \frac{f(q')}{(q'-a)^{n+1}(q'-q)} dq'.$$

Let the function $f(q)$ be developed as given below:

$$f(q) = f(a) + \frac{q-a}{1!} c_1 + \frac{(q-a)^2}{2!} c_2 + \ldots + \frac{(q-a)^n}{n!} c_n(q) + R_n(q). \tag{11}$$

or

$$f(q) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (q-a)^n. \tag{12}$$

Performing successive derivations on function $f(q)$, then we obtain:

$$c_n = \frac{f^n(q)}{n!} (q-a)^n. \tag{13}$$

Moreover, using the theorem given in [9], which is derived from quaternionic, it follows that:

$$f^n(q) = \frac{n!}{\pi(i+j+2k)} \int_\Lambda \frac{f(q')}{(q'-q)^{n+1}} dq'. \tag{14}$$

It now remains to prove that the limit below:

$$\lim_{n\to\infty} R_n(q) = 0.$$

Using the facts below:

$$|q'-q| > 0,$$

$$|\frac{f(q')}{q'-q}| < K$$

and

$$|q'-a| = r.$$

taking into account that $f(q')$ is analytic inside Λ and $q' \in \Lambda$. Since then the area of the sphere $4\pi r^2$, it follows that:

$$|R_n| = \frac{|q-a|^{n+1}}{6\pi}|\int_\Lambda \frac{f(q')}{(q'-a)^{n+1}(q'-q)}dq'| < \frac{|q-a|^{n+1}}{6\pi}K\frac{1}{r^{n+1}}2\pi^2 r^3$$
$$= \frac{1}{3}\pi r^3 K|\frac{q-a}{r}|^{n+1}$$

but as $|\frac{q}{q'-a}| = |\frac{q}{r}| < 1$ when $n \to \infty$ the expression on the right approaches zero.

The results are resumed in the following Theorem:

Theorem 2. *If $f(q)$ analytic in P and $q = a$ one point in P. Exists, one power series with center in a such that:*

$$f(q) = \sum_{n=0}^{\infty} c_n(q-a)^n$$

where

$$c_n = \frac{1}{n!}f^n(q), \qquad n = 0, 1, 2, ...$$

and

$$\lim_{n\to\infty} R_n = 0.$$

4. Conclusion

In (12) and (13) was seen a generalization to power series, in the case of quaternionic functions. Moreover, we noted that the Liouville theorem is valid from the second derivative of the function, differently from what happens in case of functions of one complex variable. It is worth mentioning that this work can be extended to the determination of Power series as well as to the extension of Liouville's theorem for the octonionic case. These are main purpose of future works.

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References

[1] S. Marques, C. de Oliveira, An extension of quaternionic metrics to octonions, *J. Math. Phys.*, **26** (1985), 3131-3139.

[2] J.M. Machado, M.F. Borges, Hypercomplex functions and conformal mappings, *Internacional Journal of Applied Math.*, **1** (2002), 27-38.

[3] J.B. Maricato, J.M. Machado, M.F. Borges, Quasiconformal transformations and hypercomplex functions, *Internacional Journal of Applied Math.*, **20** (2007), 691-702.

[4] M.F. Borges, L.F. Benzatti, Quasiconformal mappings in octonionic algebra: A graphical an analytical compareson, *Far East Journal of Mathematical Sciences*, **33** (2009), 355-361.

[5] M.F. Borges, J.A. Marão, R.C. Barreiro, A Cauchy-like theorem for hypercomplex functions, *Journal of Geometry and Topology*, **3** (2009), 263-271.

[6] M.F. Borges, J.A. Marão, J.M. Machado, Geometrical octonions I: On trigonometric and logarithmic functions of octonionic type, *Internacional Journal of Applied Math.*, **21** (2008), 461-471.

[7] V.I. Smirnov, *Corso di Matematica Superiore*, Volume Primo, Editori Riutini (1977).

[8] P. Loskot, N.C. Beaulieu, On monotonicity of the hypersphere volume and area, *Journal of Geometry*, **87** (2007), 96-98.

[9] M.F. Borges, A.D. Figueiredo, J.A. Marão, Hypercomplex geometric derivative from a Cauchy-like integral formula, *International Journal of Pure and Applied Mathematics*, **68** (2011), 55-59.

[10] G. Warnault, Liouville Theorems for stable radial solutions for the biharmonic operator, *Asymptot. Anal.*, **69** (2010), 87-98.

[11] X. Zhu, M. Wang, Liouville theorems for quasi-harmonic functions, *Non-Linear Anal.*, **73**, No. 9 (2010), 2890-2896.

[12] S. Della, Giuseppe Liouville-type theorem's for foliations with complex leaves, *Ann. Inst. Fourier*, **60**, No. 2 (2010), 711-725.

[13] D. Pascali, Quaternion holomorphes, *Acad. R. P. Romîne Stud. Cerc. Mat.*, **12** (1961), 163-186.

[14] M.F. Borges, J.A. Marão, J.M. Machado, Geometrical octonions II: Hyper regularity and hyper periodicity of the exponential function, *Internacional Journal of Pure and Applied Math.*, **48** (2008), 495-500.