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GNAN MEAN AND ITS DUAL IN n VARIABLES

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Abstract: In this paper, we define a weighted Gnan mean and its dual form in n variables, and prove their monotonicities.

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1. Introduction

For positive numbers a, b, let

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a < b; \\ a, & a = b; \end{cases} \tag{1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{2}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{3}$$

These are respectively called the identric, logarithmic and Heron means (see

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[1]).

In [3] and [4], Zhang et al. gave the generalization of Heron mean, similar product type mean and their dual forms. For two variables, there are respectively as follows

$$I(a,b;k)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a+ib}{k+1}\right)^{\frac{1}{k}}, \tag{4}$$

$$I^{*}(a,b;k)=\prod_{i=0}^{k}\left(\frac{(k-i)a+ib}{k}\right)^{\frac{1}{k+1}}, \tag{5}$$

and

$$H(a,b;k)=\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}}b^{\frac{i}{k}}, \tag{6}$$

$$h(a,b;k)=\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}}b^{\frac{i}{k+1}}, \tag{7}$$

where k is a natural number. Authors proved that $H(a,b;k)$ and $I^{*}(a,b;k)$ are monotone decreasing functions and $h(a,b;k)$ and $I(a,b;k)$ are monotone increasing functions with k, and

$$\lim_{k\to+\infty}I(a,b;k)=\lim_{k\to+\infty}I^{*}(a,b;k)=I(a,b),$$

and

$$\lim_{k\to+\infty}H(a,b;k)=\lim_{k\to+\infty}h(a,b;k)=L(a,b).$$

For n variables, let $a=(a_0,a_1,\cdots,a_n)$ and r be a nonnegative integer, where a_i for $0\le i\le n$ are nonnegative real numbers, there are respectively defined by

$$I_n^{[r]}(a)=\prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\ge 1}}\left[\sum_{k=0}^{n}\frac{i_k}{n+r}a_k\right]^{\frac{1}{\binom{n+r-1}{r-1}}}, \tag{8}$$

$$I_n^{*[r]}(a)=\prod_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\ldots,i_n\ge 0}}\left[\sum_{k=0}^{n}\frac{i_k}{r}a_k\right]^{\frac{1}{\binom{n+r}{r}}}, \tag{9}$$

and

$$H_n^{[r]} = H_n^{[r]}(a) = \frac{1}{\binom{n+r}{r}} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k/r}, \tag{10}$$

$$h_n^{[r]} = h_n^{[r]}(a) = \frac{1}{\binom{n+r-1}{r-1}} \sum_{\substack{i_1+i_2+\cdots+i_n=n+r,\\ i_1,i_2,\ldots,i_n\geq 1 \text{ are integers}}} \prod_{k=1}^{n} a_k^{i_k/(n+r)}. \tag{11}$$

Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geq 0$ and $\lambda_i > 0$ for $0 \leq i \leq n$, then

$$I_n^{[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\geq 1 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k\right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}}, \tag{12}$$

$$I_n^{*[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\ldots,i_n\geq 0 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{r} a_k\right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i}}, \tag{13}$$

and

$$\begin{aligned} &H_n^{[r]}(a,\lambda) \\ &\quad= \frac{1}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right) \prod_{k=0}^{n} a_k^{i_k/r}, \end{aligned} \tag{14}$$

$$\begin{aligned} &h_n^{[r]}(a,\lambda) \\ &\quad= \frac{1}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}} \left(\sum_{k=0}^{n}(i_k-1)\lambda_k\right) \prod_{k=0}^{n} a_k^{i_k/(n+r)}. \end{aligned} \tag{15}$$

In [5]-[7], authors researched that $I_n^{[r]}(a,\lambda)$ and $H_n^{[r]}(a,\lambda)$ are monotone decreasing functions, $I_n^{*[r]}(a,\lambda)$ and $h_n^{[r]}(a,\lambda)$ is a monotone increasing functions with k, and

$$\lim_{r\to\infty} I_n^{[r]}(a,\lambda)$$

$$= \lim_{r\to\infty} I_n^{*[r]}(a,\lambda) = \exp\left\{ \frac{\int_E \left(\sum_{i=0}^n \lambda_i x_i\right) \ln(\sum_{i=0}^n a_i x_i) dx}{\int_E \left(\sum_{i=0}^n \lambda_i x_i\right) dx} \right\}, \quad (16)$$

$$\lim_{r\to\infty} h_n^{[r]}(a,\lambda) = \lim_{r\to\infty} H_n^{[r]}(a,\lambda) = \frac{\int_E \left(\sum_{i=0}^n \lambda_i x_i\right)(\prod_{i=0}^n a_i^{x_i}) dx}{\int_E \left(\sum_{i=0}^n \lambda_i x_i\right) dx}, \quad (17)$$

where $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E :

$$E = \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^n x_i \le 1, x_i \ge 0, i = 1, 2, \cdots, n \right\}, \quad (18)$$

and $x_0 = 1 - \sum_{i=1}^n x_i$.

For a, b are positive, k be a non negative integer and α, β two real numbers. Then Gnan mean $GN(a,b,k;\alpha,\beta)$ and its dual mean $gn(a,b,k;\alpha,\beta)$ in two variables are defined as follows:

$$GN(a,b,k;\alpha,\beta) = \left[\frac{1}{k} \left(\sum_{k=1}^k \frac{(k+1-i)a^\alpha + ib^\alpha}{k+1} \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}}, \quad (19)$$

$$GN(a,b,k;0,\beta) = \left(\frac{1}{k} \sum_{k=1}^k a^{\frac{(k+1-i)\beta}{k+1}} b^{\frac{i\beta}{k+1}} \right)^{\frac{1}{\beta}}, \quad (20)$$

$$GN(a,b,k;\alpha,0) = \prod_{i=1}^k \left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1} \right)^{\frac{1}{k\alpha}}, \quad (21)$$

$$GN(a,b,k;0,0) = \sqrt{ab}, \quad (22)$$

and

$$gn(a,b,k;\alpha,\beta) = \left[\frac{1}{k+1} \left(\sum_{k=0}^k \frac{(k-i)a^\alpha + ib^\alpha}{k} \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}}, \quad (23)$$

$$gn(a,b,k;0,\beta) = \left(\frac{1}{k+1} \sum_{k=0}^k a^{\frac{(k-i)\beta}{k}} b^{\frac{i\beta}{k}} \right)^{\frac{1}{\beta}}, \quad (24)$$

$$gn(a,b,k;\alpha,0) = \prod_{i=0}^k \left(\frac{(k-i)a^\alpha + ib^\alpha}{k} \right)^{\frac{1}{k\alpha}}, \quad (25)$$

$$gn(a,b,k;0,0) = \sqrt{ab}, \quad (26)$$

The properties, limitations and monotonities of Gnan mean and its dual mean for two variables are studied and similar results on weighted Gnan mean and its dual mean for two variables are stated in (see [2]).

In this paper, the weighted Gnan mean and its dual mean in n variables are defined. Further, the properties, monotonities and limitations are proved.

2. Definition and Properties

Definition 1. Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geq 0$ and $\lambda_i > 0$ for $0 \leq i \leq n$, then we introduce the following two cases of means $GN_n^{[r]}(a, \lambda; \alpha, \beta)$ and $gn_n^{[r]}(a, \lambda; \alpha, \beta)$:

$$gn_n^{[r]}(a, \lambda; \alpha, \beta) = \left[\frac{1}{\binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r, \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n} (1+i_k)\lambda_k \right) \left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{r} \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}}, \tag{27}$$

$$gn_n^{[r]}(a, \lambda; 0, \beta) = \left[\frac{1}{\binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r, \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n} (1+i_k)\lambda_k \right) \prod_{k=0}^{n} a_k^{\beta i_k / r} \right]^{\frac{1}{\beta}}, \tag{28}$$

$$gn_n^{[r]}(a, \lambda; \alpha, 0) = \prod_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0, i_1, \ldots, i_n \geq 0 \text{ are integers}}} \left[\sum_{k=1}^{n} \frac{i_k a_k^{\alpha}}{r} \right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\alpha \binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i}}, \tag{29}$$

$$gn_n^{[r]}(a, \lambda; 0, 0) = \prod_{k=0}^{n} a_k^{\sum_{i=0}^{n}(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^{n}\lambda_i]}; \tag{30}$$

and

$$GN_n^{[r]}(a, \lambda; \alpha, \beta) = \left[\frac{1}{\binom{n+r-1}{r-2} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=n+r, \\ i_0, i_1, \cdots, i_n \geq 1 \text{ are integers}}} \left(\sum_{k=0}^{n} (i_k - 1)\lambda_k \right) \left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{n+r} \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}}, \tag{31}$$

$$GN_n^{[r]}(a,\lambda;0,\beta) = \left[\frac{1}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}}\left(\sum_{k=0}^{n}(i_k-1)\lambda_k\right)\prod_{k=0}^{n}a_k^{\beta i_k/(n+r)}\right]^{\frac{1}{\beta}}, \tag{32}$$

$$GN_n^{[r]}(a,\lambda;\alpha,0) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\geq 1 \text{ are integers}}}\left[\sum_{k=1}^{n}\frac{i_k a_k^{\alpha}}{n+r}\right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\alpha\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}}, \tag{33}$$

$$GN_n^{[r]}(a,\lambda;0,0) = \prod_{k=0}^{n} a_k^{\sum_{i=0}^{n}(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^{n}\lambda_i]}. \tag{34}$$

According to Definition 1, we easily find the following remark and characteristic properties for $GN_n^{[r]}(a,\lambda;\alpha,\beta)$ and $gn_n^{[r]}(a,\lambda;\alpha,\beta)$.

Remark 2. We call that $GN_n^{[r]}(a,\lambda;\alpha,\beta)$ and $gn_n^{[r]}(a,\lambda;\alpha,\beta)$ the generalized weighted Gnan mean and its dual form of a for λ, respectively.

Proposition 3. *If r is a natural number, then*

(a) $GN_n^{[r]}(a,\lambda;0,0) = gn_n^{[r]}(a,\lambda;0,0)$.

(b) $gn_n^{[r]}(a,\lambda;1,0) = I_n^{[r]}(a,\lambda),$ *and* $gn_n^{[r]}(a,\lambda;0,1) = H_n^{[r]}(a,\lambda)$.

(c) $GN_n^{[r]}(a,\lambda;1,0) = I_n^{*[r]}(a,\lambda),$ *and* $GN_n^{[r]}(a,\lambda;0,1) = h_n^{[r]}(a,\lambda)$.

(d) $p \leqslant GN_n^{[r]}(a,\lambda;\alpha,\beta) \leqslant q,$ *and* $p \leqslant gn_n^{[r]}(a,\lambda;\alpha,\beta) \leqslant q$.

(e) $GN_n^{[r]}(a,\lambda;\alpha,\beta) = gn_n^{[r]}(a,\lambda;\alpha,\beta) = a_0$ *if and only if* $a_0 = a_1 = \cdots = a_n$.

(f) $GN_n^{[r]}(ta,\lambda;\alpha,\beta) = tGN_n^{[r]}(a,\lambda;\alpha,\beta),$ *and*

$$gn_n^{[r]}(ta,\lambda;\alpha,\beta) = tgn_n^{[r]}(a,\lambda;\alpha,\beta), \text{ if } t>0.$$

where $p = \min_{0\leqslant k\leqslant n}\{a_k\},\ q = \max_{0\leqslant k\leqslant n}\{a_k\}$, *and* $ta = (ta_0, ta_1, \cdots, ta_n)$.

3. Monotonicities and Limitations

Theorem 4. *The Gnan mean $GN_n^{[r]}(a,\lambda;\alpha,\beta)$ and $gn_n^{[r]}(a,\lambda;\alpha,\beta)$ in n variables is a monotone increasing function with respect to α or β.*

Proof. From the well known weighted power mean inequality, we have that

$$M_\alpha(a,x)=\begin{cases}\left[\dfrac{\sum_{k=0}^n a_k^\alpha x_k}{\sum_{k=0}^n x_k}\right]^{\frac{1}{\alpha}} & (\alpha\neq 0),\\ =\prod_{k=0}^n a_k^{x_k/\sum_{k=0}^n x_k} & (\alpha=0)\end{cases}$$

is a monotone increasing function with respect to r or a_i for $1\leqslant i\leqslant n$. It is easily to obtain Theorem 4. □

Theorem 5. *Let $r\in\mathbb{N}$. Then $gn_n^{[r]}(a,\lambda;\alpha,\beta)$ is a monotone decreasing function and $GN_n^{[r]}(a,\lambda;\alpha,\beta)$ is a monotone increasing function with r, i.e. the following two inequalities*

$$gn_n^{[r]}(a,\lambda;\alpha,\beta)\geqslant gn_n^{[r+1]}(a,\lambda;\alpha,\beta). \tag{35}$$

$$GN_n^{[r]}(a,\lambda;\alpha,\beta)\leqslant GN_n^{[r+1]}(a,\lambda;\alpha,\beta). \tag{36}$$

hold if $\alpha<\beta$, and inequalities (35) *and* (36) *inverse if $\alpha>\beta$. The equalities in* (35) *and* (36) *are valid if and only if $a_0=a_1=\cdots=a_n$.*

Proof. For $\alpha=0$ or $\beta=0$, the proofs of inequalities (35) and (36) are obtained in [5]-[7].

If $\alpha\neq 0$ and $\beta\neq 0$, we will only prove inequality (35), the proof of the inequality (36) is similar.

From Definition 1, we get

$$\binom{n+r+2}{r+1}\sum_{k=0}^n\lambda_k\left[gn_n^{[r]}(a,\lambda;\alpha,\beta)\right]^\beta \tag{37}$$

$$=\frac{n+r+2}{r+1}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0\text{ are integers}}}\left(\sum_{k=0}^n(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^n i_k a_k^\alpha}{r}\right)^{\frac{\beta}{\alpha}}$$

$$=\frac{n+r+1}{r+1}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0\text{ are integers}}}\frac{n+r+2}{n+r+1}\left(\sum_{k=0}^n(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^n i_k a_k^\alpha}{r}\right)^{\frac{\beta}{\alpha}}$$

$$=\frac{\sum_{j=0}^n(1+i_j)}{r+1}$$

$$\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0\text{ are integers}}}\frac{\sum_{j=0}^n(1+i_j)+1}{n+r+1}\left(\sum_{k=0}^n(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^n i_k a_k^\alpha}{r}\right)^{\frac{\beta}{\alpha}}$$

$$= \frac{\sum_{j=0}^{n}\nu_j}{r+1}$$

$$\sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1,\\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \frac{\sum_{j=0}^{n}\nu_j+1}{n+r+1}\left[\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\lambda_j\right]\left(\frac{\sum_{k=0}^{n}i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\frac{\beta}{\alpha}}.$$

When $\sum_{j=0}^{n}\nu_j = r+1$, we have

$$\begin{aligned} &\sum_{j=0}^{n}(1+\nu_j)\left[\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\lambda_j\right] \\ &= \sum_{j=0}^{n}(1+\nu_j)\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\sum_{j=0}^{n}(1+\nu_j)\lambda_j \\ &= (n+r+2)\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\sum_{k=0}^{n}(1+\nu_k)\lambda_k \\ &= (n+r+1)\sum_{k=0}^{n}(\nu_k-1)\lambda_k. \end{aligned} \tag{38}$$

For $(\alpha/\beta)(\alpha/\beta-1)>0$, by using the weighted arithmetic-geometric mean inequality and a simple fact that

$$\frac{\nu_j}{r+1}\left(\frac{\sum_{k=0}^{n}i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\frac{\beta}{\alpha}}=0$$

for $\nu_j=0$, from (37) and (38), we find

$$\begin{aligned} &\binom{n+r+2}{r+1}\sum_{k=0}^{n}\lambda_k\left[gn_n^{[r]}(a,\lambda;\alpha,\beta)\right]^{\beta} \\ &= \frac{\sum_{j=0}^{n}\nu_j}{r+1}\sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1,\\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n}i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\frac{\beta}{\alpha}} \\ &= \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1,\\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\frac{\sum_{j=0}^{n}\nu_j}{r+1}\left(\frac{\sum_{k=0}^{n}i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\frac{\beta}{\alpha}} \\ &\geqslant \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1,\\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{j=0}^{n}\nu_j}{r+1}\cdot\frac{\sum_{k=0}^{n}i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\frac{\beta}{\alpha}} \end{aligned} \tag{39}$$

$$= \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1,\\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{r+1}\right)^{\frac{\beta}{\alpha}}$$

$$= \binom{n+r+2}{r+1}\sum_{k=0}^{n}\lambda_k\left[gn_n^{[r+1]}(a,\lambda;\alpha,\beta)\right]^{\beta},$$

that follows

$$\left[gn_n^{[r]}(a,\lambda;\alpha,\beta)\right]^{\beta} \geqslant \left[gn_n^{[r+1]}(a,\lambda;\alpha,\beta)\right]^{\beta}, \tag{40}$$

and inequalities (40) inverses if $(\alpha/\beta)(\alpha/\beta-1)<0$. The equalities above are valid if and only if

$$\left(\sum_{k=0}^{n} i_k a_k^{\alpha} - a_0^{\alpha}\right)/r = \left(\sum_{k=0}^{n} i_k a_k^{\alpha} - a_1^{\alpha}\right)/r = \cdots = \left(\sum_{k=0}^{n} i_k a_k^{\alpha} - a_n^{\alpha}\right)/r$$

which is equivalent to $a_0 = a_1 = \cdots = a_n$.

If $(\alpha/\beta)(\alpha/\beta-1)>0$, that is $\alpha/\beta<0$ or $\alpha/\beta>1$, then $\beta>\alpha$ and $\beta>0$, we immediately find inequality (35) from (40). If $(\alpha/\beta)(\alpha/\beta-1)<0$, then $\alpha<\beta<0$, and we also obtain inequality (35) from inverses (40). That is to say that inequality (35) holds if $\alpha>\beta$. Similarly, we have that inequality (35) inverses if $\alpha<\beta$.

The proof of Theorem 4 is completed. □

Theorem 6. *We have*

$$\lim_{r\to\infty} GN_n^{[r]}(a,\lambda;\alpha,\beta) = H(a,\lambda;\alpha,\beta) = \lim_{r\to\infty} gn_n^{[r]}(a,\lambda;\alpha,\beta),$$

where

$$H(a,\lambda;\alpha,\beta) = \left[\frac{\int_E (\sum_{i=0}^{n}\lambda_i x_i)(\sum_{i=0}^{n} a_i^{\alpha} x_i)^{\beta/\alpha}dx}{\int_E (\sum_{i=0}^{n}\lambda_i x_i)dx}\right]^{\frac{1}{\beta}}, \tag{41}$$

$$H(a,\lambda;0,\beta) = \left[\frac{\int_E (\sum_{i=0}^{n}\lambda_i x_i)(\prod_{i=0}^{n} a_i^{\beta x_i})dx}{\int_E (\sum_{i=0}^{n}\lambda_i x_i)dx}\right]^{\frac{1}{\beta}}, \tag{42}$$

$$H(a,\lambda;\alpha,0) = \exp\left\{\frac{\int_E (\sum_{i=0}^{n}\lambda_i x_i)\ln(\sum_{i=0}^{n} a_i^{\alpha} x_i)dx}{\alpha\int_E (\sum_{i=0}^{n}\lambda_i x_i)dx}\right\}, \tag{43}$$

$$H(a,\lambda;0,0) = \prod_{k=0}^{n} a_k^{\sum_{i=0}^{n}(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^{n}\lambda_i]}. \tag{44}$$

Proof. This follows straightforward computation from Definition 1. □

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