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THE SHARP COMBINATION BOUNDS OF ARITHMETIC AND LOGARITHMIC MEANS FOR SEIFFERT'S MEAN

Yefang Qiu[1], Miaokun Wang[2], Yuming Chu[3 §]
[1,2]Department of Mathematics
Zhejiang Sci-Tech University
Zhejiang, Hangzhou, 310018, P.R. CHINA
[2]Department of Mathematics
Huzhou Teachers College
Zhejiang, Huzhou, 313000, P.R. CHINA

Abstract: For $a, b > 0$, the arithmetic mean $A(a,b)$, logarithmic mean $L(a,b)$ and Seiffert's mean $P(a,b)$ are defined by

$$A(a,b) = \frac{a+b}{2}, \quad L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a \end{cases}$$

and

$$P(a,b) = \begin{cases} \frac{a-b}{4\arctan\sqrt{\frac{a}{b}} - \pi}, & b \neq a, \\ a, & b = a, \end{cases}$$

respectively.

In this paper we find the greatest value α and least value β such that inequality $A^{\alpha}(a,b)L^{1-\alpha}(a,b) < P(a,b) < A^{\beta}(a,b)L^{1-\beta}(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

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1. Introduction

For $a, b > 0$, the arithmetic mean $A(a,b)$, logarithmic mean $L(a,b)$ and Seiffert's

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§Correspondence author

mean $P(a,b)$ are defined by

$$A(a,b) = \frac{a+b}{2}, \tag{1}$$

$$L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a \end{cases} \tag{2}$$

and

$$P(a,b) = \begin{cases} \frac{a-b}{4\arctan\sqrt{\frac{a}{b}} - \pi}, & b \neq a, \\ a, & b = a, \end{cases} \tag{3}$$

respectively. In the recent past, the logarithmic mean $L(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for $L(a,b)$ can be found in the literature [1-7]. It might be suprising that the logarithmic mean has applications in physics [8], ecnomic [9], and even in meteorology [10].

The Seiffert's mean was introduced by Seiffert in [11], it has attracted the attention of many mathematicians [12-15] since 1993.

Let $I(a,b) = \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}$ and $G(a,b) = \sqrt{ab}$ be the identric and geometric means of two positive real numbers a and b with $a \neq b$, then it is well known that

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}. \tag{4}$$

In [11], Seiffert proved that

$$L(a,b) < P(a,b) < I(a,b) \tag{5}$$

for all $a, b > 0$ with $a \neq b$.

Later, Seiffert [12] established that

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b) + 2G(a,b)} \tag{6}$$

and

$$P(a,b) > \frac{A(a,b)G(a,b)}{L(a,b)} \tag{7}$$

for all $a, b > 0$ with $a \neq b$.

In [15], the author Sándor presented the following lower bound for $P(a,b)$ in terms of the geometric combination of $A(a,b)$ and $G(a,b)$:

$$P(a,b) > A^{\frac{2}{3}}(a,b)G^{\frac{1}{3}}(a,b) \tag{8}$$

for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to present the best possible geometric combination bounds of arithmetic and logarithmic means for the Seiffert's mean. Our main result is the following Theorem 1.

Theorem 1. *The inequality*

$$A^{\alpha}(a,b)L^{1-\alpha}(a,b) < P(a,b) < A^{\beta}(a,b)L^{1-\beta}(a,b) \tag{9}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{2}$ and $\beta \geq 1$.

2. Lemmas

For the convenience of the readers, we establish several lemmas which will be used in next section.

Lemma 1. *If $x > 1$, then*

$$8\arctan x - \frac{4x^2\log x + x^4 - 1}{x^3 + x} - 2\pi < 0. \tag{10}$$

Proof. Let $g_1(x) = 8\arctan x - \frac{4x^2\log x + x^4 - 1}{x^3 + x} - 2\pi$, then simple computations lead to

$$\lim_{x\to 1} g_1(x) = 0, \tag{11}$$

$$g_1'(x) = \frac{x^2 - 1}{x^2(x^2+1)^2}h_1(x), \tag{12}$$

where $h_1(x) = 4x^2\log x - x^4 + 1$,

$$\lim_{x\to 1} h_1(x) = 0, \tag{13}$$

$$h_1'(x) = 4xh_2(x), \tag{14}$$

where $h_2(x) = 2\log x + 1 - x^2$,

$$\lim_{x\to 1} h_2(x) = 0, \tag{15}$$

$$h_2'(x) = \frac{2}{x}(1 - x^2) < 0 \tag{16}$$

for $x > 1$.

Therefore, Lemma 1 follows from (11)-(16).

Lemma 2. *If $a, b > 0$ and $a \neq b$, then*

$$\sqrt{A(a,b)L(a,b)} > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)}.$$

Proof. Without loss of generality, we assume that $a > b$. Let $x = \sqrt{\frac{a}{b}} > 1$, then from (1)-(3) we have

$$\begin{aligned} & A(a,b)L(a,b) - [\frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)}]^2 \\ = \quad & \frac{b^3 A(a,b)}{[A(a,b)+2G(a,b)]^2} \frac{x^2(x^2+1)}{8\log x} g_2(x), \end{aligned} \tag{16}$$

where $g_2(x) = \frac{(x^2-1)(x^2+4x+1)^2}{x^2(x^2+1)} - 36\log x$.

Note that

$$\lim_{x\to 1} g_2(x) = 0, \tag{17}$$

$$g_2'(x) = \frac{2(x-1)^4(x^4+8x^3+10x^2+8x+1)}{x^3(x^2+1)^2} > 0 \tag{18}$$

for $x > 1$.

Therefore, Lemma 2 follows from (16)-(18).

Lemma 3. *If $a, b > 0$ and $a \neq b$, then*

$$\sqrt{A(a,b)L(a,b)} > \frac{A(a,b)G(a,b)}{L(a,b)}.$$

Proof. We clearly see that Lemma 3 is equivalent to

$$L^3(a,b) > A(a,b)G^2(a,b). \tag{19}$$

.

Without loss of generality, we suppose that $a > b$. Let $x = \frac{a}{b} > 1$, then simple computation yields that

$$\begin{aligned} & 3\log L(a,b) - \log[A(a,b)G^2(a,b)] \\ = \quad & 3\log\frac{x-1}{\log x} - \log\frac{x(x+1)}{2}. \end{aligned} \tag{20}$$

Let $g_3(x) = 3\log\frac{x-1}{\log x} - \log\frac{x(x+1)}{2}$, then

$$\lim_{x\to 1} g_3(x) = 0, \tag{21}$$

$$g_3'(x) = \frac{x^2+4x+1}{x(x-1)(x+1)\log x} h_3(x), \tag{22}$$

where $h_3(x) = \log x - \frac{3(x^2-1)}{x^2+4x+1}$. Note that

$$\lim_{x\to 1} h_3(x) = 0, \tag{23}$$

$$h_3'(x) = \frac{(x-1)^4}{x(x^2+4x+1)^2} > 0 \tag{24}$$

for $x > 1$.

Therefore, inequality (19) follows from (20)-(24).

Lemma 4. *If $a, b > 0$ and $a \neq b$, then*

$$\sqrt{A(a,b)L(a,b)} > A^{\frac{2}{3}}(a,b)G^{\frac{1}{3}}(a,b)$$

Proof. Lemma 4 follows from (19).

3. Proof of Theorem 1

We first prove that

$$P(a,b) > \sqrt{A(a,b)L(a,b)} \tag{25}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $x = \sqrt{\frac{a}{b}} > 1$, then (1)-(3) leads to

$$\begin{aligned} & A(a,b)L(a,b) - P^2(a,b) \\ = & \frac{b^2(x^2-1)(x^2+1)}{4(4\arctan x - \pi)^2 \log x} f(x), \end{aligned} \tag{26}$$

where $f(x) = (4\arctan x - \pi)^2 - \frac{4(x^2-1)\log x}{x^2+1}$. Note that

$$\lim_{x\to 1} f(x) = 0, \tag{27}$$

$$f'(x) = \frac{4}{1+x^2} g_1(x), \tag{28}$$

where $g_1(x) = 8\arctan x - \frac{4x^2\log x + x^4 - 1}{x^3+x} - 2\pi$.

Therefore, inequality (25) follows from (26)-(28) and Lemma 1.

From inequalities (4) and (5) we clearly see that

$$P(a,b) < A(a,b) \tag{29}$$

for all $a, b > 0$ with $a \neq b$.

Next, we prove the constant $\alpha = \frac{1}{2}$ and $\beta = 1$ are the best possible parameters for which inequality (9) holds.

For any $0 < \varepsilon < \frac{1}{2}$ and $x > 0$, we have

$$\begin{aligned} & A^{\frac{1}{2}+\varepsilon}(1+x,1)L^{\frac{1}{2}-\varepsilon}(1+x,1) - P(1+x,1) \\ = & \frac{J(x)}{[\log(1+x)]^{\frac{1}{2}-\varepsilon}(4\arctan\sqrt{1+x}-\pi)} \end{aligned} \tag{30}$$

and

$$\begin{aligned} & \lim_{x\to+\infty}[P(x,1) - A^{1-\varepsilon}(x,1)L^{\varepsilon}(x,1)] \\ = & \lim_{x\to+\infty} x\left[\frac{1-\frac{1}{x}}{4\arctan\sqrt{x}-\pi} - \left(\frac{1+\frac{1}{x}}{2}\right)^{1-\varepsilon}\left(\frac{1-\frac{1}{x}}{\log x}\right)^{\varepsilon}\right] \\ = & +\infty, \end{aligned} \tag{31}$$

where $J(x) = x^{\frac{1}{2}-\varepsilon}(1+\frac{x}{2})^{\frac{1}{2}+\varepsilon}(4\arctan\sqrt{1+x}-\pi) - x[\log(1+x)]^{\frac{1}{2}-\varepsilon}$.

Let $x \to 0$, making use of the Taylor expansion, we get

$$\begin{aligned} J(x) = & x^{\frac{3}{2}-\varepsilon}\left[1 + \frac{1+2\varepsilon}{4}x + \frac{(1+2\varepsilon)(2\varepsilon-1)}{32}x^2 + o(x^2)\right]\left[1 - \frac{x}{2}\right. \\ & \left. + \frac{7}{24}x^2 + o(x^2)\right] - x^{\frac{3}{2}-\varepsilon}\left[1 - \frac{1-2\varepsilon}{4}x + \frac{(13-6\varepsilon)(1-2\varepsilon)}{96}x^2\right. \\ & \left. + o(x^2)\right] \\ = & \frac{1}{12}\varepsilon x^{\frac{7}{2}-\varepsilon} + o(x^{\frac{7}{2}-\varepsilon}). \end{aligned} \tag{32}$$

Equations (30)-(32) imply that for any $0 < \varepsilon < \frac{1}{2}$ there exist $\delta = \delta(\varepsilon) > 0$ and $X = X(\varepsilon) > 1$ such that $A^{\frac{1}{2}+\varepsilon}(1+x,1)L^{\frac{1}{2}-\varepsilon}(1+x,1) > P(1+x,1)$ for $x \in (0,\delta)$ and $P(x,1) > A^{1-\varepsilon}(x,1)L^{\varepsilon}(x,1)$ for $x \in (X,\infty)$.

Remark. From Lemmas 2.2-2.4 and inequalities (6)-(8), we clearly see that the lower bound of Seiffert's mean in Theorem 1 is better than which presented in [12] and [15].

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