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THE OPTIMAL GENERALIZED HERONIAN MEAN BOUNDS FOR THE IDENTRIC MEAN

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Abstract: In this paper, we answer the question: What are the greatest value p and the least value q, such that the double inequality $H_p(a,b) < I(a,b) < H_q(a,b)$ holds for all $a,b>0$ with $a\neq b$? Here, $H_p(a,b)$ and $I(a,b)$ denote the p-th generalized Heronian mean and identric mean of two positive numbers a and b, respectively.

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1. Introduction

For $p\in\mathbb{R}$, the p-th generalized Heronian mean of two positive numbers a and b was defined by Jia and Cao [1] as follows:

$$H_p(a,b)=\begin{cases}\left(\frac{a^p+(ab)^{\frac{p}{2}}+b^p}{3}\right)^{\frac{1}{p}}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases} \tag{1}$$

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It is well-known that $H_p(a,b)$ is strictly increasing with respect to p for fixed $a,b>0$ with $a\neq b$. For $r\in\mathbb{R}$, let $M_r(a,b)=\begin{cases}(\frac{a^r+b^r}{2})^{\frac{1}{r}}, & r\neq 0,\\ \sqrt{ab}, & r=0,\end{cases}$ $A(a,b)=\frac{a+b}{2}$, $G(a,b)=\sqrt{ab}$, $H(a,b)=\frac{2ab}{a+b}$, $I(a,b)=\begin{cases}\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, & b\neq a,\\ a, & b=a\end{cases}$ and $L(a,b)=\begin{cases}\frac{b-a}{\log b-\log a}, & b\neq a,\\ a, & b=a\end{cases}$ be the r-th power, arithmetic, geometric, harmonic, identric and logarithmic means of two positive numbers a and b. Then

$$\begin{aligned}&\min\{a,b\}<H(a,b)<G(a,b)=M_0(a,b)=H_0(a,b)\\&<L(a,b)<I(a,b)<A(a,b)=M_1(a,b)<\max\{a,b\}\end{aligned}\tag{2}$$

for all $a,b>0$ with $a\neq b$.

In [2], Alzer and Janous established the following sharp double inequality (see also [3, p 350]):

$$M_{\frac{\log 2}{\log 3}}(a,b)<H_1(a,b)=\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)<M_{\frac{2}{3}}(a,b)$$

for all $a,b>0$ with $a\neq b$.

The following sharp upper generalized Heronian mean bound for the logarithmic mean was given in [1]:

$$L(a,b)<H_{\frac{1}{2}}(a,b)$$

for all $a,b>0$ with $a\neq b$.

The following comparison for generalized Heronian mean and identric mean is due to Sándor [4, 5]:

$$H_1(a,b)=\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)<I(a,b)\tag{3}$$

for all $a,b>0$ with $a\neq b$.

In [6-8] the authors presented the sharp power mean bounds for the combinations of G and H, G and L, and A and L.

$$\frac{2}{3}G(a,b)+\frac{1}{3}H(a,b)>M_{-\frac{1}{3}}(a,b),$$

$$\frac{1}{3}G(a,b)+\frac{2}{3}H(a,b)>M_{-\frac{2}{3}}(a,b),$$

$$A^{\alpha}(a,b)L^{1-\alpha}(a,b)<M_{\frac{1+2\alpha}{3}}(a,b),$$

$$G^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{\frac{1-\alpha}{3}}(a,b)$$

and

$$M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < M_{\frac{1+2\alpha}{3}}(a,b)$$

for all $\alpha \in (0,1)$ and $a,b > 0$ with $a \neq b$.

The purpose of this paper is to answer the question: What are the greatest value p and the least value q, such that the double inequality $H_p(a,b) < I(a,b) < H_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$? Our main result is the following Theorem 1.

Theorem 1. *For all $a,b > 0$ with $a \neq b$ we have*

$$H_1(a,b) < I(a,b) < H_{\log 3}(a,b),$$

and $H_{\log 3}(a,b)$ and $H_1(a,b)$ are the best possible upper and lower generalized Heronian mean bounds for the identric mean $I(a,b)$.

2. One Lemma

In order to prove our main result, we need a lemma which we present in this section.

Lemma 1. *Suppose that* $g(x) = 4x^{4p} - 4x^{4p-2} + (p-2)x^{3p+2} - 2(p-4)x^{3p} + (p-6)x^{3p-2} + 2(2p-3)x^{2p+2} - 4(2p-3)x^{2p} + 2(2p-3)x^{2p-2} + (p-6)x^{p+2} - 2(p-4)x^{p} + (p-2)x^{p-2} - 4x^2 + 4$. *If $p = \log 3 = 1.0986\cdots$, then there exists a unique $x_0 \in (1,+\infty)$ such that $g(x) > 0$ for $x \in (1,x_0)$, $g(x) < 0$ for $x \in (x_0,+\infty)$ and $g(x_0) = 0$.*

Proof. Let $g_1(x) = \frac{g'(x)}{x}$, $g_2(x) = x^{5-p}g_1'(x)$, $g_3(x) = \frac{g_2'(x)}{2px}$, $g_4(x) = \frac{g_3'(x)}{2x}$, $g_5(x) = \frac{1}{2}x^{5-p}g_4'(x)$, and $g_6(x) = \frac{g_5'(x)}{px}$. Then simple computations lead to

$$g(1) = 0, \tag{4}$$

$$\lim_{x\to+\infty} g(x) = -\infty, \tag{5}$$

$$\begin{aligned} g_1(x) = {} & 16px^{4p-2} - 8(2p-1)x^{4p-4} + (p-2)(3p+2)x^{3p} - 6p(p-4)x^{3p-2} \\ & +(p-6)(3p-2)x^{3p-4} + 4(p+1)(2p-3)x^{2p} - 8p(2p-3)x^{2p-2} \\ & +4(p-1)(2p-3)x^{2p-4} + (p-6)(p+2)x^{p} - 2p(p-4)x^{p-2} \end{aligned}$$

$$+(p-2)^2x^{p-4}-8,$$

$$g_1(1)=0, \tag{6}$$

$$\lim_{x\to+\infty} g_1(x)=-\infty, \tag{7}$$

$$\begin{aligned} g_2(x) &= 32p(2p-1)x^{3p+2}-32(p-1)(2p-1)x^{3p}+3p(p-2)(3p+2)\\ &\quad\times x^{2p+4}-6p(p-4)(3p-2)x^{2p+2}+(p-6)(3p-2)(3p-4)x^{2p}\\ &\quad+8p(p+1)(2p-3)x^{p+4}-16p(p-1)(2p-3)x^{p+2}+8(p-1)\\ &\quad\times(p-2)(2p-3)x^{p}+p(p-6)(p+2)x^4-2p(p-2)(p-4)x^2\\ &\quad+(p-2)^2(p-4), \end{aligned}$$

$$g_2(1)=144(p-1)>0, \tag{8}$$

$$\lim_{x\to+\infty} g_2(x)=-\infty, \tag{9}$$

$$\begin{aligned} g_3(x) &= 16(2p-1)(3p+2)x^{3p}-48(p-1)(2p-1)x^{3p-2}+3(p-2)(p+2)\\ &\quad\times(3p+2)x^{2p+2}-6(p-4)(p+1)(3p-2)x^{2p}+(p-6)(3p-2)\\ &\quad\times(3p-4)^{2p-2}+4(p+1)(p+4)(2p-3)x^{p+2}-8(p-1)(p+2)\\ &\quad\times(2p-3)x^{p}+4(p-1)(p-2)(2p-3)x^{p-2}+2(p-6)(p+2)x^2\\ &\quad-2(p-2)(p-4), \end{aligned}$$

$$g_3(1)=360(p-1)>0, \tag{10}$$

$$\lim_{x\to+\infty} g_3(x)=-\infty, \tag{11}$$

$$\begin{aligned} g_4(x) &= 24p(2p-1)(3p+2)x^{3p-2}-24(p-1)(2p-1)(3p-2)x^{3p-4}+3\\ &\quad\times(p-2)(p+1)(p+2)(3p+2)x^{2p}-6p(p-4)(p+1)(3p-2)x^{2p-2}\\ &\quad+(p-1)(p-6)(3p-2)(3p-4)x^{2p-4}+2(p+1)(p+2)(p+4)\\ &\quad\times(2p-3)x^{p}-4p(p-1)(p+2)(2p-3)x^{p-2}+2(p-1)(p-2)^2\\ &\quad\times(2p-3)x^{p-4}+2(p-6)(p+2), \end{aligned}$$

$$g_4(1)=636p^2-684p+24=40.168\cdots>0, \tag{12}$$

$$\lim_{x\to+\infty} g_4(x)=-\infty, \tag{13}$$

$$g_5(x) = 12p(2p-1)(3p-2)(3p+2)x^{2p+2}-12(p-1)(2p-1)(3p-2)$$

$$\begin{aligned}&\times(3p-4)x^{2p}+3p(p-2)(p+1)(p+2)(3p+2)x^{p+4}-6p(p-4)\\&\times(p-1)(p+1)(3p-2)x^{p+2}+(p-1)(p-2)(p-6)(3p-2)\\&\times(3p-4)x^{p}+p(p+1)(p+2)(p+4)(2p-3)x^{4}-2p(p-1)\\&\times(p-2)(p+2)(2p-3)x^{2}+(p-1)(p-2)^{2}(p-4)(2p-3),\end{aligned}$$

$$g_5(1)=1038p^3-1950p^2+1092p-240=-17.510\cdots<0 \tag{14}$$

and

$$\begin{aligned}g_6(x) &= 24(p+1)(2p-1)(3p-2)(3p+2)x^{2p}+24(p-1)(2p-1)\\&\quad\times(3p-2)(4-3p)x^{2p-2}-3(2-p)(p+1)(p+2)(p+4)\\&\quad\times(3p+2)x^{p+2}+6(p-1)(4-p)(p+1)(p+2)(3p-2)x^{p}\\&\quad-(p-1)(2-p)(6-p)(3p-2)(4-3p)x^{p-2}-4(p+1)\\&\quad\times(p+2)(p+4)(3-2p)x^{2}-4(p-1)(2-p)(p+2)\\&\quad\times(3-2p) \qquad (15)\\&< 24(p+1)(2p-1)(3p-2)(3p+2)x^{p+2}+24(p-1)(2p-1)\\&\quad\times(3p-2)(4-3p)x^{p+2}-3(2-p)(p+1)(p+2)(p+4)\\&\quad\times(3p+2)x^{p+2}+6(p-1)(4-p)(p+1)(p+2)(3p-2)x^{p+2}\\&= (-9p^5+99p^4+1896p^3-2628p^2+528p-96)x^{p+2}\\&= (-43.944\cdots)x^{p+2}<0\end{aligned}$$

for $x\geq 1$.

From inequalities (14) and (15) we clearly see that $g_5(x)<0$ for $x\in[1,+\infty)$, hence $g_4(x)$ is strictly decreasing in $[1,+\infty)$.

Form (12) and (13) together with the monotonicity of $g_4(x)$ we know that there exists $\lambda_1\in(1,+\infty)$, such that $g_4(x)>0$ for $x\in[1,\lambda_1)$ and $g_4(x)<0$ for $x\in(\lambda_1,+\infty)$. Hence $g_3(x)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,+\infty)$.

The monotonicity of $g_3(x)$ together with (10) and (11) imply that there exists $\lambda_2\in(1,+\infty)$, such that $g_3(x)>0$ for $x\in[1,\lambda_2)$ and $g_3(x)<0$ for $x\in(\lambda_2,+\infty)$. Hence $g_2(x)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,+\infty)$.

From (8) and (9) together with the monotonicity of $g_2(x)$ we clearly see that there exists $\lambda_3\in(1,+\infty)$, such that $g_2(x)>0$ for $x\in[1,\lambda_3)$ and $g_2(x)<0$ for $x\in(\lambda_3,+\infty)$. Hence $g_1(x)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,+\infty)$.

The monotonicity of $g_1(x)$ together with (6) and (7) lead to that there exists $\lambda_4\in(1,+\infty)$, such that $g_1(x)>0$ for $x\in(1,\lambda_4)$ and $g_1(x)<0$ for $x\in(\lambda_4,+\infty)$. Hence $g(x)$ is strictly increasing in $[1,\lambda_4]$ and strictly decreasing in $[\lambda_4,+\infty)$.

Therefore, Lemma 1 follows from (4) and (5) together with the monotonicity of $g(x)$.

3. Proof of Theorem 1

From (3) we clearly see that $H_1(a,b) < I(a,b)$ for all $a,b>0$ with $a \neq b$.

Next, we prove that

$$I(a,b) < H_{\log 3}(a,b) \tag{16}$$

for all $a,b>0$ with $a\neq b$.

Without loss of generality, we assume that $a>b$, and put $x=\sqrt{\frac{a}{b}}>1$ and $p=\log 3$. Then

$$\begin{aligned} &\log[H_p(a,b)] - \log[I(a,b)] \\ =\ & \tfrac{1}{p}\log(1+x^p+x^{2p}) - \tfrac{2x^2}{x^2-1}\log x. \end{aligned} \tag{17}$$

Let

$$f(x) = \frac{1}{p}\log(1+x^p+x^{2p}) - \frac{2x^2}{x^2-1}\log x. \tag{18}$$

Then simple computations lead to

$$\lim_{x\to 1} f(x) = \lim_{x\to +\infty} f(x) = 0, \tag{19}$$

$$f'(x) = \frac{4x}{(x^2-1)^2} f_1(x), \tag{20}$$

where $f_1(x) = \log x - \frac{(x^2-1)(2x^{2p-2}+x^p+x^{p-2}+2)}{4(1+x^p+x^{2p})}$, and

$$f_1(1) = 0, \tag{21}$$

$$\lim_{x\to +\infty} f_1(x) = -\infty, \tag{22}$$

$$f_1'(x) = \frac{g(x)}{4x(1+x^p+x^{2p})^2}, \tag{23}$$

where $g(x)$ is defined as in Lemma 1.

From (23) and Lemma 1 we clearly see that there exists $x_0 \in (1,+\infty)$ such that $f_1(x)$ is strictly increasing in $[1,x_0]$ and strictly decreasing in $[x_0,+\infty)$. Then (21) and (22) together with the monotonicity of $f_1(x)$ imply that there exists $\lambda \in (1,+\infty)$ such that $f_1(x)>0$ for $x\in(1,\lambda)$ and $f_1(x)<0$ for $x\in(\lambda,+\infty)$, this result and (20) lead to that $f(x)$ is strictly increasing in $(1,\lambda]$ and strictly decreasing in $[\lambda,+\infty)$.

Therefore, inequality (16) follows from (17)-(19) and the monotonicity of $f(x)$.

At last, we prove that $H_{\log 3}(a,b)$ and $H_1(a,b)$ are the best possible upper and lower generalized Heronian mean bounds for the identric mean $I(a,b)$.

For, any $0<\varepsilon<\log 3$ and $x>0$ one has

$$\begin{aligned}&\lim_{x\to+\infty}\frac{H_{\log 3-\varepsilon}(1,x)}{I(1,x)}=e\lim_{x\to+\infty}\frac{[\frac{1}{3}(1+x^{\frac{\varepsilon-\log 3}{2}}+x^{\varepsilon-\log 3})]^{\frac{1}{\log 3-\varepsilon}}}{x^{\frac{1}{x-1}}}\\&=\frac{e}{3^{\frac{1}{\log 3-\varepsilon}}}<\frac{e}{3^{\frac{1}{\log 3}}}=1,\end{aligned}\tag{24}$$

$$\begin{aligned}&H_{1+\varepsilon}(1+x,1)-I(1+x,1)\\=\ &[\frac{1+(1+x)^{\frac{1+\varepsilon}{2}}+(1+x)^{1+\varepsilon}}{3}]^{\frac{1}{1+\varepsilon}}-\frac{1}{e}(1+x)^{\frac{1+x}{x}}.\end{aligned}\tag{25}$$

Let $x\to 0$, making use of the Taylor expansion we get

$$\begin{aligned}&[\frac{1+(1+x)^{\frac{1+\varepsilon}{2}}+(1+x)^{1+\varepsilon}}{3}]^{\frac{1}{1+\varepsilon}}-\frac{1}{e}(1+x)^{\frac{1+x}{x}}\\=\ &\frac{\varepsilon}{12}x^2+o(x^2).\end{aligned}\tag{26}$$

Inequality (24) implies that for any $0<\varepsilon<\log 3$ there exists $X=X(\varepsilon)>1$ such that $I(1,x)>H_{\log 3-\varepsilon}(1,x)$ for $x\in(X,+\infty)$.

Equations (25) and (26) imply that for any $0<\varepsilon<\log 3$ there exists $\delta=\delta(\varepsilon)>0$ such that $H_{1+\varepsilon}(1+x,1)>I(1+x,1)$ for $x\in(0,\delta)$.

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