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ABUNDANT SEMIGROUPS WITH QUASI-MEDIAL IDEMPOTENTS

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Abstract: In this paper, the concept of a medial idempotent is generalized, called a quasi-medial idempotent. After that, some properties about an abundant semigroup with a quasi-medial idempotent is studied. In particular, a structure theorem for an abundant semigroup with a quasi-medial idempotent is given and some special cases are considered.

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1. Introduction and Preliminaries

Since Blyth and McFadden in [2] introduced the concept of *normal medial idempotent* of a regular semigroup, much attention has been paid to this class of regular semigroups. Recall that an idempotent u of a regular semigroup S is a *medial idempotent* if for any element x of the regular semigroup $\overline{E}$ generated by the set E of idempotents of S, $xux = x$. A medial idempotent u is called *normal* if $u\overline{E}u$ is a semillatice. Loganathan [9] got a structure theorem for regular semigroups with a medial idempotent in via of an idempotent-generated regular semigroup with a medial idempotent and an orthodox semigroup. El-Qallali [8] introduced the notion of (normal) medial idempotent to abundant semigroups and studied the abundant semigroup with a normal idempotent.

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Jing [10] describe a method of constructing all abundant semigroups with a medial idempotent by an idempotent-generated regular semigroup with a medial idempotent and a quasi-adequate semigroup, which generalize the result described in [8]. However, it is somehow confused that the construction of abundant semigroups with a medial idempotent is based on a medial idempotent. Guo [1] called an idempotent u of an abundant semigroup S is a *weak medial idempotent* if for any idempotent x of S, $xux = x$ and then a weak medial is called a *weak normal idempotent* if uSu is an adequate semigroup. Furthermore, he constructed an abundant semigroup with a weak normal idempotent by a left normal bands and a right normal bands and a adequate semigroup.

In this paper, the generalization of a medial idempotent is introduced, named a quasi-medial idempotent. After some preliminary results, an abundant semigroup with a quasi-medial idempotent is constructed without the embarrassment confronted in [10].

The reader is referred to [4], [5], [6], and [7] for all the notation and terminology not defined in this paper.

We shall list some basic results which are used in the sequel. The following lemma is due to Fountain [4] which providing us an alternative description for $\mathcal{L}^*$ ($\mathcal{R}^*$).

Lemma 1.1. (see [4]) *Let S be a semigroup and $a, b \in S$. Then the following conditions are equivalent:*

(1) $a\ \mathcal{L}^*\ b$ ($a\ \mathcal{R}^*\ b$);

(2) For all $x, y \in S^1$, $ax = ay$ ($xa = ya$) $\Leftrightarrow$ $bx = by$ ($xb = yb$).

The condition (2) in Lemma 1.1 is somewhat simplified when one of the elements concerned is an idempotent.

Corollary 1.2. (see [4]) *Let S be a semigroup, $a \in S$ and e be an idempotent of S. Then the following conditions are equivalent:*

(1) $a\ \mathcal{L}^*\ e$ ($a\ \mathcal{R}^*\ e$);

(2) $a = ae$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) $\Rightarrow ex = ey$ ($xe = ye$).

Let S be an abundant semigroup and $a \in S$. Denote by a^* (a^+) a typical idempotent in $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of S containing a.

Lemma 1.3. (see [1]) *Let S be abundant semigroup and u be a weak medial idempotent for S. Then for any $a \in S, e \in E$*

(1) $ue, eu, ueu \in E$;

(2) $a^+u\ \mathcal{R}^*\ a\ \mathcal{L}^*\ ua^*$;

(3) $ua^+u\ \mathcal{R}^*\ uau\mathcal{L}^*\ ua^*u$.

Definition 1.4. Let S be a semigroup with the set of idempotents E. An idempotent u of S is said to be quasi-medial idempotent if for any $e \in E$, $eue = e$ and uEu is a band.

Recall from [10] that if u is a medial idempotent then uEu is a band. It means a quasi-medial idempotent is also a medial idempotent.

In what follows, we always suppose that S is an abundant semigroup with the set of idempotents E without mention.

Lemma 1.5. *Let u be a weak medial idempotent for S. Then for any $x \in S, e \in E$,*
(1) $ux^+u \ \mathcal{L} \ x^+u, ux^*u \ \mathcal{R} \ ux^*$;
(2) $x^+u \ \mathcal{R}^* \ xu, \ ux\mathcal{L}^* \ ux^*$.

Proof. (1) As u is a weak medial idempotent, from $ux^+u(x^+u) = u(x^+ux^+)u = ux^+u$ and $(x^+u)ux^+u = (x^+ux^+)u = x^+u$ follows that $ux^+u \ \mathcal{L} \ x^+u$. Dually, $ux^*u \ \mathcal{R} \ ux^*$.

(2) Obviously, $x^+uxu = xu$. Let $a, b \in S$ be such that $axu = bxu$. Notice that $axu = ax^+uxu$ and $bxu = bx^+uxu$. From $uxu \ \mathcal{R}^* \ ux^+u$ and $uyu \ \mathcal{R}^* \ uy^+u$ follows that $ax^+ux^+u = bx^+ux^+u$. It means $ax^+u = bx^+u$. Hence by Corollary 1.2, $x^+u \ \mathcal{R}^* \ xu$. As a dual, $ux \ \mathcal{L}^* \ ux^*$. □

Let U be an abundant subsemigroup of S. U is called a left (right) $*$-subsemigroup if for all $a \in U$, there exists $e \in U \cap E$ such that $a\mathcal{L}^*(S)e(a \ \mathcal{R}^*(S) \ e)$. If U is both a left and a right $*$-subsemigroup, then we call it a $*$-subsemigroup.

The following two Lemmas describe the relationship between a quasi-medial idempotent with *an abundant transversal* (see[6]).

Lemma 1.6. *Let u be a quasi-medial idempotent for S. For any $x \in S$,*
(1) uSu is a $*$-subsemigroup with a band uEu
(2) $uxu \in C_{uSu}(x)$;
(3) $x^+u \in I_x$ and $ux^* \in \Lambda_x$;
(4) Eu and uE are bands;
(5) $I = Eu$ and $\Lambda = uE$.

Proof. (1) Obviously, u is a weak medial idempotent. Then by Lemma 1.3, uSu is an abundant $*$-subsemigroup of S. It is trivial to check that $E(uSu) = uEu$.

(2), (3) By Lemma 1.5, $x = (x^+u)uxu(ux^*)$ and

$$x^+u \ \mathcal{L} \ ux^+u \ \mathcal{R}^* \ uxu \text{ and } ux^* \ \mathcal{R} \ ux^*u \ \mathcal{L}^* \ uxu.$$

Hence $uxu \in C_{uSu}(x)$ and $x^+u \in I_x$, $ux^* \in \Lambda_x$.

(4) By Lemma 1.3, $uE(Eu)$ is a set of idempotents. Let $s, t \in uE$. Then $(st)^2 = (usut)^2 = usutusut = (usutu)^2t = usutut = st$. Hence $uE(Eu)$ is a band.

(5) For any $g \in I$, there exists $x \in S$ and $x^\circ \in C_{uSu}(x)$ such that $g \mathcal{L} x^{\circ+}$ for some $x^{\circ+} \in uEu$. It follows that $g = gx^{\circ+} = gx^{\circ+}u = gu \in Eu$. On the other hand, let $h \in Eu$. Then $h = hu \mathcal{L} uhu \in uEu$. It means $h \in I_h \subseteq I$. Hence $I = Eu$. The remained proof is a dual. □

Lemma 1.7. *For any $x \in S$, $i \in I_x$ and $\lambda \in \Lambda_x$, there exists unique $s \in C_{uSu}(x)$ such that $ui \mathcal{R}^* s \mathcal{L}^* \lambda u$.*

Proof. By Lemma 1.6, $uxu \in C_{uSu}(x)$. As $i \mathcal{R}^* x$ and $iu = u$, $ui = uiu \mathcal{R}^* uxu$. Dully, $\lambda u = u\lambda u \mathcal{L}^* uxu$. On the other hand, let $t \in C_{uSu}(x)$ be such that $ui \mathcal{R}^* t \mathcal{L}^* \lambda u$. Then there exist $e \in I_x, f \in \Lambda_x$ such that $e \mathcal{L} t^+, f \mathcal{R} t^*$ for some $t^+, t^* \in uEu$ and $x = etf$. It follows that $t = t^+xt^* = (t^+u)x(ut^*) = t^+(uiu)(uxu)(u\lambda u)t^* = uiuxu\lambda u = uxu$. □

Corollary 1.8. *Let u be a quasi-medial idempotent for S. Then uSu is an abundant transversal for S.*

2. A Structure Theorem

The objective of this section is to obtain a construction for all abundant semigroups with a quasi-medial idempotent. It will be seen that the "building bricks" are two bands and quasi-adequate semigroup with an identity.

Let S° be a quasi-adequate semigroup with the band of idempotents E° and let I and Λ be bands such that $I \cap \Lambda = E^\circ$. The triple (I, S°, Λ) is said to be *strong permissible* if

$(SP1)$ E° is a right-ideal and a left-ideal for I and Λ respectively;

$(SP2)$ There exists $u \in E^\circ$ such that $Iu = I$ and $u\Lambda = \Lambda$.

A mapping $*$: $\Lambda \times I \to S^\circ$ with $(\lambda, i) \mapsto \lambda * i$ is called *quasi-perfect* if it satisfies the following conditions:

$(N1)$ $(\forall e \in E^\circ)$ $e(\lambda * i) = (e\lambda) * i$;

$(N2)$ $(\forall f \in E^\circ)(\lambda * i)f = \lambda * (if)$;

$(N3)$ $(\forall \lambda \in E^\circ$ or $\forall i \in E^\circ)$ $(\lambda * i) = \lambda i$;

$(N4)$ For any $s^\circ \in S^\circ$ and $\lambda \in \Lambda, i \in I$, if $s^\circ(\lambda * i)s^\circ = s^\circ$ and $ui \mathcal{R} s^\circ \mathcal{L} \lambda u$ then $s^\circ \in E^\circ$.

A strong permissible triple (I, S°, Λ) with a quasi-normal mapping $* : \Lambda \times I \to S^\circ$ is called a QM-system and is denoted by $(I, S^\circ, \Lambda, *)$. In this case, let

$$QM = QM(I, S^\circ, \Lambda, *) = \{(R_i, x, L_\lambda) : \; ui \, \mathcal{R}^* \, x \, \mathcal{L}^* \, \lambda u\}$$

be a subset of $I/\mathcal{R} \times S^\circ \times \Lambda/\mathcal{L}$.

Define a multiplication on QM by:

$$(R_i, x, L_\lambda)(R_j, y, L_\mu) = (R_{ia^+}, a, L_{a^*\mu}),$$

where $a = x(\lambda * j)y$.

Lemma 2.1. *The multiplication on QM is well-defined.*

Proof. Let $(R_i, x, L_\lambda), (R_j, y, L_\mu) \in QM$. As $uix(\lambda*j)y = x(\lambda*j)y$, $uia^+ = a^+$. Then $uia^+ \, \mathcal{R}^* \, a$. Dually, $a^*\mu u \, \mathcal{L}^* \, a$. It meas $(R_{ia^+}, a, L_{a^*\mu}) \in QM$ for any $a^+, a^* \in E^\circ$.

Now we prove that the multiplication on QM dose not depend on the choice of λ, j. Let $\sigma \in \Lambda, t \in I$ be such that $\lambda \, \mathcal{L} \, \sigma$ and $j \, \mathcal{R} \, t$. Then

$$x^*\lambda = x^*\lambda\sigma = x^*\lambda u\sigma = x^*\sigma$$

and

$$jy^+ = jy^+ = tjy^+ = tujy^+ = ty^+.$$

It follows that $x^*(\lambda * j)y^+ = (x^*\lambda) * (jy^+) = (x^*\sigma) * (ty^+) = x^*(\sigma * t)y^+$. Hence $x(\lambda * j)y = x(\sigma * t)y$. □

Lemma 2.2. *QM is a semigroup.*

Proof. Let $a = (R_i, x, L_\lambda), b = (R_j, y, L_\mu), c = (R_s, z, L_\sigma) \in QM$. Then

$$(ab)c = (R_{im^+}, m, L_{m^*\mu})(R_s, z, L_\sigma) = (R_{im^+n^+}, n, L_{n^*\sigma})$$

and

$$a(bc) = (R_i, x, L_\lambda)(R_{jp^+}, p, L_{p^*\sigma}) = (R_{iq^+}, q, L_{q^*p^*\sigma}),$$

where $m = x(\lambda * j)y$, $n = m((m^*\mu) * s)z$, $p = y(\mu * s)z$, $q = x(\lambda * (jp^+))p$ and $m^+, m^*, n^+, n^*, p^+, p^*, q^+, q^* \in E^\circ$. Hence $n = m(\mu*s)z = x(\lambda*j)y(\mu*s)z = q$. It follows that $m^+n^+ = n^+$ and $q^*p^* = q^*$. Therefore $(ab)c = a(bc)$ and Q is a semigroup. □

Proposition 2.3. *QM is an abundant semigroup with the set of idempotents*

$$E(QM) = \{(R_i, x, L_\lambda) \in Q \mid x(\lambda * i)x = x \; \}.$$

Proof. It is trivial to check that

$$E(QM) = \{(R_i, x, L_\lambda) \in Q \mid x(\lambda * i)x = x \}.$$

Let $(R_i, x, L_\lambda) \in QM$. Then $(R_i, ui, L_{ui}) \in E(QM)$ and $(R_i, ui, L_{ui})(R_i, x, L_\lambda) = (R_i, x, L_\lambda)$ since $ui \in E^\circ$. If $(R_j, y, L_\tau)(R_i, x, L_\lambda) = (R_k, z, L_\mu)(R_i, x, L_\lambda)$, then $y(\tau * i)x = z(\mu * i)x$. From $ui\ \mathcal{R}^*\ x$ follows that $y(\tau * i)ui = z(\mu * i)ui$, which together with $y(\tau * i)x\ \mathcal{R}^*\ y(\tau * i)ui$ and $z(\mu * i)x\ \mathcal{R}^*\ z(\mu * i)ui$ implies that $(R_j, y, L_\tau)(R_i, ui, L_{ui}) = (R_k, z, L_\mu)(R_i, ui, L_{ui})$. Hence by Lemma 1.2, $(R_i, x, L_\lambda)\ \mathcal{R}^*\ (R_i, ui, L_{ui})$. Similarly, we have $(R_{\lambda u}, \lambda u, L_\lambda) \in E(QM)$ and $(R_i, x, L_\lambda)\ \mathcal{L}^*\ (R_{\lambda u}, \lambda u, L_\lambda)$. Therefore QM is an abundant semigroup. □

Proposition 2.4. *(1) $(R_u, u, L_u)QM(R_u, u, L_u) \cong S^\circ$ and*

$$QM^\circ = (R_u, u, L_u)QM(R_u, u, L_u) = \{(R_{x^+}, x, L_{x^*}) \in QM | x \in S^\circ,\ x^+, x^* \in E^\circ\};$$

(2) (R_u, u, L_u) is a quasi-medial idempotent for QM.

Proof. (1) It is easy to check that

$$QM^\circ = (R_u, u, L_u)QM(R_u, u, L_u) = \{(R_{x^+}, x, L_{x^*}) \in QM |\, x^+, x^* \in E^\circ\}.$$

For convenience, denote (R_u, u, L_u) by $\bar{u}$. Define a mapping ϕ from $\bar{u}Q\bar{u}$ to S° by: for $(R_i, x, L_\lambda) \in QM$,

$$[\bar{u}(R_i, x, L_\lambda)\bar{u}]\phi = (ui)x(\lambda u) = x.$$

Obviously, the map is well defined. Let $[\bar{u}(R_i, x, L_\lambda)\bar{u}]\phi = [\bar{u}(R_j, y, L_\tau)\bar{u}]\phi$. Then $x = y$ and $\bar{u}(R_i, x, L_\lambda)\bar{u} = \bar{u}(R_j, y, L_\tau)\bar{u}$. It means ϕ is injective. Let $s \in S^\circ$ and $s^+ \in E^\circ$. Then $[\bar{u}(R_{s^+}, s^+, L_{s^+})\bar{u}]\phi = us^+ss^*u = s$. It means ϕ is surjective. Next we shall show that ϕ is a homomorphism. Let $(R_i, x, L_\lambda), (R_j, y, L_\tau) \in QM$. Then

$$\bar{u}(R_i, x, L_\lambda)\bar{u} = (R_{x^+}, x, L_{x^*})$$

and

$$\bar{u}(R_j, y, L_\tau)\bar{u} = (R_{y^+}, y, L_{y^*}).$$

It follows that

$$\bar{u}(R_i, x, L_\lambda)\bar{u}\bar{u}(R_j, y, L_\tau)\bar{u} = \bar{u}(R_{(xy)^+}, xy, L_{(xy)^*})\bar{u}$$

Hence

$$[\bar{u}(R_i, x, L_\lambda)\bar{u}\bar{u}(R_j, y, L_\tau)\bar{u}]\phi = xy = [\bar{u}(R_i, x, L_\lambda)\bar{u}]\phi[\bar{u}(R_j, y, L_\tau)\bar{u}]\phi.$$

Therefore $(R_u, u, L_u)QM(R_u, u, L_u) \cong S^\circ$.

(2) Let $e = (R_i, x, L_\lambda) \in E(QM)$. By (N4), $x \in E^\circ$. Then

$$\begin{aligned} e\bar{u}e &= (R_i, x, L_\lambda)(R_u, u, L_u)(R_i, x, L_\lambda) \\ &= (R_{i(x\lambda u)^+}, x\lambda u, L_{(x\lambda u)^* u})(R_i, x, L_\lambda) \\ &= (R_{i(x\lambda u)^+(x\lambda uix)^+}, x\lambda uix, L_{(x\lambda uix)^*(x\lambda u)^* u}) \\ &= (R_{i(x\lambda uix)^+}, x\lambda uix, L_{(x\lambda uix)^* u}) \\ &= (R_{ix^+}, x, L_{x^*\lambda}) = (R_i, x, L_\lambda) = e. \end{aligned}$$

It means $\bar{u}$ is a weak medial idempotent for QM. Hence $\bar{u}E(QM)\bar{u} \subseteq E(QM^\circ)$. On the other hand, if $(R_{x^+}, x, L_{x^*}) \in E(QM^\circ)$ then

$$\bar{u}(R_{x^+}, x, L_{x^*})\bar{u} = (R_{x^+}, x, L_{x^*}),$$

i.e, $E(QM^\circ) \subseteq \bar{u}E(QM)\bar{u}$. From (1) follows that $\bar{u}E(QM)\bar{u} = E(QM^\circ)$ is a band. Therefore (R_u, u, L_u) is a quasi-medial idempotent for QM. □

Theorem 2.5. *Let $(I, S^\circ, \Lambda, *)$ be an QM-system. Then $QM(I, S^\circ, \Lambda, *)$ is an abundant semigroup with a quasi-medial idempotent (R_u, u, L_u). Conversely, any such abundant semigroup can be constructed in this way.*

Proof. ($\Rightarrow$) By Proposition 2.3 and 2.4.

($\Leftarrow$) Let S be an abundant semigroup with a quasi-medial idempotent u. Then $I = Eu, \Lambda = uE, Eu \cap uE = uEu = E(uSu)$ and uSu is a quasi-adequate semigroup. Obviously, $u \in uEu$, $Euu = Eu$, $uuE = uE$ and $uEuEu \subseteq uEu$. Hence (Eu, uSu, uE) is strong permissible. Define a mapping from $\Lambda \times I$ to S° by

$$\begin{aligned} * : \quad uE \times Eu &\to uSu \\ (\lambda, i) &\mapsto \lambda i \end{aligned}$$

If $s^\circ \lambda i s^\circ = s^\circ$ and $ui \ \mathcal{R}^* \ s^\circ \ \mathcal{L}^* \ \lambda u$, then $e = is^\circ\lambda = is^\circ\lambda is^\circ\lambda = e^2$. Since $i \ \mathcal{L} \ ui$ and $\lambda \ \mathcal{R} \ \lambda u$, $s^\circ \in C_{uSu}(e)$. By Lemma 1.7, $s^\circ = ueu \in uEu$. It is clear that $*$ is a quasi-perfect mapping..

Next we prove that S is isomorphic to $QM = QM(Eu, uSu, uE, *)$. As u is a quasi-medial idempotent, for any $x \in S, x^+, x^* \in E$, $ux^+u \ \mathcal{R}^* \ uxu \ \mathcal{L}^* \ ux^*u$. Define a mapping θ by:

$$\begin{aligned} \theta : \quad & S \to QM \\ & x \mapsto (R_{x^+u}, uxu, L_{ux^*}). \end{aligned}$$

Let $(R_i, s, L_\lambda) \in QM$. For convenience, take $x = is\lambda$. As $i \ \mathcal{L} \ ui \ \mathcal{R}^* \ s \ \mathcal{L}^* \ u\lambda \ \mathcal{R} \ \lambda$, $i \ \mathcal{R}^* \ x \ \mathcal{L}^* \ \lambda$. Then $x\theta = (R_i, s, L_\lambda)$. It means θ is surjective. Suppose that

$x\theta = y\theta$. Then $(R_{x^+u}, uxu, L_{ux^*}) = (R_{y^+u}, uyu, L_{yx^*})$. It follows that

$$\begin{aligned} x &= (x^+u)uxu(ux^*) = (x^+u)uyu(ux^*) \\ &= (x^+u)(y^+u)uyu(uy^*)(ux^*) \\ &= (y^+u)uyu(uy^*) = y. \end{aligned}$$

It means θ is injective. For any $x, y \in S$ and $x^*, y^+ \in E$, $uxu(ux^*y^+u)uyu = ux(x^*uux^*)(y^+uuy^+)yu = uxyu$. Hence for any $(uxyu)^+, (uxyu)^* \in uEu$,

$$\begin{aligned} x\theta y\theta &= (R_{x^+u}, uxu, L_{ux^*})(R_{y^+u}, uyu, L_{uy^*}) \\ &= (R_{x^+u(uxyu)^+}, uxyu, L_{(uxyu)^*ux^*}) \\ &= (R_{(uxyu)^+}, uxyu, L_{(uxyu)^*}) \\ &= (R_{(xy)^+u}, uxyu, L_{u(xy)^*}) \\ &= (xy)\theta. \end{aligned}$$

Therefore $QM(I, S^\circ, \Lambda, *) \cong S$. □

3. Applications to Some Special Cases

In this section, it shall be concerned with some special cases of structure theorem on abundant semigroups with any other kinds of medial idempotents. In particular, we shall describe a method of constructing all abundant semigroups with a medial idempotent, which generalize the results obtained in [9] and [10].

3.1. Medial Case

Lemma 3.1. *Let u be a quasi-medial idempotent for S. Then u is a medial idempotent if and only if $u\overline{E}u = uEu$ where $\overline{E}$ is a regular semigroup generated by E.*

Proof. ($\Rightarrow$) Obviously.

($\Leftarrow$) For any $x \in S$, $xu \mathcal{R}^* x$ since $x = (xu)ux^*$. Suppose that $s \in \overline{E}$. As uE is a band, $su = s^+u(usu)us^*u \in Eu$. From $s \mathcal{R}^* su \in E$ follows that $sus = s$. Therefore u is a medial idempotent. □

A quasi-perfect $* : \Lambda \times I \to S^\circ$ is a *normal mapping*(see [6]) if the range of $*$ is a subset of E°. Since $\lambda * i \in E^\circ$, (N4) is a natural result. For convenience, A strong permissible triple (I, S°, Λ) with a normal mapping $*$ is called a *SQM*-system.

Lemma 3.2. *Let $(I, S^\circ, \Lambda, *)$ be an SQM-system.*

(1) If $\overline{E(QM)}$ is the subsemigroup of QM generated by $E(QM)$, then

$$\overline{E(QM)} = \{(R_i, x, L_\lambda) \in Q \mid x \in E^\circ\};$$

(2) $(R_u, u, L_u)\overline{E(QM)}(R_u, u, L_u) = (R_u, u, L_u)E(QM)(R_u, u, L_u)$;

(3) $QM(I, S^\circ, \Lambda, *)$ is an abundant semigroup with a medial idempotent (R_u, u, L_u).

Proof. (1) Suppose $(R_i, x, L_\lambda), (R_j, y, L_\tau) \in E(QM)$. Since $\lambda * i \in E^\circ$ and $x(\lambda * i)x = x$, $x(\lambda * i), (\lambda * i)x \in E^\circ$. Then $x \in E^\circ$. Similarly, $y \in E^\circ$. As $x(\lambda * j)y \in E^\circ$,

$$(R_i, x, L_\lambda)(R_j, y, L_\tau) \in \{(R_i, x, L_\lambda) \in QM \mid x \in E^\circ\}.$$

Conversely, let $(R_i, x, L_\lambda) \in QM$ and $x \in E^\circ$. Then $(R_x, x, L_\lambda), (R_i, x, L_x) \in E(QM)$ and $(R_i, x, L_\lambda) = (R_i, x, L_x)(R_x, x, L_\lambda) \in \overline{E(QM)}$.

(2) Let $(R_i, x, L_\lambda) \in \overline{E(QM)}$. From $(ui)x(\lambda u) = x$ follows that

$$(R_u, u, L_u)(R_i, x, L_\lambda)(R_u, u, L_u) = (R_u, u, L_u)(R_x, x, L_x)(R_u, u, L_u).$$

Hence $(R_u, u, L_u)\overline{E(QM)}(R_u, u, L_u) = (R_u, u, L_u)E(QM)(R_u, u, L_u)$.

(3) Follows from Theorem 2.5 and Lemma 3.1. □

Theorem 3.3. *Let $(I, S^\circ, \Lambda, *)$ be an SQM-system. Then $QM(I, S^\circ, \Lambda, *)$ is an abundant semigroup with a medial idempotent (R_u, u, L_u). Conversely, any such abundant semigroup can be constructed in this way.*

Proof. By Theorem 2.5 and Lemma 3.2. □

3.2. Weak Normal Medial Case

Lemma 3.4. *Let $(I, S^\circ, \Lambda, *)$ be an QM-system. If S° is adequate, then*

$$QM = QM(I, S^\circ, \Lambda, *) = \{(i, x, \lambda) \mid ui\ \mathcal{R}^*\ x\ \mathcal{L}^*\ \lambda u.\}$$

Proof. Let $(R_i, x, L_\lambda) \in QM$ and $R_i = R_j$. Then $ui\ \mathcal{R}^*\ uj$. Since S° is adequate, $ui = ui$ and $i\ \mathcal{L}\ ui = ui\ \mathcal{L}\ j$. Hence $i = j$. Dully, if $L_\lambda = L_\tau$ then $\lambda = \tau$. □

Lemma 3.5. *Let $(I, S^\circ, \Lambda, *)$ be an QM-system. Then (u, u, u) is a weak normal idempotent if and only if S° is adequate.*

Proof. By Proposition 2.4. □

Theorem 3.6. *Let $(I, S^\circ, \Lambda, *)$ be an SQM-system and S° be adequate. Then $QM(I, S^\circ, \Lambda, *)$ is an abundant semigroup with a weak normal medial idempotent (u, u, u). Conversely, any such abundant semigroup can be constructed in this way.*

3.3. Regular Case

Lemma 3.7. *Let $(I, S^\circ, \Lambda, *)$ be a QM-system. Then $QM(I, S^\circ, \Lambda, *)$ is regular if and only if S° is regular*

Proof. $(\Rightarrow)$ Let $s \in S^\circ$. For any $s^+, s^* \in E^\circ$, there exists $(R_i, t, L_\lambda) \in QM$ such that

$$(R_{s^+}, s, L_{s^*})(R_i, t, L_\lambda)(R_{s^+}, s, L_{s^*}) = (R_{s^+}, s, L_{s^*}).$$

Obviously, $s = s(s^* * i)t(\lambda * s^+)s$. Hence S° is regular.

$(\Rightarrow)$ Suppose that S° is regular. For any $(R_i, s, L_\lambda) \in QM$. Let $t \in S^\circ$ be such that $s = sts$. Then for any $t^+, t^* \in E^\circ$,

$$\begin{aligned} & (R_i, s, L_\lambda)(R_{t^+}, t, L_{i^*})(R_i, s, L_\lambda) \\ = \; & (R_{i(sts)^+}, sts, L_{(sts)^*\lambda}) = (R_{is^+}, s, L_{s^*\lambda}) \\ = \; & (R_i, s, L_\lambda) \end{aligned}$$

Therefore QM is regular. □

Theorem 3.8. *Let $(I, S^\circ, \Lambda, *)$ be a QM-system. If S° is regular, then QM is a regular semigroup with a quasi-medial idempotent (R_u, u, L_u). Conversely, any such regular semigroup can be constructed in this way.*

Theorem 3.9. *Let $(I, S^\circ, \Lambda, *)$ be a SQM-system. If S° is regular, then QM is a regular semigroup with a medial idempotent (R_u, u, L_u). Conversely, any such regular semigroup can be constructed in this way.*

Theorem 3.10. *Let $(I, S^\circ, \Lambda, *)$ be a QM-system. If S° is inverse, then QM is a regular semigroup with a weak normal medial idempotent (u, u, u). Conversely, any such regular semigroup can be constructed in this way.*

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