International Journal of Pure and Applied Mathematics
Volume 72 No. 2 2011, 249-258

ON PSEUDO M-PROJECTIVE RICCI SYMMETRIC MANIFOLDS

Füsun Özen Zengın

Department of Mathematics
Faculty of Sciences and Letters
Istanbul Technical University
Maslak, 34469, Istanbul, Turkey

Abstract: The object of the present paper is to study pseudo M-projective Ricci symmetric manifolds denoted by $(PMRS)_n$. Several properties of $(PMRS)_n$ are established and it is proved that if the scalar curvature is constant then $(n+1-r)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P given by $g(X,P) = A(X)$. In the section 3, assuming that the manifold $(PMRS)_n$ is conformally flat, it is shown that if the M-projective Ricci tensor of this manifold is Codazzi type then this manifold becomes a quasi-Einstein manifold. In addition, it is proved that if P is a torse-forming vector field with constant energy then P must be a concircular.

AMS Subject Classification: 53B05, 53B15
Key Words: pseudo Ricci symmetric manifold, M-projective Ricci tensor, codazzi tensor, cyclic Ricci tensor, quadratic conformal Killing tensor, torse-forming vector field, concircular vector field

1. Introduction

A Riemannian manifold is Ricci symmetric if its Ricci tensor S of type $(0,2)$ satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. The notion of Ricci symmetric manifolds have been weakened by many authors in the last five decades such as Ricci-recurrent manifolds [6], Ricci semi-symmetric manifolds [11], pseudo Ricci symmetric manifolds [1].

Received: October 6, 2011

A non-flat Riemannian manifold (M_n, g) is said to be pseudo Ricci symmetric if the Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{1.1}$$

where A is a non-zero 1-form,

$$g(X,P) = A(X), \tag{1.2}$$

for every vector field P and ∇ denotes the operator of covariant differentiation with respect to the metric g. This manifold is denoted by $(PRS)_n$. The name "Pseudo Ricci symmetric" is chosen, because if the 1-form A in (1.1) is taken as zero, then the equation assumes the form $(\nabla_X S)(Y,Z) = 0$ and the manifold thus becomes Ricci symmetric.

Let (M_n, g) be an n-dimensional differentiable manifold of class C^∞ with the metric tensor g and the Riemannian connection ∇. The M-projective curvature tensor of the manifold defined by G.P. Pokhariyal and R.S. Mishra in 1971, [7], is the following form

$$\begin{aligned} M(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.3}$$

where $R(X,Y)Z$ and $S(X,Y)$ are the curvature tensor and the Ricci tensor of M_n, respectively. Some authors studied the properties of this tensor, [2], [4], [5].

In this paper, extending the notion of pseudo Ricci symmetric manifold, the author introduces a type of non-flat Riemannian manifold (M_n, g), $(n > 3)$ whose M-projective Ricci tensor $\overline{M}$ of type $(0,2)$ satisfies the condition

$$(\nabla_X \overline{M})(Y,Z) = 2A(X)\overline{M}(Y,Z) + A(Y)\overline{M}(X,Z) + A(Z)\overline{M}(Y,X), \tag{1.4}$$

where A and ∇ have the meanings already mentioned. Such a manifold shall be called a pseudo M-projective Ricci symmetric manifold. A shall be its associated 1-form and an n-dimensional manifold of this kind shall be denoted by $(PMRS)_n$.

The object of the present paper is to study $(PMRS)_n$. The paper is organized as follows:

Section 2 is devoted to the study of some properties of $(PMRS)_n$. It is shown that in a $(PMRS)_n$, $(n+1-r)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if the scalar curvature of this manifold is constant. After, it is proved that in a $(PMRS)_n$, $(r-n+1)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if the M-projective Ricci tensor of this manifold is Codazzi type. In addition, if the pseudo M-projective Ricci tensor is cyclic then it is shown that $(PMRS)_n$ becomes an Einstein manifold.

Section 3 deals with the study of conformally flat $(PMRS)_n$. It is proved that in this manifold if the M-projective Ricci tensor is Codazzi type then this manifold becomes a quasi-Einstein manifold. Moreover, in a conformally flat $(PMRS)_n$, if we choose the vector field P as a torse-forming vector field given by $\nabla_X P = aX + w(X)P$ for all vector fields X, where a is a non-zero scalar and w is a 1-form, then the scalar a must be equal $-A(P)$. In the end of this section, it is shown that if the energy of the torse-forming vector field P is constant then the vector field P is concircular.

2. Pseudo M-Projective Ricci Symmetric Manifolds

In this section, if we consider the M-projective curvature tensor given by (1.3), we obtain the M-projective Ricci tensor $\overline{M}$ of type (0,2) as

$$\overline{M}(X,Y) = \frac{1}{2}(S(X,Y) - g(X,Y)). \tag{2.1}$$

We assume that our manifold (M_n, g) is $(PMRS)_n$ then we have the form (1.4).

Theorem 2.1. *In a $(PMRS)_n$, $(n+1-r)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if the scalar curvature of this manifold is constant.*

Proof. By taking the covariant derivative of (2.1) and using the equations (1.4) and (2.1), we find

$$\begin{aligned}(\nabla_X S)(Y,Z) = 2A(X)(S(Y,Z) - g(Y,Z)) + A(Y)(S(X,Z) - g(X,Z))\\ + A(Z)(S(Y,X) - g(Y,X)).\end{aligned} \tag{2.2}$$

Contracting the equation (2.2) on Y and Z, we get

$$A(QX) = (n+1-r)A(X) + \frac{1}{2}dr(X). \tag{2.3}$$

If the scalar curvature of this manifold is constant then by the aid of (2.3), it is seen that

$$A(QX) = (n+1-r)A(X). \tag{2.4}$$

Thus, the proof is completed. □

Theorem 2.2. *In a $(PMRS)_n$, $(r-n+1)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if the pseudo M-projective Ricci tensor of this manifold is Codazzi type.*

Proof. In $(PMRS)_n$, if we assume that the M-projective Ricci tensor $\overline{M}$ is Codazzi type, we have

$$(\nabla_X \overline{M})(Y,Z) - (\nabla_Z \overline{M})(Y,X) = 0. \tag{2.5}$$

With the help of (1.4) and (2.5), we find

$$A(X)\overline{M}(Y,Z) - A(Z)\overline{M}(Y,X) = 0. \tag{2.6}$$

Contracting on Y and Z and putting (2.1) in the equation (2.6), we get

$$A(QX) = (r-n+1)A(X). \tag{2.7}$$

This completes the proof. □

Corollary 2.1. *In a $(PMRS)_n$, if the scalar curvature is constant and the pseudo M-projective Ricci tensor is Codazzi type then it must be $r = n$.*

Proof. The result is clear from Theorem 2.1 and Theorem 2.2. □

Definition 2.1. A Riemannian manifold (M_n, g), $(n > 2)$ is said to be cyclic Ricci symmetric if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following, [9]

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{2.8}$$

Theorem 2.3. *For a $(PMRS)_n$, if the pseudo M-projective Ricci tensor is cyclic then this manifold must be an Einstein manifold.*

Proof. By assuming the pseudo M-projective Ricci tensor $\overline{M}$ is cyclic, from (1.4) and (2.8), we can obtain

$$A(X)\overline{M}(Y,Z) + A(Y)\overline{M}(X,Z) + A(Z)\overline{M}(Y,X) = 0. \tag{2.9}$$

In Walker's Lemma, [13], it is said that if $a(X,Y)$ and $b(X)$ are the numbers satisfying $a(X,Y) = a(Y,X)$ and

$$b(X)a(Y,Z) + b(Y)a(Z,X) + b(Z)a(X,Y) = 0, \tag{2.10}$$

for all X,Y,Z. Then either all the $a(X,Y)$ are zero or all the $b(X)$ are zero. Hence, by the above Lemma, from (2.9) and (2.10), it must be either $A(X) = 0$ or $\overline{M}(Y,Z) = 0$. Since M_n is pseudo M-projective Ricci symmetric manifold then $A(X) \neq 0$. In this case, we find $\overline{M}(Y,Z) = 0$. Thus, by using (2.1), we finally get that $S(Y,Z) = g(Y,Z)$. This completes the proof. □

Definition 2.2. A quadratic conformal Killing tensor is defined as a second order symmetric tensor T satisfying the condition, [10]-[13], [14]

$$\begin{aligned}(\nabla_X T)(Y,Z)+(\nabla_Y T)(Z,X) &+ (\nabla_Z T)(X,Y)=\lambda(X)g(Y,Z) \\ &+ \lambda(Y)g(Z,X)+\lambda(Z)g(X,Y).\end{aligned} \quad (2.11)$$

for a smooth 1-form λ on M_n. The above equation is equivalent to the requirement that $T(l,l)$ be constant along null geodesic with parallely propagated tangent vector.

Theorem 2.4. *In a $(PMRS)_n$ with constant scalar curvature, if the M-projective Ricci tensor is a quadratic conformal Killing tensor then this manifold must be an Einstein manifold.*

Proof. We suppose that, in a $(PMRS)_n$, the M-projective Ricci tensor is a quadratic conformal Killing tensor then from (2.11)

$$\begin{aligned}(\nabla_X \overline{M})(Y,Z)+(\nabla_Y \overline{M})(Z,X) &+ (\nabla_Z \overline{M})(X,Y)=\lambda(X)g(Y,Z) \\ &+ \lambda(Y)g(Z,X)+\lambda(Z)g(X,Y).\end{aligned} \quad (2.12)$$

By putting the equation (1.4) in (2.12) and contracting the last equation on Y and Z, from (2.1), we can easily seen that

$$2A(X)(r-n)+4A(QX)-4A(X)=(n+2)\lambda(X), \quad (2.13)$$

Now, let us suppose that the scalar curvature of this manifold is constant. Thus, from (2.4) the equation (2.13) reduces to

$$A(X)=\frac{(n+2)}{2(n-r)}\lambda(X), \qquad (r\neq n) \quad (2.14)$$

If the equation (2.14) puts in the equation (2.12) and use the equations (1.4) and (2.1), we find, from Walker's Lemma, [13],

$$either \quad \lambda(X)=0 \quad or \quad \left(\frac{n+2}{n-r}S(Y,Z)-\left(\frac{2n+2-r}{n-r}\right)g(Y,Z)\right)=0 \quad (2.15)$$

where $(r\neq n)$. Since $\overline{M}$ is a quadratic conformal Killing tensor,i.e., $\lambda(X)\neq 0$ then we obtain

$$S(Y,Z)=\frac{2n+2-r}{n+2}g(Y,Z), \qquad r\neq 2(n+1) \quad (2.16)$$

The proof is completed. □

Corollary 2.2. *In a $(PMRS)_n$ with constant scalar curvature, if the M-projective Ricci tensor is a quadratic conformal Killing tensor and $r = 2(n+1)$ then this manifold is Ricci-flat.*

Proof. From (2.16), the proof is clear. □

3. Conformally Flat $(PMRS)_n$

In this section, we suppose that our manifold $(PMRS)_n$ is conformally flat.

Let B be the 1-form defined by

$$B(X) = A(QX) \tag{3.1}$$

where Q is the symmetric endomorphism of the tangent space at each point of (M_n, g) corresponding to the Ricci tensor S. Then

$$g(QX, Y) = S(X, Y) \tag{3.2}$$

for all vector fields X and Y.

Theorem 3.1. *In a conformally flat $(PMRS)_n$, if the M-projective Ricci tensor is Codazzi type then this manifold is a quasi-Einstein manifold.*

Proof. In a conformally flat Riemannian manifold, we have, [3]

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = \frac{1}{2(n-1)}(dr(X)g(Y, Z) - dr(Z)g(X, Y)). \tag{3.3}$$

We also have from (1.4)

$$(\nabla_X \overline{M})(Y, Z) - (\nabla_Z \overline{M})(Y, X) = A(X)\overline{M}(Y, Z) - A(Z)\overline{M}(Y, X) \tag{3.4}$$

If we put (2.1) in (3.4), we get

$$\begin{aligned}(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) &= A(X)(S(Y, Z) - g(Y, Z)) \\ &- A(Z)(S(Y, X) - g(Y, X)).\end{aligned} \tag{3.5}$$

Now, contracting (3.5) on Y, Z and by using (1.4), (2.1), (2.7), (3.1) and (3.5), it follows that

$$dr(X) = 2A(X)(r - n - 1) + 2B(X) \tag{3.6}$$

If $\overline{M}(X,Y)$ is Codazzi tensor then Theorem 2.2, we get

$$B(X) = (r - n + 1)A(X) \tag{3.7}$$

Comparing (3.6) with (3.7), we find

$$dr(X) = 4(r - n)A(X) \tag{3.8}$$

By putting (3.8) in (3.3), one can obtain that

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{2(r-n)}{n-1}(A(X)g(Y,Z) - A(Z)g(X,Y)) \tag{3.9}$$

In virtue of (3.5), (3.9) becomes

$$A(X)(S(Y,Z) - \frac{(2r-n-1)}{n-1}g(Y,Z)) \\ = A(Y)(S(X,Z) - \frac{(2r-n-1)}{n-1}g(X,Z)) \tag{3.10}$$

Putting $X = P$ in (3.10) and using (1.2), (3.1) and (3.7), it can be obtained that

$$S(Y,Z) = ag(Y,Z) + bu(Y)u(Z), \qquad (r \neq n \;\; and \;\; r \neq \frac{n+1}{2})$$

where $a = \frac{2r-n-1}{n-1}$, $b = \frac{n(r+3)-n^2-3r}{n-1}$ and $u(Y) = \frac{A(Y)}{\sqrt{A(P)}}$. Thus, the proof is completed. □

Theorem 3.2. *In a conformally flat $(PMRS)_n$, P is a torse-forming vector field given by $\nabla_X P = aX + w(X)P$ then the scalar a must be equal to $-A(P)$.*

Proof. We suppose that our space $(PMRS)_n$ is a conformally flat. After contracting (3.4) on Y and Z, by using (1.4), (2.1) and (3.1), we get

$$dr(X) = 2(r - n - 1)A(X) + 2B(X). \tag{3.11}$$

Since $(PMRS)_n$ is conformally flat, by contracting (3.3) on Y and Z and by using (1.4), (2.1) and (3.1) in the last equation, it follows that

$$dr(X) = 2(r - n - 3)A(X) + 6B(X). \tag{3.12}$$

Comparing (3.11) and (3.12), we get

$$A(X) = B(X) \tag{3.13}$$

and from (3.11) and (3.13), it follows that

$$dr(X) = 2(r-n)A(X) \quad , \qquad (r \neq n) \tag{3.14}$$

In this case, by taking $Y = P$ in (2.1) and by using (1.2), (3.1) and (3.13), we find

$$\overline{M}(X, P) = 0. \tag{3.15}$$

Moreover, we have

$$(\nabla_X \overline{M})(Y, Z) = \nabla_X \overline{M}(Y, Z) - \overline{M}(\nabla_X Y, Z) - \overline{M}(Y, \nabla_X Z) \tag{3.16}$$

By virtue of (1.4) and (3.16), we get

$$\begin{aligned} \nabla_X \overline{M}(Y, Z) - \overline{M}(\nabla_X Y, Z) - \overline{M}(Y, \nabla_X Z) \\ = 2A(X)\overline{M}(Y, Z) + A(Y)\overline{M}(X, Z) + A(Z)\overline{M}(Y, X) \end{aligned} \tag{3.17}$$

If we put $Z = P$ in (3.17), by the aid of (3.15) then we obtain

$$\overline{M}(Y, \nabla_X P) = -A(P)\overline{M}(Y, X). \tag{3.18}$$

Let us now suppose that P is a torse forming vector field,[8]

$$\nabla_X P = aX + w(X)P \tag{3.19}$$

for all vector fields X, where a is a non-zero scalar and w is a 1-form. Thus, from (3.15), (3.18) and (3.19), we get

$$\overline{M}(Y, X)(a + A(P)) = 0. \tag{3.20}$$

Since $\overline{M}(Y, X) \neq 0$, from (3.20), it follows that

$$a = -A(P). \tag{3.21}$$

This completes the proof. □

Theorem 3.3. *In a conformally flat $(PMRS)_n$, the non-zero 1-form A of the manifold is closed.*

Proof. We suppose that our manifold $(PMRS)_n$ is conformally flat. Thus, in virtue of (3.3) and (3.14), we find

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{r-n}{n-1}(A(X)g(Y,Z) - A(Z)g(X,Y)) \qquad (r \neq n). \quad (3.22)$$

By putting $Y = P$ in (3.22) and using (1.2), (3.1), (3.2) and (3.13), it follows that

$$dA(X,Z) = 0. \quad (3.23)$$

In this case, we can say that the non-zero 1-form A is closed. Thus, the proof is completed. □

Theorem 3.4. *In a conformally flat $(PMRS)_n$, if P is a torse-forming vector field such that its energy is constant then the vector field P is concircular.*

Proof. Let

$$f = \frac{1}{2}g(P,P) \quad (3.24)$$

be the energy of the torse-forming vector field P given by (3.19) and let

$$g(\xi, Y) = w(Y) \quad (3.25)$$

for all vector fields Y. By using (1.2), (3.21), (3.24) and (3.25), we obtain

$$df(Y) = g(aP + 2f\xi, Y) \quad (3.26)$$

Thus,

$$gradf = aP + 2f\xi = -A(P).P + A(P)\xi \quad (3.27)$$

If f is constant, then it follows from (3.27)

$$A(P)(\xi - P) = 0$$

Since $A(P) \neq 0$ thus, it must be $\xi = P$. In this case, from (1.2) and (3.25), we finally get that

$$w(Y) = A(Y) \quad (3.28)$$

for all vector fields Y. By Theorem 3.3, since A is closed in a conformally flat $(PMRS)_n$, therefore from (3.28), the 1-form w is closed. It is known [8] that when w is also closed in (3.19), the vector field P is concircular. Thus, this completes the proof. □

References

[1] M.C. Chaki, On pseudo Ricci symmetric manifolds, *Bulg. J. Phys.*, **15** (1099), 526-531.

[2] S.K. Chaubey, R.H. Ojha, On the M-projective curvature tensor of a Kenmotsu manifold, *Diff. Geom. Dyn. Sys.*, **12** (2010), 52-60.

[3] L.P. Eisenhart, *Riemannian Geometry*, Princeton University Press (1926).

[4] R.H. Ojha, A note on the M-projective curvature tensor, *Indian J. Pure Appl. Math.*, **8**, No. 12 (1975), 1531-1534.

[5] R.H. Ojha, M-projectively flat Sasakian manifolds, *Indian J. Pure Appl. Math.*, **17**, No. 4 (1986), 481-484.

[6] E.M. Patterson, Some theorems on Ricci-recurrent spaces, *J. London Math. Soc.*, **27** (1952), 287-295.

[7] G.P. Pokhariyal, R.S. Mishra, Curvature tensors and their relativistic significance, *Yokohama Math. Jour.*, **19**, No. 2 (1971), 97-103.

[8] J.A. Schouten, *Ricci-Calculus*, Springer-Verlag, Berlin (1954).

[9] A.A. Shaikh, S.K. Hui, On pseudo cyclic Ricci symmetric manifolds, *Asian-European J. of Math.*, **2**, No. 2 (2009), 227-237.

[10] R. Sharma, A. Ghosh, Perfect fluid space-times whose energy-momentum tensor is conformal Killing, *Journal of Math. Phys.*, **51** (2010), 1-5.

[11] Z.I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, The local version, *J. Diff. Geom.*, **17** (1982), 531-582.

[12] R.M. Wald, *General Relativity*, The University of Chicago Press, Chicago (1984).

[13] A.G. Walker, On Ruse's spaces of recurrent curvature, *Proc. London Math. Soc.*, **52** (1950), 36-64.

[14] M. Walker, R. Penrose, On quadratic first integrals of the geodesic equations for type $\{22\}$ spacetimes, *Commun. Math. Phys.*, **18** (1970), 265-274.