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PICONE-TYPE IDENTITIES AND INEQUALITIES FOR GENERAL QUASILINEAR ELLIPTIC EQUATIONS. PART II: COMPARISON RESULTS AND EXTENSION TO SOME NON AUTONOMOUS EQUATIONS

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Abstract: In Part I we saw that when the main coefficients $a(x)$ and $c(x)$ fulfil the conditions (1.7) for the equation (1.2) or the conditions (1.7) and (2.1) for the equation (1.1) (see [2]), the oscillation of the solutions of those equations follows.

In this part, we show using some comparison results how, those results can be extended to more general equations via some minor hypotheses on the added terms including some non autonomous cases.

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1. Extension via Comparison Results

In this part of the work we display some examples where some of our results can apply.

1.1. Some Direct Applications

Inspired by some equations displayed in [4, 1] we consider for $\beta > \alpha > \gamma > 0$ the equation

$$\nabla\left\{a(x)\Phi(\nabla w)\right\} + a(x)B(x).\Phi(\nabla w) + g(x,w) = 0 \quad \text{in } \mathbb{R}^n \tag{4.1}$$

where

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$$g(x,t) = C(x)|t|^{\beta-1}t + D(x)|t|^{\gamma-1}t.$$

With

$$f(x,t) := C(x)|t|^{\beta-1}t + D(x)|t|^{\gamma-1}t - c(x)|t|^{\alpha-1}t, \tag{4.2a}$$

the equation becomes

$$\nabla\Big\{a(x)\Phi(\nabla w)\Big\} + a(x)B(x).\Phi(\nabla w) + c(x)\phi(w) + f(x,w) = 0. \tag{4.2b}$$

Obviously if $\forall x \in \mathbb{R}^n \quad C(x), D(x) \geq 0$ and $c(x) \leq \min\{C(x), D(x)\}$ then $tf(x,t) \geq 0$. In application of B) of the Theorem 3.2 we have

Theorem 4.1. *Assume that (2.1) holds and a satisfies*

$$\lim_{t\to\infty}\int_1^t \Big[r^{n-1}\max_{|x|=r} a(x)\Big]^{-1/\alpha} dr = \infty. \tag{4.2c}$$

Then if there is a non negative Holder-continuous function μ such that

$$\forall x \in \mathbb{R}^n \qquad \min\{C(x), D(x)\} = 2\mu(x) > 0,$$

any non trivial solution of (4.1) is strongly oscillatory.

Proof. Consider for $c(x) := \mu(x)$ the equation (4.2b). Then (4.2a) reads

$$tf(x,t) := C(x)|t|^{\beta+1} + D(x)|t|^{\gamma+1} - \mu(x)(x)|t|^{\alpha+1}. \tag{4.2c}$$

When $|t| \geq 1$,

$$C(x)|t|^{\beta+1} - \mu(x)|t|^{\alpha+1} = |t|^{\beta+1}[C(x) - \mu(x)|t|^{\alpha-\beta}] > 0,$$

and for $|t| < 1$,

$$D(x)|t|^{\gamma+1} - \mu(x)|t|^{\alpha+1} = |t|^{\gamma+1}[D(x) - \mu(x)|t|^{\alpha-\gamma}] > 0.$$

These inequalities make $tf(x,t) \geq 0$ and B) of the Theorem 3.2 applies. $\square$

Consider for $\eta, \alpha, \nu > 0$ the equation

$$\nabla.\Big\{\frac{1}{|x|+\eta}|\nabla u|^{\alpha-1}\nabla u\Big\} + \frac{1}{|x|^2+\nu}|u|^{\alpha-1}u = 0, \quad x \in \mathbb{R}^n. \tag{4.3}$$

Theorem 4.2. *For any $\alpha, \eta, \nu > 0$, if $3 \leq n < 2+\alpha$ or $2+\alpha \leq n < 2+\alpha+\alpha^2$, then the classical solutions of (4.3) are strongly oscillatory.*

Proof. As $A(r) = r^{-1}$ and $\gamma(r) = \dfrac{1}{|x|^2+\nu}$, if $3 \le n < 2+\alpha$ then (Cia) of (1.7) is satisfied. In fact $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \int_\tau^t s^{(2-n)/\alpha} ds = \infty$ and

$$\lim_{r\to\infty} \int_\tau^r q(r)dr = \lim_{r\to\infty} \int_\tau^r \frac{s^{n-1}}{s^2+\nu} ds = \infty.$$

If $2+\alpha \le n < 2+\alpha+\alpha^2$ then (Ciia) of (1.7) is satisfied. □

If we reduce the domain $\mathbb{R}^n$ to any exterior domain $\Omega^c_{r_0} := \{x \in \mathbb{R}^n \| x| > r_0 > 0\}$ then the result remains true even with $\eta = \nu = 0$. This applies to the problem (2.7) in [3]. In fact for

$$\nabla . \left\{ \frac{1}{|x|} |\nabla u|^{\alpha-1} \nabla u \right\} + \frac{1}{|x|^2} |u|^{\alpha-1} u = 0 \quad \text{in} \quad |x| \ge 1 \tag{4.3a}$$

where $\alpha = 3$, as $A(r) = r^{-1}$ and $\gamma(r) = r^{-2}$, Theorem 4.2 applies. □

Theorem 4.3. *Let g_0 be a positive function such that*

$$\int_a^\infty r^{n-1} \{\min_{|x|=r} g_0(x)\} dr = \infty.$$

Then if $a(x)$ satisfies (Cia), $\forall \gamma \in (0,\alpha)$ and $\phi_\gamma(t) = |t|^{\gamma-1}t$ any non trivial solution of

$$\nabla \left\{ a(x)\Phi(\nabla v) \right\} + g_0(x)\phi_\gamma(v) = 0 \quad \textit{in} \quad \mathbb{R}^n \tag{4.4}$$

is strongly oscillatory and that solution is unique.

Proof. For a $\beta > 0$, rewrite the equation as

$$\nabla \left\{ a(x)\Phi(\nabla v) \right\} + \beta g_0(x)\phi(v) + f(x,v) = 0,$$

where $f(x,t) = g_0(x)|t|^{\gamma-1}t\{1 - \beta|t|^{\alpha-\gamma}\}$.

The non trivial solutions of $\nabla \left\{ a(x)\Phi(\nabla v) \right\} + g_0(x)\phi(v) = 0$ are oscillatory and by the homogeneous character of the equation, if u is such a solution, so is $\lambda u, \forall \lambda \in \mathbb{R}$. For a solution v of (4.4), the normalized associated function $w(x) := \dfrac{v(x)}{\lambda}$ where $\lambda := |v|_\infty$ satisfies for

$$f_\gamma(x,t) = \lambda^{\gamma-\alpha} f(x,t), \quad \nabla \left\{ a(x)\Phi(\nabla w) \right\} + \beta g_0(x)\phi(w) + f_\gamma(x,w) = 0.$$

If we take $\beta \le \lambda^{\gamma-\alpha}$ then as $t f_\gamma(x,t) \ge 0$ for $|t| \le 1$, Theorem 3.2 applies. □

2. Comparison Results

Let $u, v, w \in \mathbb{R}^n$ and assume that

$$P_0 v = \nabla.\{a(x)\Phi(\nabla v)\} + c(x)\phi(v) = 0 \geq (\neq) P_o u \quad \text{in} \quad \mathbb{R}^n. \tag{4.5}$$

$$P_0 v = \nabla.\{a(x)\Phi(\nabla v)\} + c(x)\phi(v) = 0 \leq (\neq) P_o w \quad \text{in} \quad \mathbb{R}^n. \tag{4.6}$$

Then we have the following inequalities :

$$\nabla.\left\{a(x)v\Phi(\nabla v) - a(x)v\phi(\frac{v}{u})\Phi(\nabla u)\right\} \geq a(x)Z(v,u) \tag{4.7}$$

$$\text{and} \quad \nabla.\left\{a(x)w\Phi(\nabla w) - a(x)w\phi(\frac{w}{v})\Phi(\nabla v)\right\} \geq a(x)Z(w,v). \tag{4.8}$$

Moreover for

$$\begin{array}{ll} (a) & P_i v = \nabla.\{a(x)\Phi(\nabla v)\} + c_i(x)\phi(v) = 0 \quad \text{in } \mathbb{R}^n; \quad i = 1,2, \\ & \qquad i \neq j \Longrightarrow \\ (b) & \nabla.\left\{a(x)v_i\Phi(\nabla v_i) - a(x)v_j\phi(\frac{v_i}{v_j})\Phi(\nabla v_j)\right\} = \\ & \qquad = a(x)Z(v_i,v_j) + |v_i|^{\alpha+1}\{c_j - c_i\}. \end{array} \tag{4.9}$$

Then we get the following result:

Theorem 4.4. *Assume that $v, u, w \in \mathbb{C}^1(\mathbb{R}^n)$ are those in (4.5)-(4.8). Then if v is strongly oscillatory in $\mathbb{R}^n$ then:*

(i) u has a zero inside any connected component of supp(v);

(ii) v has a zero inside any connected component of supp.(w), where

$$supp(W) := \overline{\{x \in \mathbb{R}^n | W(x) \neq 0\}}.$$

Moreover if u_i and u_j are differentiable in $\mathbb{R}^n$ and satisfy (4.9), then if $c_j(x) > c_i(x)$, v_j has a zero inside any connected component of supp.(v_i).

Proof. In fact if we integrate over any component of supp.(v) any of (4.7) and (4.8), we get a contradiction unless e.g. for (4.6) $\exists k \in \mathbb{R}$ such that $u = kv$ violating (4.5). This holds even if we suppose that u is null on the boundary of that component. □

We now consider for $\lambda > 0$ the equation

$$P_\lambda u := \nabla.\left\{a(x)\Phi(\nabla u)\right\} + \lambda c(x)\phi(u) = g(x) \quad \text{in } \mathbb{R}^n. \tag{4.10λ}$$

Theorem 4.5. *Assume that c satisfy the conditions (Cia) of (1.7).*

i) If $\forall x \in \mathbb{R}^n$ $g(x) \leq (\neq)0$, then any bounded and non trivial classical solution of (4.10λ) is strongly for any $\lambda > 0$.

ii) Assume that in addition, there is an $\nu, r_0 > 0$ such that for any bounded $D \subset \Omega^c_{r_0}$ of non zero measure

$$\begin{array}{lll} \int_{\mathbb{R}^n} |g(x)|dx < +\infty & \text{and} & \int_D c(x)dx > 0 \quad \text{or} \\ |g| \quad \text{is bounded and} & \forall x \in \mathbb{R}^n & c(x) > \nu > 0. \end{array} \tag{4.11}$$

Then for large $\lambda > 0$ any bounded and non trivial classical solution u_λ of (4.10λ) is strongly oscillatory unless it satisfies $\lim_{|x|\to\infty} \inf |u_\lambda(x)| = 0$.

Proof. For any $\lambda > 0$,

$$\nabla.\left\{a(x)\Phi(\nabla v)\right\} + \lambda c(x)\phi(v) = 0 \quad \text{in } \mathbb{R}^n \tag{4.11λ}$$

has a strongly oscillatory solution v_λ, say.

i) Any classical and bounded solution u_λ of (4.10λ) satisfies $\nabla.\left\{a(x)\Phi(\nabla v_\lambda)\right\} + \lambda c(x)\phi(v_\lambda) = 0 \geq P_\lambda(u_\lambda)$ and from Theorem 4.4 (i), u_λ has a zero inside any connected component of supp.(v_λ).

ii) Let v_1 be a strongly oscillatory solution of

$$\nabla.\left\{a(x)\Phi(\nabla u)\right\} + c(x)\phi(u) = 0 \quad \text{in } \mathbb{R}^n.$$

We rewrite the equation in (4.10λ) as

$$\nabla.\left\{a(x)\Phi(\nabla u)\right\} + c(x)\phi(u) + (\lambda - 1)c(x)\phi(u) - g(x) = 0. \tag{4.12λ}$$

For a classical and bounded solution of (4.12λ) u_λ say,

$$\nabla.\left\{a(x)v\Phi(\nabla v) - a(x)v\phi(\frac{v}{u_\lambda})\Phi(\nabla u_\lambda)\right\} = aZ(v, u_\lambda) + |v|^{\alpha+1}\left\{(\lambda - 1)c(x) - \frac{g(x)}{\phi(u_\lambda)}\right\}. \tag{4.13λ}$$

Assume that $\exists R_0, \lambda_0, \beta > 0$ such that $\forall \lambda > \lambda_0 + 1$ any classical and bounded solution u_λ of (4.10λ) satisfies $|u_\lambda(x)| > \beta \quad \forall x \in \Omega^c_{R_0}$.

The solution $v = v_1$ defined above has a nodal set G, say, inside $\Omega^c_{R_0}$.From (4.13λ), as $u_\lambda > \beta$ in $\Omega^c_{r_0}, \forall \lambda > \lambda_0$

$$0 \geq \int_G \left\{ a(x)Z(v, u_\lambda) + |v|^{\alpha+1} \left[(\lambda - 1)c(x) - \frac{g(x)}{\beta} \right] \right\} dx. \tag{4.14}$$

But from (4.11), for large λ, $\int_G |v|^{\alpha+1} \left[(\lambda - 1)c(x) - \frac{g(x)}{\beta} \right] dx > 0$ conflicting with (4.14). □

Assume the (2.1) holds. Let u be a classical non trivial and bounded solution of (1.1) in $\mathbb{R}^n$ and concider for $\lambda > 0$ a solution $w := w_\lambda$ of the equation

$$\nabla \left\{ a(x)\Phi(\nabla w) \right\} + a(x)B(x).\Phi(\nabla w) + \lambda c(x)\phi(w) = g(x) \quad \text{in } \mathbb{R}^n. \tag{4.15}$$

As before,

$$\begin{aligned} &\nabla. \left\{ a(x)b(x)u\Phi(\nabla u) - a(x)b(x)u\phi(\frac{u}{w})\Phi(\nabla w) \right\} = a(x)uB(x).\Gamma(u, w) \\ &+ b(x) \left[a(x)Z(u, w) - a(x)uB(x).\Gamma(u, w) + |u|^{\alpha+1} \left\{ (\lambda - 1)c(x) - \frac{g(x)}{\phi(w)} \right\} \right]. \end{aligned} \tag{4.15λ}$$

Theorem 4.6. *Assume that a and c satisfy (Cia) of (1.7).*

(i) If $g(x) \leq 0 \quad \forall x \in \mathbb{R}^n$ then $\forall \lambda \geq 1$ any non trivial classical solution of (4.15) is strongly oscillatory.

ii) Assume that in addition, there is an $\nu, r_0 > 0$ such that for any bounded $D \subset \Omega^c_{r_0}$ of non zero measure

$$\begin{aligned} &\int_{\mathbb{R}^n} |g(x)|dx < +\infty \quad \textit{and} \quad \int_D c(x)dx > 0 \quad \textit{or} \\ &|g| \quad \textit{is bounded and} \quad \forall x \in \mathbb{R}^n \quad c(x) > \nu > 0. \end{aligned} \tag{4.11}$$

Then for large $\lambda > 0$ any bounded and non trivial classical solution u_λ of (4.15) is strongly oscillatory unless it satisfies $\lim_{|x|\to\infty} \inf |u_\lambda(x)| = 0$.□

Proof. From the hypotheses on a and c, u is strongly oscillatory in $\mathbb{R}^n$.

(i) If we replace in (4.15λ) b by $b(x) + k$ for any $k > 0$ and integrate the result over any nodal set G of u , we get $\forall k > 0$,

$$0 = \int_G a(x)uB(x).\Gamma(u,w)dx + \int_G (b(x)+k)\Big[a(x)Z(u,w) - a(x)uB(x).\Gamma(u,w) + |u|^{\alpha+1}\Big\{(\lambda-1)c(x) - \frac{g(x)}{\phi(w)}\Big\}\Big]dx \quad (4.16)$$

which holds only if among other

$$\int_G \Big[a(x)Z(u,w) + |u|^{\alpha+1}\Big\{(\lambda-1)c(x) - \frac{g(x)}{\phi(w)}\Big\} dx = 0. \quad (4.17)$$

This is absurde if g is a negative function and $\lambda \geq 1$. Therefore w cannot be strictly positive in G.

(ii) The proof of this part follows the process used for (ii) of the last theorem, using (4.17). □

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