International Journal of Pure and Applied Mathematics
Volume 72 No. 2 2011, 173-194

# CONDITION OF NON-LINEAR STABILITY OF DUMBELL SATELLITE IN ELLIPTICAL ORBIT

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**Abstract:** Condition of Non-linear stability of dumbell satellites, in elliptical orbit in the central gravitational field of force under the combined effects of various perturbing forces has been studied. The system comprises of two satellites connected by a light, flexible and inextensible cable, moves with tight cable like a dumbell satellite in elliptical orbit, in the central gravitational field of force. The gravitational field of the Earth is the main force governing the motion and various perturbing forces like magnetic field of the Earth, oblateness of the Earth and forces of general nature are considered to be perturbing forces, disturbing in nature. Non-linear oscillations of dumbell satellites about the equilibrium position in the neighbourhood of main resonance $\omega = \nu$, under the influence of perturbing forces which is suitable for exploiting the method of Bogoliubov-Krilov and Mitropoloskey, has been discussed, considering 'e' to be samll parameter and the condition of stability of dumbell satellites has been discussed using poicare method. It has been observed that the discontinuing in the amplitude of oscillations occur at a frequency less than the natural frequency.This discontinuity of amplitude of oscillations of the system is important for analysing the condition of stability of dumbell satellite in elliptical orbit.

**AMS Subject Classification:** 70F15
**Key Words:** condition of stability, perturbing forces, dynamical system, non-linear oscillation

Received: July 6, 2011



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## 1. Introduction

The present paper is devoted to the analysis of condition of stability of cable connected satellites system, connected by a light, flexible and inextensible cable moving in the central gravitational field of the Earth under the combined effects of the Earth magnetic field, oblateness of the Earth and the perturbing force of general nature in non-resonance and resonance cases. The satellites are cosidered to be charged material particles and the motion of the system in studied, relative to their centre of mass, under the assumption that the later moves along elliptical orbit. The cable connecting the two satellites is taut and nonelastic in nature, so that the system moves like a dumbell satellite. Many space configurations of cable connected satellites system have been proposed and analysed like two satellites are connected by a rod (dumbell satellite), see [4], two or more satellites are connected by a tether, M. Krupa et al [6,7], Beletsky and E.H. Levin [2], A.K. Mishra and V.J. Modi [9] and spring connected satellites, see [13]. All these authors have mentioned numerous important applications of system and stability of relative equilibrium, if the system moves in a circular and elliptical orbit. Beletsky and Novikova (see [3]) studied the motion of a system of two satellites connected by a light, flexible and inextensible string in the central gravitational field of force relative to the centre of mass, which is itself assumed to move along a keplerian elliptical orbit, under the assumption that the two satellites are moving in the plane of motion of the centre of mass. The same problem in its general form was further investigated Singh see [18,19], these works conducts the analysis of relative motion of the system for the elliptcal orbit of the centre of mass in two dimensional as well as three dimensional cases. Narayan and Singh see [10,11,12], studies non-linear oscillations due to solar radiation pressure the centre of mass of the system moves along an elliptical orbit. Sharma and Narayan [16,17], studies the combined effects of the solar radiation pressure and the forces of general nature on the motion and stability of cable connected satellites system in elliptical orbit. Singh et al [20,21], studied the non-linear effects of the Earth's oblateness in the motion and stability of cable connected satellites system in elliptical orbit Das et al [5] and Narayan et al [29] studied the non-linear effects of Earth's magnetic field on the stability of cable connected system in inclined and equitorial orbit.

The different aspects of the problem of stability of a satellites in low and high altitude orbit with different perturbation forces have been studied by many scientists. Special references are mentioned, Sarychev et al [22,23] studied the problem determining all equilibria of a satellite subjected to gravititational and acrodynamic torque in circular orbit. All bifurcation values of the parameters corresponding to qualitative changes of stability domain are determined. Palacian in [24], studied the dynamics of a satellites orbiting are Earth like planet at low altitude orbit and perturbation is caused by in homogenious potential due to the Earth. C. Labort in [26] studied bifurcation of relative equilibria in the main problem of artificial satellite theory for a prolate body. Markeev et al [25] studied the planar oscillations of a satellite in a circular orbit. Ayub Khan etal [14] investigated Choatic motion in problem of dumbell satellite.

In the present paper, we have studied the combined effects of magnetic field of the Earth, oblateness of the Earth and the external periodic forces of general nature on conditon of stability of dumbell satellite in elliptical orbit.

The perturbing forces due to the Earth's magnetic field results from the interaction between space craft's residual magnetic field and the geomagnetic field. The perturbing force is arising due to magnetic moment, eddy current and hysteresis, out of these the space craft magnetic moment is usually the dominant source of disturbing effects. Nevertheless a distant satellite beyond gravitational field of the Earth in addition to magnetic field of earth, it could still expected to be affected by general nature of external forces could arise due to dissipation of the energy generated on account of friction of bodies in the atmosphere by tidal forces, gravitational radiation etc. these forces though small can significantly affect the oscillations of the system under consideration. These forces could be modelled as frictional forces with small dissipation coefficient. Further more the forces generates by the multipole moments and absportion of gravitational waves at resonance frequency, could be characterized as external periodic forces having a slowly varying frequency and these forces could be estimated by certain model assumption.

Thus in order to study the condition of non-linear stability of dumbell satellite system on realistic basis, it is essential to consider the combined influence of the Earth's magnetic field, oblateness of the Earth and periodic forces of general nature.

## 2. Equation of Motion

The combined effects of the geomagnetic field and oblatness of the Earth on the motion and stability of the satellites connected by a light, flexible and inextensible cable, under the influence of the central gravitational field of the Earth has been considered.

The analysis of the motion and stability of the cable connected satellites system has been restricted to two dimensional case, we have assumed that the

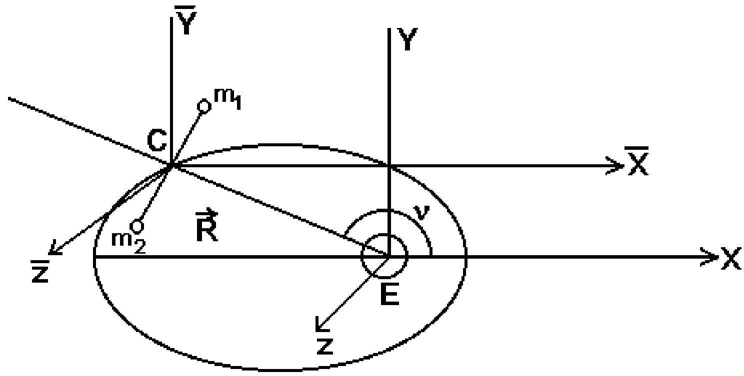



Figure 1: Rotating frame of reference

satellites are moving in the orbital plane of the centre of mass of the system. The analysis of stability and instability of motion of the system under the influence of the above mentioned perturbing forces has been simulated in the two dimensional plane. The motion and stability of cable connected satellites system under the effects of Earth's magnetic field Das et al [5], Narayan et al [29] and combined effects of earth magnetic field and oblateness of the Earth, Narayan and Pandey [30], in elliptical and in low altitude oribit have been studied.

The equation of two dimensional motion of one of the satellites under the rotating frame of reference in Nechville's co-ordinate system (see [15]) relative to the centre of mass which moves along equitorial orbit under the combined influence of the Earth's magnetic field and oblateness of the Earth can be deduced and represented in (2.1).

$$\begin{aligned} x'' - 2y' - 3x\rho &= \lambda_\alpha x + \frac{4Ax}{\rho} - \frac{B}{\rho}\cos\delta, \\ y'' - 2x' &= \lambda_\alpha y - \frac{Ay}{\rho} - \frac{B\rho'}{\rho^2}\cos\delta. \end{aligned} \tag{2.1}$$

Here the x-axis is in the direction of the position vector joining the centre of mass and the attracting centre and $Y$-axis is along the normal to the position vector in the orbital plane of the centre of mass in the direction of motion of satellite $m_1$, where $A$ is the oblateness due to Earth and B is the magnetic field of the Earth. Moreover

$$\begin{aligned}
A =& \frac{3k_2}{\rho^2}, \quad \lambda = \frac{p^3\rho^4}{\mu}\frac{(m_1+m_2)\,\lambda}{m_1 m_2}, \\
B =& -\left(\frac{m_2}{m_1+m_2}\right)\left[\frac{Q_1}{m_1}-\frac{Q_2}{m_2}\right]\frac{\mu E}{\sqrt{\mu\rho}}, \\
\rho =& \frac{R}{P} = \frac{1}{1+e\cos v}.
\end{aligned} \tag{2.2}$$

The dipole of the Earth has its axis inclined from the polar axis of the Earth by a value of $11^0 4'$. The angle $\phi$ and $\Omega$ completely defined the position of $k_e$, the unit vector along the axis of magnetic dipole of the Earth.

In this case the condition for the constraint is given by the inequality:

$$x^2+y^2 \leq \frac{1}{\rho^2}. \tag{2.3}$$

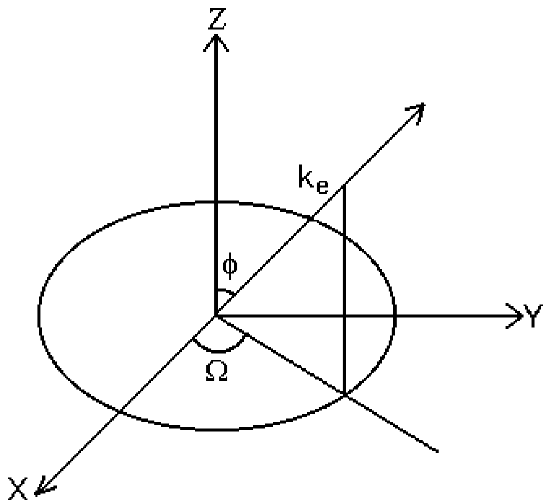



Figure 2: Orientation of $k_e$

where $\lambda$ denotes the Lagrange's multiplier and $\mu$ denotes product of the gravitational contant and the mass of the Earth. where

$$Q_i = \frac{\text{change}\, q_i \text{of the}\, ith\, \text{particle}}{\text{velocity of light}}, \quad i=1,2,$$

on the mass $m_1$ and $m_2$, where $v$ is the true anomaly of the centre of mass of the system in elliptical orbit.

$$\text{and } \rho = \left(\frac{R}{P}\right) = \left(\frac{1}{1+e\cos v}\right),$$

where $p$ and $e$ are the focal parameter and the eccentricity of the orbit of the centre of mass. In equation (2.1)the prime denotes differentiation with respected to v. When the motion of the satellite $m_1$ is determines with the help of equation (2.1), the motion of the satellite $m_2$ is easily determined with the help of the identify

$$m_1\vec{\rho_1} + m_2\vec{\rho_2} = 0, \tag{2.4}$$

where $\vec{\rho_1}$ and $\vec{\rho_2}$ are the radius vectors of the satellites of masses $m_1$ and $m_2$ respectively with respect to the centre of mass of the system.

In order to discuss the non-linear oscillations of the system, we transform the equation (2.1) into polar forms by substituting

$$\begin{aligned} x &= (1+e\cos v)\cos\psi, \\ y &= (1+e\cos v)\sin\psi, \end{aligned} \tag{2.5}$$

where $\psi$ is the angular deviation of the line joining the centre of mass and the cable connected satellites with the stable position of equilibrium joining with respect to $\psi$ and $\lambda_\alpha$ we obtain:

$$\begin{aligned} &(1+e\cos v)\,\psi'' - 2e\psi'\sin v + 3\sin\psi\cdot\cos\psi + 5A\,(1+e\cos v)^2\sin\psi\cdot\cos\psi \\ &\quad = B\cos\delta\,(1+e\cos v)\cdot\sin\psi - B\cos\delta\cdot\sin v\cdot\cos\psi + 2e\sin v. \end{aligned} \tag{2.6}$$

The equation (2.6) is the equation of motion of a dumbell satellite in the central gravitational field of the Earth's magnetic field and oblateness due to Earth. The equation determining the Lagrange's multiplier is given by:

$$\begin{aligned} &(1+e\cos v)^4\left(\psi'+1\right)^2 + (1+e\cos v)^3\left(3\cos^2\psi - 1\right) \\ &\qquad - B\cos\phi\,(1+e\cos v)^3\left(\cos\psi + e\cos(\psi+v)\right) \\ &\qquad - A\,(1+e\cos v)^3\left(4\cos^2\psi - \sin^2\psi\right) = \lambda_\alpha \geq 0. \end{aligned} \tag{2.7}$$

The non-linear osillations described by (2.6) take place as long as inequality given below is satisfied.

$$(1+e\cos v)^4\left(\psi'+1\right)^2 + (1+e\cos v)^3\left(3\cos^2\psi - 1\right)$$

$$
\begin{aligned}
&- B\cos\delta\,(1+e\cos v)^3\,(\cos\psi + e\cos(\psi+v)) \\
&- A\,(1+e\cos v)^3\left(4\cos^2\psi - \sin^2\psi\right) \geq 0, \qquad (2.8)
\end{aligned}
$$

where $v$ and $e$ are respectively true anomaly of the centre of mass of the system and the eccentricity of the orbit of the system. The prime denotes differentiation with respect to true anomaly $v$.The system of equation (2.1) oscillates about the stable position of equilibrium in which it lies wholly along the radius vector joining the centre of mass and the centre of force Narayan et al (2010), substituting $2\psi = \eta$, the equation (2.1) can be expressed as follows:

$$
\begin{aligned}
\eta'' + 3\sin\eta = {}& 4e\sin v + 2e\eta'\sin v - 5A\,(1+e\cos v)^2\sin\eta - \eta'' e\cos v \\
& + 2B\cos\delta\sin\frac{\eta}{2} + 2eB\cos\delta\sin\left(\eta - \frac{v}{2}\right). \qquad (2.9)
\end{aligned}
$$

Equation (2.9), describes non-linear oscillations of the dumbell satellites in elliptical orbit, in central gravitational field of the oblate Earth's together with the magnetic field of the Earth.

The presence of the friction force $\gamma\psi'$ and the small periodic force $E\sin\nu v$ results in the following equation rather than (2.9).

$$
\begin{aligned}
\eta'' + 3\sin\eta = {}& 4e\sin v + 2e\eta'\sin v - \eta'' e\cos v - 5A\,(1+e\cos v)^2\sin\eta \\
& + e2B\cos\delta\sin\frac{\eta}{2} + 2eB\cos\delta\sin\left(\eta - \frac{\gamma}{2}\right) + E\sin\nu v + \gamma\eta', \qquad (2.10)
\end{aligned}
$$

where $\gamma$ and $E$ are some phenomenological parameters characterizing the tidal and periodic forces acting on the system and have been assumed to be the order of $e$, where $\nu$ is the frequency of the external periodic forces. However, these parameters can be determines from specific model assumptions concerning these problems.

## 3. Non-Linear Non-Resonance Oscillations of the System about the Position of Equilibrium for Small Eccentricity

The non-linear oscillations of the system of cable connected satellites under the influence of above mentioned forces described (2.10), will be investigated for non-resonance cases on the assumption that $\gamma$ and $E$ are of order of $e$. This is justified on account of the fact that these estimates are always concerned with a certain model assumptions. Hence setting $E = eE_1$ and $\lambda = e\lambda_1$ and $B\cos\delta = eB\cos\delta$, these equation (2.10) can be put in the form:

$$
\eta'' + \omega^2\eta = e\Big[B\,(\eta - \sin\eta) + 2\eta'\sin v + 4\sin v - \eta''\cos v
$$

$$+ E_1 \sin \nu v + \gamma' \eta' + 2B \cos \delta \sin \frac{\eta}{2} - 5A \sin \eta \Big]$$
$$+ e^2 \left[ 10A \cos v \sin \eta + 2B \cos \delta \sin \left( v - \frac{\eta}{2} \right) \right]. \tag{3.1}$$

In the equation (3.1) $\omega^2 = 3$ and $\beta = \dfrac{\omega^2}{e}$. Moreover the non-linearity will be assumed to be the order of $e$.

The system described by the equation (3.1) moves under the forced vibration due to the presence of the magnetic field of the Earth and the external periodic forces of general nature on the right hand side of the equation. We are benifitted by the smallness of the eccentricity $'e'$ in equation (3.1) and hence solution may be obtained by exploiting the Bogoliubov, Krilov and Mitropolsky method. For $e = 0$, the generating solution of zeroth order are:

$$\eta = a \cos \theta; \;\; \theta = \omega v + \theta^*,$$

where the amplitude $'a'$ and phase $'\theta'$ are constant, which can be determine by the intial conditions. The solution of equation (3.1) is obtained in the form.

$$\eta = a \cos \theta + e u_1 (a, \theta, v) + e^2 u_2 (a, \theta, v) + \cdots , \tag{3.2}$$

where the amplitude $'a'$ and phase $'\theta'$ are determine by the differential equations

$$\frac{da}{dv} = e A_1 (a_r) + e^2 A_2 (a), \tag{3.3}$$
$$\frac{d\theta}{dv} = \omega + e B_1 (a) + e^2 B_2 (a). \tag{3.4}$$

From equation (3.2), we find $\dfrac{d\eta}{dv}$ and $\dfrac{d^2\eta}{dv^2}$ and substituting the values of $\eta$, $\dfrac{d\eta}{dv}$ and $\dfrac{d^2\eta}{dv^2}$ in eqution (3.1). In this final equation, equating the coefficients of like powers of $'e'$, we get:

$$\omega^2 \frac{\partial^2 u_1}{\partial \theta^2} + 2\omega \frac{\partial^2 u_1}{\partial \theta \partial v} + \frac{\partial^2 u_1}{\partial v^2} - 2\omega A_1 \sin \theta - 2\omega B_1 \cos \theta + \omega^2 u_1$$
$$= 4 \sin v + \beta (\eta - \sin \eta) + 2\eta' \sin v - \eta'' \cos v + E_1 \sin \nu v + \gamma' \eta'$$
$$+ 2B \cos \delta \sin \left( \frac{\eta}{2} \right) - 5A \sin \eta, \tag{3.5}$$

$$\omega^2 \frac{\partial^2 u_2}{\partial \theta^2} + 2\omega \frac{\partial^2 u_2}{\partial \theta \partial v} + \frac{\partial^2 u_2}{\partial v^2} + \omega^2 u_2$$

$$
\begin{aligned}
&= (2\sin v + \gamma_1)\left[\left(A_1\cos\theta - aB_1\sin\theta + \omega\frac{\partial u_1}{\partial\theta} + \frac{\partial u_1}{\partial v}\right)\right] \\
&\quad - \cos v\left[\frac{\partial^2 u_1}{\partial\theta^2}\omega^2 + 2\omega\frac{\partial^2 u_1}{\partial\theta\partial v} + \frac{\partial^2 u_1}{\partial v^2} - 2a\omega B_1\cos\theta\right. \\
&\quad \left. - 2\omega a_1\sin\theta\right] + \left(aB_1^2 - A_1\frac{\partial A_1}{\partial a}\right)\cos\theta + \left(A_1 a\frac{\partial B}{\partial a} + 2A_1B_1\right)\sin\theta \\
&\quad - 2\omega B_1\frac{\partial^2 u_1}{\partial\theta^2} - A_1\frac{\partial^2 u_1}{\partial a\partial v} - 2B_1\frac{\partial^2 u_1}{\partial\theta\partial v} - 2\omega A_1\frac{\partial^2 u_1}{\partial a\partial\theta} - -2B_1\frac{\partial^2 u_1}{\partial\theta\partial v} \\
&\quad - 2\omega A_1\frac{\partial^2 u_1}{\partial a\partial\theta} + 10A\cos v\cdot\sin\eta + 2B_1\cos\delta\sin\left(v + \frac{\eta}{2}\right) \\
&\quad + 2a\omega B_2\cos\theta + 2\omega A_2\sin\theta. \qquad (3.6)
\end{aligned}
$$

Using Fourier expansion given by:

$$
\begin{aligned}
\sin(a\cos\theta) &= 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(a)\cdot\cos(2k+1)\theta, \\
\cos(a\cos\theta) &= J_0(a) + 2\sum_{k=0}^{\infty}(-1)^k J_{2k}(a)\cdot\cos 2k\theta,
\end{aligned}
\qquad (3.7)
$$

where $J_k$,$k = 0, 1, 2, 3....$ stands for Bessel's function. Substituting of these values in equation (3.5) and determines $A_1(a)$ and $B_1(a)$ in such a way as $u_1(a,\theta,v)$, should not contain resonance terms and hence, equating the coefficients of $\sin\theta$ and $\cos\theta$ to zero, seperately we obain:

$$
\begin{aligned}
A_1(a) &= \left[\frac{\gamma_1}{2}, a\right], \\
B_1(a) &= \frac{-\beta_1}{2\omega a}(a - J_1(a)) + \frac{10AJ_1(a)}{2\omega a} - \frac{2B\cos\delta}{2\omega a}J_1\left(\frac{a}{2}\right).
\end{aligned}
\qquad (3.8)
$$

With the help of equation (3.8) it is not difficult to obtain $u_1(a,\theta,v)$ in the form

$$
\begin{aligned}
u_1(a,\theta,v) &= \frac{4}{\omega^2 - 1}\cdot\sin v + \frac{3a}{2(2\omega+1)}\cdot\cos(v+\theta) \\
&\quad + \frac{a}{2(2\omega-1)}\cdot\cos(v-\theta) + \frac{E_1\sin\nu v}{\omega^2-\nu^2} \\
&\quad - \frac{\beta}{2\omega^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\cdot J_{2k+1}(a)\cdot\cos(2k+1)\theta \\
&\quad + \frac{\beta\cos\delta}{2\omega^2}\sum_{k=1}^{\infty}\frac{(-1)^k J_{2k+1}\left(\frac{a}{2}\right)}{k(k+1)}\cdot\cos(2k+1)\theta.
\end{aligned}
\qquad (3.9)
$$

In order to obtain the second approximation of the solution, we need to determine $A_2(a)$, $B_2(a)$ and $u_1(a,\theta,v)$ as obtained in (3.8) and (3.9) in equation (3.6) and equating the coefficients of $\sin\theta$ and $\cos\theta$ to zero with a view to eliminate resonance terms from $u_1(a,\theta,v)$, we obtain:

$$\begin{aligned}
A_2(a) =& \frac{\gamma_1}{4\omega^3 a}\Bigg[(a\omega-1)(2\beta+10A)J_1(a)\\
&+\left(1-2\omega^2 a\right)\left(2B\cos\delta J_1\left(\frac{a}{2}\right)+\beta a\right)\\
&-\frac{\gamma_1 a}{4\omega^2}\left\{(2\beta+10A)J_1'(a)-B\cos\delta J'\left(\frac{a}{2}\right)\right\},\\
B_2(a) =& \Bigg[-\frac{\gamma_1^2}{4a\omega^2}+\frac{\omega(1-4\omega)}{4(4\omega^2-1)}+\frac{\omega(\omega-1)}{4(4\omega^2-1)}-\frac{1}{8\omega^2 a}\\
&\left\{(2\beta+10A)^2 J_1^2(a)+4B^2\cos^2\delta J_1^2\left(\frac{a}{2}\right)-\beta^2 a^2\right.\\
&\left.-4(2\beta+\omega A)\beta J_1(a)J_1\left(\frac{a}{2}\right)\right\}\\
&-2\beta a(2\beta+10A)J_1(a)+4\beta aB\cos\delta J_1\left(\frac{a}{2}\right)\Bigg].
\end{aligned}\tag{3.10}$$

Thus, in the first approximation, the solution is given by

$$\eta = a\cos\theta,\tag{3.11}$$

where, the amplitude $'a'$ and the phase $'\theta'$ are given by:

$$\begin{aligned}
\frac{da}{dv} =& \frac{e\gamma_1 a}{2}=\frac{\gamma a}{2},\\
\frac{d\theta}{dv} =& \omega+\frac{1}{2\omega a}(a-2J_1(a))+\frac{10eAJ_1(a)}{2\omega a}-\frac{e2B\cos\delta J_1\left(\frac{a}{2}\right)}{2\omega a}\\
=& \omega+\frac{1}{2\omega a}(2J_1(a)-a)+\frac{5eAJ_1(a)}{\omega a}-\frac{eB\cos\delta J_1\left(\frac{a}{2}\right)}{\omega a},
\end{aligned}\tag{3.12}$$

and in the second approximation, the solution is obtaines as:

$$\begin{aligned}\eta = {}& a\cos\theta + \frac{4e\sin v}{\omega^2-1} + \frac{3ea}{2\left(2\omega+1\right)}\cos\left(v+\theta\right) + \frac{ea}{2\left(2\omega-1\right)}\cos\left(v-\theta\right)\\ &+ \frac{E\sin\nu v}{\omega^2-v^2} - \frac{e\beta}{2\omega^2}\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k\left(k+1\right)}J_{2k+1}\left(a\right)\cos\left(2k+1\right)\theta\\ &+ \frac{eB\cos\delta}{2\omega^2}\sum_{k=1}^{\infty}\frac{J_{2k+1}\left(\frac{a}{2}\right)}{k\left(k+1\right)}\cos\left(2k+1\right)\theta\end{aligned} \tag{3.13}$$

where, the amplitude $a$ and phase $'\theta'$ are given by:

$$\begin{aligned}\frac{da}{dv} ={}& \frac{\gamma a}{2} + \frac{e\gamma}{4\omega^3 a}\Bigg[\Big\{\left(a\omega-1\right)\left(2\beta+10A\right)J_1\left(a\right)\\ &+\left(1-2\omega^2 a\right)\left(2B\cos\delta J_1\left(\frac{a}{2}\right)\right)+\beta a\\ &-\frac{e}{4\omega^2}\left\{\left(2\beta+10A\right)J_1'\left(a\right)-B\cos\delta J_1'\left(\frac{a}{2}\right)\right\}\Bigg],\\ \frac{d\theta}{dv} ={}& \omega+\frac{1}{2\omega a}\left(2J_1\left(a\right)-a\right)+\frac{5eAJ_1\left(a\right)}{\omega a}-\frac{eB\cos\delta J_1\left(\frac{a}{2}\right)}{\omega a}\\ &+e^2\left[\frac{-r_1^2}{4a\omega^2}+\frac{\omega\left(1-4\omega\right)}{4\left(4\omega^2-1\right)}+\frac{\left(\omega-1\right)}{4\left(4\omega^2-1\right)}\right]\\ &-\frac{1}{8\omega^2a^2}\Bigg[\left(2\beta+10A\right)^2J_1^2\left(a\right)\\ &\Bigg[+4B^2\cos^2\delta J_1^2\left(\frac{a}{2}\right)-\beta^2a^2-4\left(2\beta+10A\right)\beta J_1\left(a\right)J_1\left(\frac{a}{2}\right)\\ &-2\beta a\left(2\beta+10A\right)J_1\left(a\right)+4\beta aB\cos\delta J_1\left(\frac{a}{2}\right)\Bigg].\end{aligned} \tag{3.14}$$

Thus, the amplitude of oscillation varies in the first approximation on account of the presence of dissipative force, which was constant in the central gravitational field of oblate Earth and magnetic field of the Earth. Moreover, from equation (3.2), it is clear that for positive $\gamma$, the system is self excited and the amplitude increases monotonically. However, if $\gamma$ is negative $a\left[v\right]\to 0$ as $v\to\infty$ and hence oscillations will damp down with passage of time an the equilibrium regime $a=0$ is stable.

In the second approximation, it follows from equation (3.14) that for positive $\gamma$, amplitude of oscillations $a\left(v\right)$ increases monotonically and also variation of amplitude is of order of eccentricity. the conditions are reversed for negative value of $\gamma$.

Moreover it is obvious that the presence of sine forces give rise to the existance of resonance $\omega = v$ and $\omega = 1$ and parametric resonance appear only for $\omega = \pm\frac{1}{2}$ upto the second approximation of the solution.

## 4. Non-Linear Planar Oscillation of Cable Connected Satellites System in Elliptical Orbit at the Main Resonance $\omega = v$

The non-linear oscillations of the system of cable connected satellites under the influence of above mentioned forces described by the equation (2.10) will be investigated for the main resonance case on the assumption that $\gamma$ and $E$ are of oder of $e$. This is justified on account of the fact that these estimations are always concerned with a certain model assumptions. Hence setting $E = eE_1$ and $r = e\gamma_1$ and $B\cos\delta = eB\cos\delta$, then the equation (2.10) can be put in the form:

$$\begin{aligned} \eta'' + \omega^2\eta =& e\Big[\beta\left(\eta - \sin\eta\right) + 2\eta'\sin v + 4\sin v - \eta''\cos v \\ & + E_1\sin\nu v + \gamma'\eta' + 2B\cos\delta\sin\left(\frac{\eta}{2}\right) - 5A\sin\eta\Big] \\ & + e^2\left[10A\cos v\sin\eta + 2B\cos\delta\cdot\sin\left(v - \frac{\eta}{2}\right)\right]. \end{aligned} \tag{4.1}$$

In equation (4.1), $\omega^2 = 3$ and $\beta = \frac{\omega^2}{e}$. Moreover, the non-linearly $(\eta - \sin\eta)$ will be assumes to be the order of $e$.

Now, we construct the asymptotic solution of the system representing (4.1) in the most general case which is valid at and near the main resonancce $\omega = \nu$, exploiting the well known Bogoliubov-Krilov and Mitropolsky method. The solution of equation (4.1) in the first approximation willbe sought in the form:

$$\eta = a\cos\left(\nu v + \theta\right), \tag{4.2}$$

$$\frac{da}{dv} = eA_1\left(a,\theta\right), \tag{4.3}$$

$$\frac{d\theta}{dv} = \left(\omega - v\right) + eB_1\left(a,\theta\right), \tag{4.4}$$

where $A_1\left(a,\theta\right)$ and $B_1\left(a,\theta\right)$ are particular solution periodic with respect to $\theta$ of the system:

$$(\omega - v)\frac{\partial A_1}{\partial \theta} - 2a\omega B_1$$
$$= \frac{1}{2\pi^2}\sum_{\sigma=-\infty}^{\sigma=+\infty}\int_0^{2\pi}\int_0^{2\pi} e^{-i\sigma\theta} f_0\left(a,\eta,\eta',\eta''\right) e^{-i\sigma\theta'}\cos k\, dv\, dk,$$

$$a(\omega - v)\frac{\partial B_1}{\partial \theta} + 2\omega A_1$$
$$= -\frac{1}{2\pi^2}\sum_{\sigma=-\infty}^{\sigma=+\infty} e^{-i\sigma\theta}\int_0^{2\pi}\int_0^{2\pi} f_0\left(a,\eta,\eta',\eta''\right) e^{-i\sigma\theta}\sin k\, dv\, dk, \quad (4.5)$$

where $k - v = \theta'$ and $f_0\left(a\eta\eta'\eta''\right)$ is the coefficient of $e$ on the right hand side of equation (4.1), simple integration gives us

$$(\omega - v)\frac{\partial A_1}{\partial \theta} - 2a\omega B_1$$
$$= \beta\left(a - 2J_1(a) - E_1\sin\theta - 2B\cos\delta J_1\left(\frac{a}{2}\right) - 10AJ_1(a)\right),$$

$$a(\omega - v)\frac{\partial B_1}{\partial \theta} - 2\omega A_1 = \gamma_1 a - E_1\cos\theta, \quad (4.6)$$

where $J_1(a)$ is the Bessel's function of the first order. The periodic solution of the system given by equation (4.5) can be easily obtaines as:

$$A_1 = \frac{\gamma_1 a}{2\omega} - \frac{E_1\cos\theta}{(\omega + v)}, \quad (4.7)$$

$$B_1 = -\frac{\alpha}{2a\omega}(a - 2J_1(a)) + \frac{E_1\sin\theta}{a(\omega + v)} + \frac{2B_1\cos\delta}{2a\omega}J_1\left(\frac{a}{2}\right) + \frac{10A_1J_1(a)}{2a\omega},$$

where the amplitude $'a'$ and phase $'\theta'$ are given by the system of differential equations:

$$\frac{da}{dv} = \frac{\gamma a}{2} - \frac{E\cos\theta}{(\omega + v)}, \quad (4.8)$$

$$\frac{d\theta}{dv} = \omega - v - \frac{1}{2a\omega}(a - 2J_1(a)) + \frac{E sin\theta}{a(\omega + v)} + \frac{2B\cos\delta}{2a\omega}J_1\left(\frac{a}{2}\right) + \frac{10AJ_1(a)}{2\omega a}.$$

The system (4.8) in independent of the eccentricity of the orbit of the centre of mass. Hence, we conclude that the oscillations of the system in the first approximation is independent of the form of the orbit of the centre of mass.

The equation (4.9) cannot be integated in a closed form due to dependence of right hand side $'a'$ and $'\theta'$. However qualitative aspects of the solution can be examined with the help of poincare theory [28].

$$\begin{aligned}\frac{da}{dv} &= a\delta_e(a) - \frac{E}{(\omega+v)}\cos\theta,\\ \frac{d\theta}{dv} &= \omega_e(a) - v + \frac{E}{(\omega+v)}\sin\theta,\end{aligned} \tag{4.9}$$

where $\omega_e(a) = \omega - \frac{1}{2a\omega}(a - 2J_1(a)) + \frac{2B\cos\delta}{2a\omega}J_1\left(\frac{a}{2}\right) + \frac{10AJ_1(a)}{2a\omega}$.

The parameter $\delta_e(a)$ and $\omega_e(a)$ introduced here denoted, respectively the equivalent damping decrement and equivalent frequency of non-linear oscillations of the dumbell system when the impressed force is absent.

We now examine the stationary regime of oscillations of the system in the first approximation. the stationary state of oscillation is defined by:

$$\frac{da}{dv} = 0, \quad \frac{d\theta}{dv} = 0.$$

Hence, from the set of equation (4.9) retaining upto second order term in the amplitude, we obtain:

$$\begin{aligned}2\omega\delta_e(a) - Ecos\theta &= 0,\\ \left[\omega_e^2(a) - v^2\right]a + E\sin\theta &= 0.\end{aligned} \tag{4.10}$$

Eliminating the phase $\theta$, we obtain:

$$\left[\omega_e^2(a) - v^2\right]^2 = \left[\frac{E^2}{a^2} - \gamma^2\right]. \tag{4.11}$$

In order to obtain this relation in the neighbourhood of the resonance frequency, we get:

$$v = \omega + \delta.$$

Assuming that the quantity $\delta$ is small, we obtain the relation (3.6) into a more convenient form:

$$\delta = -La^2 + \frac{1}{2\omega}\sqrt{\frac{E^2}{a^2} - \gamma^2}, \tag{4.12}$$

where

$$L = \left[\frac{1}{8} + \frac{B\cos\delta}{128} + \frac{5A}{8} + \frac{B\cos\delta}{32\omega^2} + \frac{B^2\cos^2\delta}{512\omega^2} + \frac{5A}{16\omega^2}\right].$$

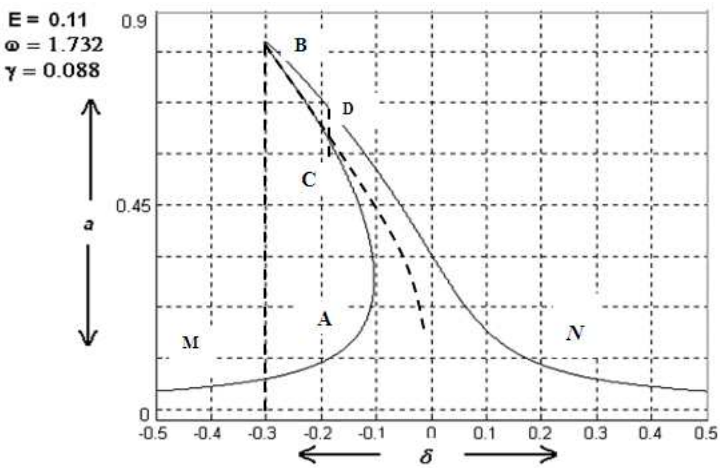



Figure 3: Resonance curve in the case of main resonance $\omega = \nu$ under the combined influence of earth magnetic field and small periodic forces

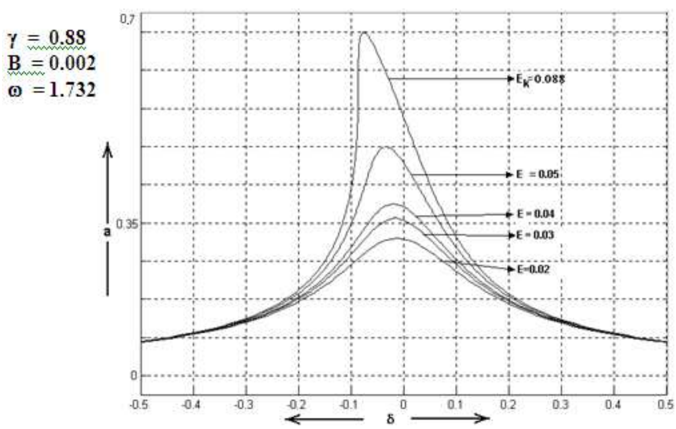



Figure 4: Resonance curve in the case of main resonance $\omega = \nu$ under the combined influence of earth magnetic field & small periodic forces

A schematic representation of the behaviours of the relation (4.12) in the range of the parameter $E$ and $\gamma$ is given in figure (3). The dotted line in the figure represents the skelton curve $v = \omega_e(a)$. This after using the relation $v = \omega + \delta$, takes the form

$$\delta = -La^2 \tag{4.13}$$

We notice here, that as $\delta$ increases, the amplitude of oscillation increases along MA but NC is increased discontinuously from C to D and further decresed along DN with the increase in $\delta$ but at B is falls abruptly to A. Thus the section BC corresponds to the unstable amplitude, which the remaining portion of the response course corresponds to the stable amplitude. The specifid properly of the curve is the fact that three staionary amplitude of oscillations situated in the region ABDC corresponds to the same frequency of external force over some frequency range when the parameter $E$ and $\gamma$ are connected by certain relationship. Two of the amplitudes are stable while the third which corresponds to the section BC of the curve is unstable.

We shall determine the relation that must exist between the parameters of the necessary condition for instability (Jump an fall) found at A and B is $\dfrac{d\delta}{da} = \infty$.

Proceeding with the equation (4.13), we obtain:

$$4L^2a^6 + 8\delta La^4 + 4\delta^2a^2 + \frac{\gamma^2a^2}{\omega^2} = \left(\frac{E^2}{\omega^2}\right). \tag{4.14}$$

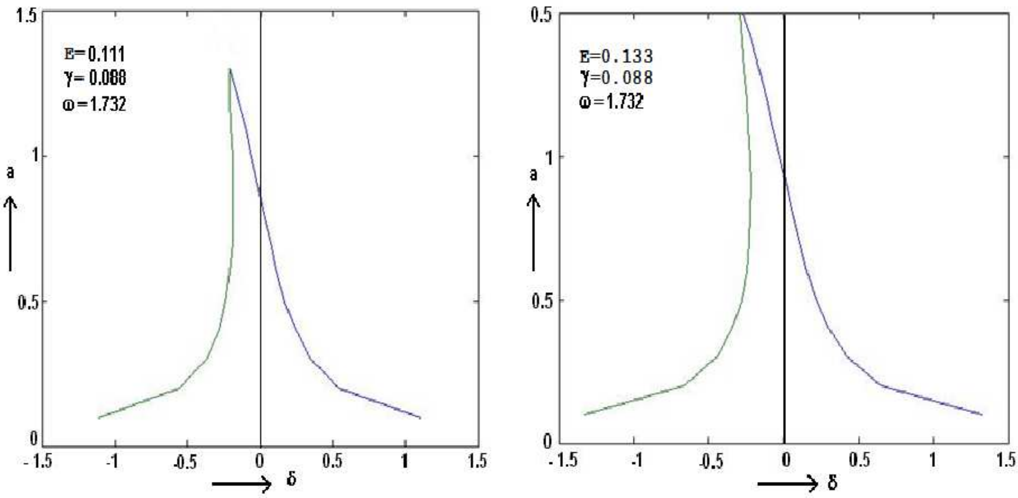
Differenciating with respect to $\delta$, the condition of instability takes the form:

$$4\delta^2 + 12\delta La^4 + 16\delta La^2 + \frac{\gamma^2}{\omega^2} = 0. \tag{4.15}$$

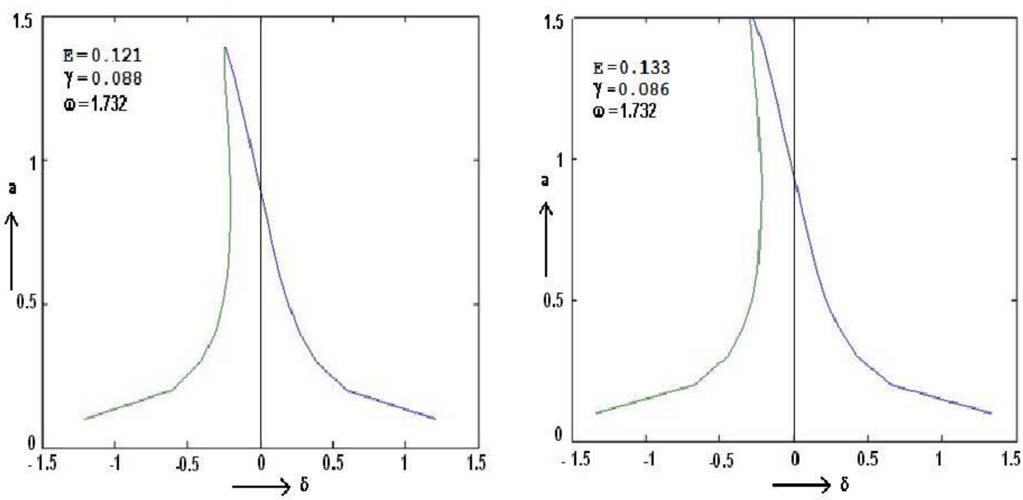
Here, we notice that both the roots of $\delta$ are negative, that is the effects occur at a frequency of the external periodic force which is less than the frequency of natural oscillation of the system. the maximum value of the amplitude is defined by the condition $\dfrac{da}{d\delta} = 0$. Thus we obtain $a_{max} = \left(\dfrac{E}{\gamma}\right)$. Also, we obtain, from (4.15) the critical value $E_k$ of the amplitude of the external periodic force:

$$E_k^2 = \frac{\gamma^{\frac{3}{2}}}{\sqrt{2\omega^2}}.$$

The break down of the amplitude of the forced oscillation is possible only when $E > E_k$. We observe from the figure 5, figure 6, figure7.... figure 12, that a slight deformation of amplitude of oscillating system when the combined influence of magnetic field of the Earth and the oblateness of the Earth are taken into account as shown in figure 5 to figure 12. For the value of $E$ either less than or equal to $E_k$. The equation (4.12) represents symmetrical curves, which does not represent any discontinuity in the amplitude of the system.



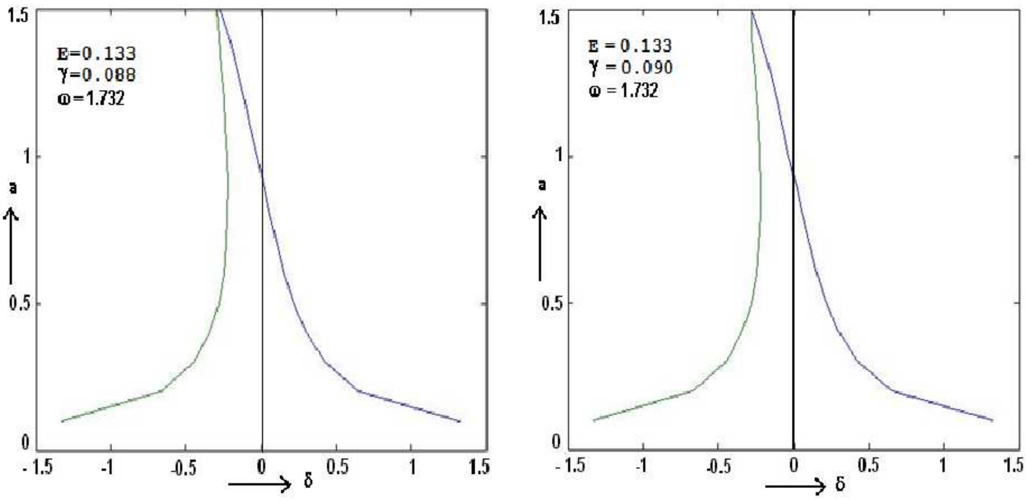

Figures 5 and 6: Behaviour of the system under perturbing forces



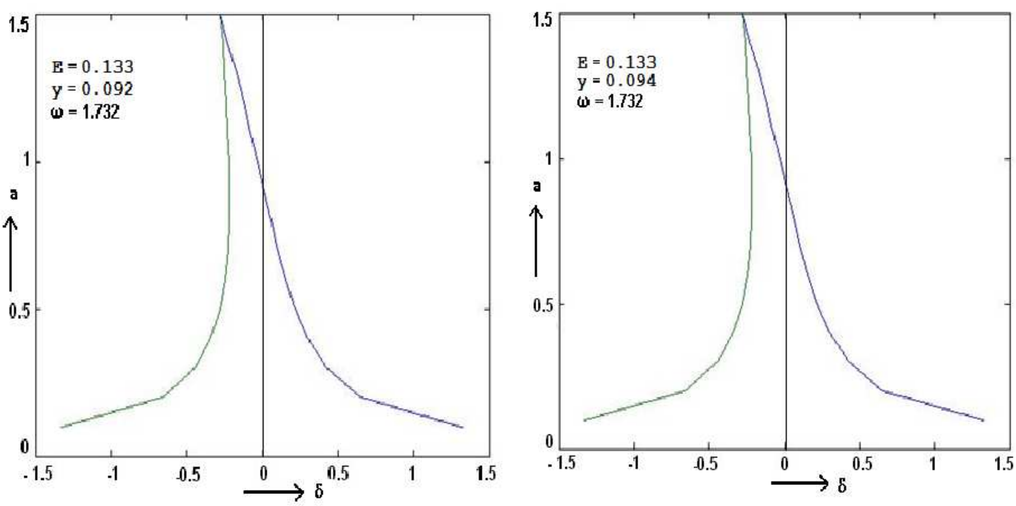

Figures 7 and 8: Behaviour of the system under perturbing forces

## 5. Discussion and Conclusion

The condition of non-linear stability of cable connected satellites system under the combined influence of the Earth's magnetic field ,oblateness of the Earth




Figures 9 and 10: Behaviour of the system under perturbing forces




Figures 11 and 12: Behaviour of the system under perturbing forces

and perturbing forces of general nature in elliptical orbit has been considered. It is assumed that the system moves like a dumbell satellite. The simulation technique has been adopted to analyze the stability of the system. We thus conclude that if a dumbell satellite in elliptical orbit is acted upon by the

Earth's magnetic field, oblateness of the Earth and perturbing forces of general nature with slowly varying frequency for which the amplitude $E$ of external periodic forces is greater than the critical value $E_k$ and there is small dissipation of energy in the system then the amplitude of oscillation of the system suffers discontinutiy in the neighbouhood of the main resoance $\omega = \nu$. When the frequency of the external periodic forces varies slowly, the amplitude of the forced oscillation breaks down at a certain point as the frequency decreses and is jumps up as the frequency increases. This discontinuity in the amplitude of the oscillation occurs at a frequency of the external periodic force, which is less than the frequency of the natural oscillation.This discontinuity in the amplitude of the oscillation of the system may affect stability of the system and is likely to change the orbit parameter of the system.

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