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NEW OPTIMAL ITERATIVE METHODS IN SOLVING NONLINEAR EQUATIONS

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Abstract: This article is concerned with numerical solution of nonlinear scalar equations by multi-point iterations. To this end, a general three-step eighth-order class of second derivative-free methods is presented, which is in fact a generalization of the three-point family of methods given by Geum and Kim in [3]. Analysis of convergence is studied and numerical examples are taken into account to support the new results in this paper.

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1. Preliminaries

Problems associated with nonlinear equations arise in many engineering fields. Perhaps, the most famous of all one-dimensional zero-finding methods is Newton's iteration, also called the Newton-Raphson method. This method requires the evaluation of both the function $f(x)$, and the derivative $f'(x)$, at arbitrary points x. The Newton-Raphson formula consists geometrically of extending the tangent line at a current point x_n until it crosses zero, then setting the next guess x_{n+1} to the abscissa of that zero-crossing. Note that algebraically, the method derives from the familiar Taylor series expansion of a function in the neighborhood of a point. Near a simple root, the number of significant digits approximately doubles with each step. This very strong convergence property

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makes Newton-Raphson the method of choice for any function, whose derivative can be evaluated efficiently/easily, and whose derivative is continuous and nonzero in the neighborhood of a root. In the last decade, many high-order iterative formulas without memory for modifying the order and efficiency of Newton's scheme have been developed, see [4] for a complete discussion. For more recent results, refer to [7, 8, 12, 13].

In 2011, Geum and Kim in [3] considered a three-point family of methods as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(u_n)\dfrac{f(y_n)}{f'(x_n)}, u_n = \dfrac{f(y_n)}{f(x_n)}, \beta, \mu, \lambda \in \mathbb{R}, \\ x_{n+1} = z_n - W_f(u_n, q_n)\dfrac{f(z_n)}{f'(x_n)}, q_n = \dfrac{f(z_n)}{f(y_n)}, \end{cases} \tag{1}$$

wherein $K_f(u_n) = \frac{1+\beta u_n+\lambda u_n^2}{1+(\beta-2)u_n+\mu u_n^2}$, $W_f(u_n, q_n) = \frac{1}{1-2u_n-q_n}$. Note that the first two steps of (1) shows a parametric extension of King's fourth-order iteration with two additional parameters λ and μ to be controlled by the users to find an efficient class. Moreover, we remark that if $\lambda = \mu = 0$, then the first two steps reduce to the King's uni-parametric family. They furthermore, investigated that (1) reaches the sixth-order of convergence and by choosing the appropriate values for the parameters, i.e. $\lambda = (1/2)(\beta - 2)$ and $\mu = (-3/2)\beta$, the iterative method will reduce to an eighth-order optimal family as comes next

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{1 + \beta u_n + (1/2)(\beta - 2)u_n^2}{1 + (\beta - 2)u_n + (-3/2)\beta u_n^2}\dfrac{f(y_n)}{f'(x_n)}, u_n = \dfrac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \dfrac{1}{1 - 2u_n - q_n}\dfrac{f(z_n)}{f'(x_n)}, q_n = \dfrac{f(z_n)}{f(y_n)}, \end{cases} \tag{2}$$

with the following error equation $e_{n+1} = (-1/2)(c_2^2(2c_2^2 - c_3)(-2c_4 + c_2^3(4 + 3\beta)))e_n^8 + O(e_n^9)$. Clearly, selecting any different values of β will result in various without memory methods of optimal convergence order eight. An example from family (2), could be given as follows ($\beta = 4$)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{1 + 4u_n + u_n^2}{1 + 2u_n - 6u_n}\dfrac{f(y_n)}{f'(x_n)}, u_n = \dfrac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \dfrac{1}{1 - 2u_n - q_n}\dfrac{f(z_n)}{f'(x_n)}, q_n = \dfrac{f(z_n)}{f(y_n)}. \end{cases} \tag{3}$$

For more information on this topic, one may refer to [1, 2, 10, 11]. This paper simply generalizes scheme (2). We present an optimal three-step three-point eighth-order class of iterations with one free parameter as (2). We also obtain (2) as an especial case from our class of without memory iterations. Our class of iterations is obtained using weight function approach in the formula (2). The convergence order of the proposed methods in this paper is found to be 2^3, being optimally consistent with the conjecture of Kung-Traub [5]. The efficiency index defined by $EI = p^{1/d}$, with p as the convergence order and d the number of new evaluations of $f(x)$ plus its derivatives per iteration for the contributed class, is $8^{1/4} \approx 1.682$, better than 1.414, the efficiency index of Newton's iteration and 1.587 of King's family efficiency. Some new methods from our class are also given in Section 3 and a numerical comparison with the existing eighth-order methods is furnished in Section 4. Finally, a short discussion will be drawn in Section 5.

2. Derivation of the New Class

In order to present a general eighth-order three-point class of methods, which are also consistent with the optimality conjecture of Kung-Traub, i.e. using only four pieces of information per full computing step, as well as being a generalization of (2), we consider the family (2) with the use of weight functions at the end of the first and third steps as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \times G(t_n), t_n = \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{1+\beta u_n + (1/2)(\beta-2)u_n^2}{1+(\beta-2)u_n + (-3/2)\beta u_n^2} \dfrac{f(y_n)}{f'(x_n)}, u_n = \dfrac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \dfrac{1}{1-2u_n-q_n} \dfrac{f(z_n)}{f'(x_n)} \times H(r_n), q_n = \dfrac{f(z_n)}{f(y_n)}, r_n = \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{4}$$

Taking into account a weight function at the end of the first step as we did above, in fact, give a generality to the celebrated Newton's iteration for the development of higher-order methods. Besides, we take into consideration a weight function at the end of the third step to present a general class. The only challenge here is to attain the weight functions such that the order arrives at eight. This is discussed in Theorem 1.

Theorem 1. *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D, which includes x_0 as the initial approximation of α. Then, the class of methods (4) is of order eight, when*

$G(0)=1$, $G'(0)=G''(0)=0$ and $|G^{(3)}(0)|\le\infty$, $H(0)=1$, $H'(0)=0$ and $|H''(0)|\le\infty$.

Proof. By defining $e_n=x_n-\alpha$ as the error of the iterative scheme in each iteration of (4), applying the Taylor's series expansion and taking into account $f(\alpha)=0$, we have $f(x_n)=f'(\alpha)[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+c_7e_n^7+c_8e_n^8+O(e_n^9)]$, where $c_k=(\frac{1}{k!})\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k\ge 2$. Furthermore, we have $f'(x_n)=f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+8c_8e_n^7+O(e_n^8)]$. Dividing these two formulas to one another gives us $\frac{f(x_n)}{f'(x_n)}=e_n-c_2e_n^2+2(c_2^2-c_3)e_n^3+(7c_2c_3-4c_2^3-3c_4)e_n^4+\ldots+O(e_n^8)$. Moreover using $G(0)=1$ and $G'(0)=G''(0)=0$, it could be attained that $z_n-\alpha=\left(2c_2^3-c_2c_3\right)e_n^4+\left(-2\left(6c_2^4-7c_2^2c_3+c_3^2+c_2c_4\right)+\frac{1}{3}c_2G^{(3)}(0)\right)e_n^5+(-88c_2^3c_3+c_2^5(50+\frac{3\beta}{2})+c_2^2(21c_4-\frac{13}{6}G^{(3)}(0))+c_3(-7c_4+\frac{5}{6}G^{(3)}(0))+c_2(30c_3^2-3c_5+\frac{1}{12}G^{(4)}(0)))e_n^6+\ldots+O(e_n^8)$. By Taylor series expanding in the third step, we get that

$$\frac{f(z_n)}{f'(x_n)}\times H(r_n)=(2c_2^3-c_2c_3)H(0)e_n^4$$

$$\frac{1}{3}H(0)(-6(8c_2^4-8c_2^2c_3+c_3^2+c_2c_4)+c_2G^{(3)}(0))e_n^5+\ldots+O(e_n^8). \qquad (5)$$

We also attain $z_n-\frac{f(z_n)}{f'(x_n)}\times H(r_n)=-c_2(2c_2^2-c_3)(-1+H(0))e_n^4+(1/3)(6c_2^2c_3(7-8H(0))+6c_3^2(-1+H(0))+12c_2^4(-3+4H(0))+c_2(-1+H(0))(6c_4-G^{(3)}(0)))e_n^5+(c_2^5(50-(3/2)\beta(-1+H(0))-82H(0))+2c_2^3c_3(-44+63H(0))-2c_2^4H'(0)+(1/6)c_3(-1+H(0))(42c_4-5G^{(3)}(0))+c_2^2(c_4(21-25H(0))+c_3H'(0)+(1/6)(-13+17H(0))(G^{(3)})(0))+(1/12)c_2(c_3^2(360-444H(0))+(-1+H(0))(36c_5-G^{(4)}(0))))e_n^6+\ldots+O(e_n^9)$.

Now, it is necessary to choose $H(0)=1$, $H'(0)=0$ and $|H''(0)|\le\infty$ in the third step. Thus, we finally obtain the following general eighth-order error equation by re-using Taylor's series expansion

$$e_{n+1}=-\frac{1}{2}\left(c_2^2\left(2c_2^2-c_3\right)\left(-2c_4+c_2^3(4+3\beta)+c_2H''(0)+G^{(3)}(0)\right)\right)e_n^8+O(e_n^9). \qquad (6)$$

This shows that our class (4) reaches the eighth-order of convergence using only three evaluations of the function and one of its first order derivatives, i.e. the same to (2) and (3), but it is general and even consist (2) as its especial element. Our novel class is also free from second derivative computation per full iteration. The proof is now complete. □

3. Some Novel Methods

This sections includes some efficient contributed methods from our class of iteration (4). It is crystal clear that choosing $G(0)=1$, $G'(0)=G''(0)=$

$G^{(3)}(0) = 0$, $H(0) = 1$, $H'(0) = H''(0) = 0$, will reduce to the family of uni-parametric methods given by Geum and Kim (2). Other methods could be defined as follows (with $t_n = \frac{f(x_n)}{f'(x_n)}$, $u_n = \frac{f(y_n)}{f(x_n)}$, $q_n = \frac{f(z_n)}{f(y_n)}$, and $r_n = \frac{f(y_n)}{f'(x_n)}$,)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}(1 + \dfrac{1}{3}t_n^3), \\ z_n = y_n - \dfrac{1-(4/3)u_n - (10/6)u_n^2}{1-(10/3)u_n + 2u_n^2}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{1}{1-2u_n - q_n}\dfrac{f(z_n)}{f'(x_n)}(1 + r_n^3), \end{cases} \tag{7}$$

with $e_{n+1} = c_2^2(2c_2^2 - c_3)(-1 + c_4)e_n^8 + O(e_n^9)$ as its error equation, and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}(1 + t_n^8), \\ z_n = y_n - \dfrac{1-u_n^2}{1-2u_n}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{1}{1-2u_n - q_n}\dfrac{f(z_n)}{f'(x_n)}(1 + \dfrac{1}{100}r_n^2), \end{cases} \tag{8}$$

with $e_{n+1} = -\frac{1}{100}(c_2^2(2c_2^2 - c_3)(c_2 + 200c_2^3 - 100c_4))e_n^8 + O(e_n^9)$ as its simple error equation.

We here remark that such iterative without memory efficient three points methods are so useful for constructing higher order methods according to the procedures given by Sargolzaei and Soleymani in [6]; and also by Soleymani and Sharifi in [14, 15].

Test Functions	Roots	Starting Points
$f_1(x) = 3x + \sin(x) - e^x$	$\alpha_1 \approx 0.360421702960324$	0.1
$f_2(x) = \sin(x) - 0.5$	$\alpha_2 \approx 0.523598775598299$	1
$f_3(x) = x^2 - e^x - 3x + 2$	$\alpha_3 \approx 0.257530285439861$	1
$f_4(x) = x^3 + 4x^2 - 10$	$\alpha_4 \approx 1.365230013414097$	2
$f_5(x) = xe^{-x} - 0.1$	$\alpha_5 \approx 0.111832559158963$	-0.3
$f_6(x) = x^3 - 10$	$\alpha_6 \approx 2.15443490031884$	3.6
$f_7(x) = 10xe^{-x^2} - 1$	$\alpha_7 \approx 1.679630610428450$	1.1

Table 1: The test functions considered in this paper

Test Functions	KT8	SM8	(3)	(7)	(8)
$\lvert f_1(x_2)\rvert$	0.3e-43	0.2e-55	0.5e-63	0.6e-55	0.2e-72
$\lvert f_2(x_2)\rvert$	0.1e-13	0.2e-18	0.4e-27	0.2e-18	0.3e-29
$\lvert f_3(x_2)\rvert$	0.4e-38	0.1e-51	0.4e-60	0.9e-29	0.1e-37
$\lvert f_4(x_2)\rvert$	0.7e-11	0.2e-27	0.5e-31	0.8e-26	0.8e-33
$\lvert f_5(x_2)\rvert$	0.2e-17	0.4e-21	0.7e-24	0.5e-28	0.4e-28
$\lvert f_6(x_2)\rvert$	0.3e-4	0.6e-15	0.9e-17	0.7e-13	0.3e-19
$\lvert f_7(x_2)\rvert$	Div.	0.3e-24	0.2e-37	0.1e-29	0.5e-36

Table 2: Results of comparisons for different methods after two full iterations

4. Numerical Testing

This section tries to provide a valid comparison among the obtained new eighth-order methods and some of the existing ones in literature. We have computed the root of each test function for the initial guess x_0, while the iterative schemes were stopped, when $|f(x_n)| \leq 10^{-600}$. We have used Div. when the iteration diverges for the considered starting point. In compared methods in this paper, it is necessary to begin with one initial approximation x_0. The test functions and their simple roots are given in Table 1.

For comparisons, we have used the eighth-order methods (7) and (8) with the eighth-order derivative-free family of Kung-Truab (KT8) with 1 as its free parameter [5], the eighth-order methods given by Soleymani and Mousavi (SM8), Equation (8) in [9], and also the eighth-order method of Geum and Kim (3). To save the space, we do not give the iterations here again and readers may consult to their papers for more information. The results are summarized in Tables 2 and 3 after two and three full iterations respectively. As they show, novel schemes are comparable with all of the methods of the same order. All numerical instances were performed by MATLAB 7.6 using 600 digits floating point arithmetic (VPA:=600). The total number of iterations for the improved methods, (7) and (8), are the same with all iterative methods considered in comaprisons.

5. Concluding Remarks

The behavior of root-finding algorithms is studied in numerical analysis. Algorithms perform best when they take advantage of known characteristics of the

Test Functions	KT8	SM8	(3)	(7)	(8)
$\lvert f_1(x_3)\rvert$	0.1e-349	0.4e-448	0.1e-510	0.3e-445	0.7e-586
$\lvert f_2(x_3)\rvert$	0.6e-112	0.8e-150	0.4e-220	0.4e-150	0.5e-238
$\lvert f_3(x_3)\rvert$	0.1e-313	0.2e-424	0.5e-492	0.2e-239	0.1e-312
$\lvert f_4(x_3)\rvert$	0.9e-94	0.1e-229	0.2e-259	0.1e-217	0.2e-274
$\lvert f_5(x_3)\rvert$	0.8e-138	0.2e-168	0.1e-190	0.1e-224	0.5e-225
$\lvert f_6(x_3)\rvert$	0.7e-40	0.1e-129	0.2e-145	0.5e-114	0.1e-165
$\lvert f_7(x_3)\rvert$	Div.	0.7e-197	0.3e-303	0.1e-240	0.1e-292

Table 3: Results of comparisons for different methods after three full iterations

given function. Toward this end, this paper contributes a general efficient class of three-step without memory iterations, which reach the optimal order eight. Furthermore, we observe from Tables 2-3 that the three-point without memory method (8) produces approximations of higher accuracy compared to the three-point methods of order eight. The contributed class has the optimal efficiency index 1.682. Accordingly, our methods from the class could be considered as new developments in this field of research.

References

[1] I.K. Argyros, S. Hilout, On the local convergence of a two-step Steffensen-type method for solving generalized equations, *Proyecciones*, **27** (2008), 319-330.

[2] A. Galantai, C.J. Hegedus, A study of accelerated Newton methods for multiple polynomial roots, *Numer. Algorithms*, **54** (2010), 219-243.

[3] Y.H. Geum, Y.I. Kim, A uniparametric family of three-step eighth-order multipoint iterative methods for simple roots, *Appl. Math. Lett.*, **24** (2011), 929-935.

[4] A. Iliev, N. Kyurkchiev, Nontrivial Methods in Numerical Analysis (Selected Topics in Numerical Analysis), Lambert Acad. Publishing (2010).

[5] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, *J. ACM*, **21** (1974), 643-651.

[6] P. Sargolzaei, F. Soleymani, Accurate fourteenth-order methods for solving nonlinear equations, *Numer. Algorithms* (2011), doi: 10.1007/s11075-011-9467-4.

[7] F. Soleymani, Concerning some sixth-order iterative methods for finding the simple roots of nonlinear equations, *Bull. Math. Anal. Appl.*, **2** (2010), 146-151.

[8] F. Soleymani, Regarding the accuracy of optimal eighth-order methods, *Math. Comput. Modelling*, **53** (2011), 1351-1357.

[9] F. Soleymani, B.S. Mousavi, A novel computational technique for finding simple roots of nonlinear equations, *Int. J. Math. Anal.*, **5** (2011), 1813-1819.

[10] F. Soleymani, A novel and precise sixth-order method for solving nonlinear equations, *Int. J. Math. Models Meth. Appl. Sci.*, **5** (2011), 730-737.

[11] F. Soleymani, On a bi-parametric class of optimal eighth-order derivative-free methods, *Int. J. Pure Appl. Math.*, **72** (2011), 27-37.

[12] F. Soleimani, F. Soleymani, Computing simple roots by a sixth-order iterative method, *Int. J. Pure Appl. Math.*, **69** (2011), 41-48.

[13] F. Soleymani, V. Hosseniabadi, New third- and sixth-order derivative-free techniques for nonlinear equations, *J. Math. Res.*, **3** (2011), 107-112.

[14] F. Soleymani, M. Sharifi, On a general efficient class of four-step root-finding methods, *Int. J. Math. Comput. Simul.*, **5** (2011), 181-189.

[15] F. Soleymani, M. Sharifi, On a class of fifteenth-order iterative formulas for simple roots, *Int. Elec. J. Pure Appl. Math.*, **3** (2011), 245-252.