International Journal of Pure and Applied Mathematics
Volume 72 No. 2 2011, 203-216

ON STRONGLY NEGATIVE DEFINITE FUNCTIONS FOR THE PRODUCT OF COMMUTATIVE HYPERGROUPS

A.S. Okb El Bab[1], Hossam A. Ghany[2], S. Ramadan[3 §]

[1,3]Department of Mathematics
Faculty of Science
Al Azhar University
Naser City, Cairo, EGYPT

[2]Department of Mathematics
Faculty of Industrial Education
Helwan University
Al-Ameraia, Cairo, EGYPT

Abstract: We study strongly negative definite functions on the product dual hypergroups and use their properties to give a proof of the Lévy-khin čin formula. Finally, as an application we give the Lévy-khinčin formula for negative definite functions defined on Jacobi polynomial hypergroups.

AMS Subject Classification: 43A62, 43A22
Key Words: hypergroup, strongly negative definite

1. Introduction

The study of harmonic analysis on hypergroups was initiated through the fundamental papers of Dunkl [8], Jewett [9] and Spector [14]. Most of the subsequent work on hypergroups has dealt with the problem of extending known results for topological groups to hypergroups. A hypergroup (see [9] and [6]) is a locally compact Hausdorff space K with a certain convolution structure $*$ on the space of complex Radon measures on K, $M(K)$. We will denote by $C(K)$, $C_b(K)$, $C_o(K)$ and $C_c(K)$ the spaces of all continuous functions on K, that are bounded, vanish at infinity and with compact support respectively. By $M(K)$, $M_b(K)$ and $M^1(K)$ we denote the spaces of all complex Radon measures, bounded Radon

Received: July 27, 2011

§Correspondence author

measures and probability Radon measures on K respectively. Let K_1, K_2 two hypergroups with convolutions $*_1, *_2$ respectively, this paper is devoted to give the Lévy-Khiňchin formula for strongly negative definite functions defined on the dual product hypergroup $\widehat{K_1 \times K_2} = \{(\chi_1, \chi_2) : \chi_i \in \widehat{K_i}, i = 1, 2\}$ with convolution $*$ defined on $M(\widehat{K_1 \times K_2})$, where the locally bounded measurable function $\chi : K \longrightarrow \mathbb{C}$ is called a semicharacter if $\chi(e) = 1$ and $\chi(x * y^-) = \chi(x)\overline{\chi(y)}$ for all $x, y \in K$. Every bounded semicharacter is called a character. If the character is not locally null then (see [3], 1.4.33) it must be continuous. The dual $\widehat{K}$ of K is just the set of continuous characters with the compact-open topology in which case K must be locally compact and a hypergroup under pointwise multiplication. We will concentrate our efforts in this paper with commutative hypergroups K_1, K_2 which admit a unique Haar measures ω_{K_1}, ω_{K_2} respectively, due to Dunkl [8].

2. Strongly Negative Definite Functions

Let $(K_1 \times K_2)$ be a commutative hypergroup such that its dual $\widehat{K_1 \times K_2}$ is a hypergroup under pointwise multiplication.

Definition 1. Let A be a non-empty subset of $\widehat{K_1 \times K_2}$. A locally bounded measurable function $\phi \in C(A)$ is called strongly positive definite on A if there exists a measure $\mu \in M_b^+(K_1 \times K_2)$ (which is necessarily unique) satisfying

$$\hat{\mu}(\chi, \rho) = \begin{cases} \varphi_1(\chi) + \varphi_2(\rho) & (\chi, \rho) \in A \\ 0 & (\chi, \rho) \notin A \end{cases}$$

A locally bounded measurable function $\psi \in C(A)$ is said to be strongly negative definite if $\psi(\mathbf{1}_1, \mathbf{1}_2) \geq 0$ and $\exp(-t\psi)$ is strongly positive definite on A for each $t > 0$, where

$$\psi(\chi, \rho) = \psi_1(\chi) + \psi_2(\rho), \qquad \text{for} \quad (\chi, \rho) \in A$$

The sets of strongly positive definite and strongly negative definite functions on A will be abbreviated by $SP(A)$ and $SN(A)$ respectively; when $A = \widehat{K_1 \times K_2}$ we omit reference to the support set.

Clearly each strongly positive (negative) definite function is positive (negative) definite but the converse implication does not hold. Negative definiteness is an analogue of one half of Schoenberg's duality result, It is not known for which hypergroups, negative definiteness implies strong negative definiteness, see [5], since the definitions of ϕ and ψ depend on ϕ_i and ψ_i which are defined on subsets of $\widehat{K}_i$, $i = 1, 2$.

Theorem 2. *Let $(\mu_t^i)_{t\succ 0}$ be a convolution semigroup on K_i, $i=1,2$ associated with the strongly negative definite function ψ_i defined on $\widehat{K}_i$. Then the function $\psi_1(\chi)+\psi_2(\rho)$ is a strongly negative definite function for $(\chi,\rho)\in\widehat{K_1\times K_2}$ and the associated convolution semigroup $(\mu_t)_{t>0}$ is given by*

$$\mu_t = \mu_t^1 * \mu_t^2.$$

Proof. From the definition of strong negative definite functions, we note that $\psi_1(\chi)+\psi_2(\rho)\in SN(\widehat{K_1\times K_2})$ for $(\chi,\rho)\in(\widehat{K_1\times K_2})$, and

$$\begin{aligned}(\widehat{\mu_t^1 * \mu_t^2}) &= \hat{\mu}_t^1\hat{\mu}_t^2 = \exp(-t\psi_1)\exp(-t\psi_2)\\ &= \exp(-t(\psi_1+\psi_2)).\end{aligned}$$

Since

$$\psi(\chi,\rho)=\psi_1(\chi)+\psi_2(\rho), \qquad for\ (\chi,\rho)\in\widehat{K_1\times K_2}, with \exp(-t\psi)=\hat{\mu}_t$$

then, $\mu_t=\mu_t^1 * \mu_t^2$. ☐

Proposition 3. *If $\phi\in SP(\widehat{K_1\times K_2})$ then*

$$\psi := \phi(\mathbf{1}_1,\mathbf{1}_2)-\phi\in SN(\widehat{K_1\times K_2}).$$

Proof. Clearly ψ is locally bounded measurable function, belongs to $C(\widehat{K_1\times K_2})$ and $\psi(\mathbf{1}_1,\mathbf{1}_2)\geq 0$.We have to show that $\exp(-t\psi)\in SP(\widehat{K_1\times K_2})$ for all $t>0$.

Consider $t>0$ fixed and, for each $n\geq 1$, write

$$\phi_n := \left(1-\frac{t(\phi(\mathbf{1}_1,\mathbf{1}_2)-\phi)}{n}\right)^n.$$

By assumption there exists a measure $\mu\in M_b^+(K_1\times K_2)$ such that $\widehat{\mu}=\phi$. For every $n\geq 1$ we introduce the measure

$$\nu_n := \left(\varepsilon_{(e_1,e_2)}-\frac{t(\|\mu\|\,\varepsilon_{(e_1,e_2)}-\mu)}{n}\right)^{*n}\in M_b(K_1\times K_2)$$

where $*n$ denotes the n-fold convolution of the term in the main brackets. It turns out that for n sufficiently large ($\geq t\,\|\mu\|$) the measures

$$\nu_n^{'} := \left(\varepsilon_{(e_1,e_2)}-\frac{t(\|\mu\|\,\varepsilon_{(e_1,e_2)}-\mu)}{n}\right)\in M^+(K_1\times K_2).$$

Moreover

$$\begin{aligned}\lim_{n\to\infty}\hat{\nu}_n(\chi,\rho) &= \lim_{n\to\infty}\left(1-\frac{t(\phi(\mathbf{1}_1,\mathbf{1}_2)-\phi(\chi,\rho))}{n}\right)^n\\ &= \exp(-t(\phi(\mathbf{1}_1,\mathbf{1}_2)-\phi(\chi,\rho)))\end{aligned}$$

for all $(\chi,\rho)\in\widehat{K_1\times K_2}$. From Levy's continuity theorem [5] we then infer that there exists a measure $\nu\in M_b^+(K_1\times K_2)$ with

$$\hat{\nu}=\exp(-t(\phi(\mathbf{1}_1,\mathbf{1}_2)-\phi))$$

whence $\psi\in SN(\widehat{K_1\times K_2})$. $\square$

Proposition 4. *Let* $\psi\in SN(\widehat{K_1\times K_2})$ *and* $\psi(\mathbf{1}_1,\mathbf{1}_2)\geq 0$, *then* $\dfrac{1}{\psi}\in SP(\widehat{K_1\times K_2})$.

Proof. By assumption $\exp(-t\psi)\in SP(\widehat{K_1\times K_2})$ for all $t>0$. Thus for all $t>0$, $(\chi,\rho)\in\widehat{K_1\times K_2}$ we obtain

$$|\exp(-t\psi)|\leq|\exp(-t\psi(\mathbf{1}_1,\mathbf{1}_2))|$$

and it follows that $\psi(\chi,\rho)>0$ as well as

$$\frac{1}{\psi(\chi,\rho)}=\int_0^\infty\exp(-t\psi(\chi,\rho))dt=\int_0^\infty\hat{\mu}_t(\chi,\rho)dt$$

where μ_t is the representing measure for $\exp(-t\psi)$. Furthermore

$$\int_0^\infty\hat{\mu}_t(\chi,\rho)dt=\int_0^\infty\mu_t(\chi,\rho)^-dt=\nu(\chi,\rho)=\hat{\nu}(\chi,\rho)$$

where

$$\nu:=\int_0^\infty\mu_t dt\geq 0.$$

Finally

$$\nu(\mathbf{1}_1,\mathbf{1}_2)=\frac{1}{\psi(\mathbf{1}_1,\mathbf{1}_2)}<\infty$$

and consequently $\nu\in M_b^+(K_1\times K_2)$, thus proving the assertion. $\square$

3. The Lévy-Khinčin Representation

Let $Z = \{\eta \in M^1(K_1 \times K_2) : \eta = \bar{\eta}, \text{supp}\ \eta \text{ compact}\}$.

Lemma 5. *Let U be a compact neighborhood of $(e_1, e_2) \in K_1 \times K_2$. Then there exists a $\eta \in Z$ such that $-\frac{1}{2} \leq \check{\eta}(x, y) \leq \frac{1}{2}$ for each $(x, y) \in (K_1 \times K_2)\backslash U$.*

Proposition 6. *Let (μ_t) be a convolution semigroup on $K_1 \times K_2$ and a locally bounded measurable function $\psi \in C(K_1 \times K_2)$ be the strongly negative definite function associated to (μ_t). The net $(\frac{1}{t} \cdot \mu_t \,|(K_1 \times K_2)\backslash \{(e_1, e_2)\})_{t>0}$ converges vaguely as $t \to 0$ to a positive measure μ on $(K_1 \times K_2)\backslash \{(e_1, e_2)\}$. For each $\eta \in Z$ the function $\psi * \eta - \psi$ is strongly positive definite and bounded. There exists a measure $\mu_\eta \in M^+(K_1 \times K_2)$ such that $\widehat{\mu_\eta} = \psi * \eta - \psi$. These measures satisfy*

$$(1 - \check{\eta})\mu = \mu_\eta \,|(K_1 \times K_2)\backslash \{(e_1, e_2)\} \text{ for } \eta \in Z. \tag{1}$$

The equation (1) determines μ uniquely, and then we have

$$\int\limits_{(K_1 \times K_2)\backslash\{(e_1,e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu(x, y) < \infty \ \text{ for each } \chi \in \widehat{K_1} \text{and } \rho \in \widehat{K_2}.$$

Proof. Let $\eta \in Z$. For $(\chi, \rho) \in \widehat{K_1 \times K_2}$ we have

$$\begin{aligned} \frac{1}{t}[\widehat{(1 - \check{\eta})\mu_t}](\chi, \rho) &= \frac{1}{t}\left[\widehat{\mu_t}(\chi, \rho) - \widehat{\mu_t} * \widehat{\bar{\eta}}(\chi, \rho)\right] \\ &= \frac{1}{t}\,[1 - \exp(-t\psi)] * (\eta - \varepsilon_{(\mathbf{1}_1, \mathbf{1}_2)})(\chi, \rho) \end{aligned}$$

and $\lim_{t\to 0} \dfrac{1}{t}\,[1 - \exp(-t\psi)] = \psi$ uniformly on compact subsets of $\widehat{K_1 \times K_2}$, Thus

$$\lim_{t\to 0} \frac{1}{t}\,[1 - \exp(-t\psi)] * (\eta - \varepsilon_{(\mathbf{1}_1, \mathbf{1}_2)}) = \psi * \eta - \psi$$

uniformly on compact subsets of $\widehat{K_1 \times K_2}$ and $\psi * \eta - \psi$ is strongly positive definite and bounded. Therefore

$$\lim_{t\to 0} \frac{1}{t}[\widehat{(1 - \check{\eta})\mu_t}]\pi = (\psi * \eta - \psi)\pi$$

vaguely in the space of all Radon measures on $\widehat{K_1 \times K_2}$. There exists a measure $\mu_\eta \in M^+(K_1 \times K_2)$ such that $\widehat{\mu_\eta} = \psi * \eta - \psi$ and $\lim_{t\to 0} \dfrac{1}{t}\,[(1 - \check{\eta})\mu_t] = \mu_\eta$

in the weak topology, see [4] in the one dimensional case. By using the above lemma and [1] now show that there exists a positive Radon measure μ on $(K_1 \times K_2) \setminus \{(e_1, e_2)\}$ such that $\mu = \lim_{t \to 0} (\frac{1}{t}) \cdot \mu_t \,|(K_1 \times K_2) \setminus \{(e_1, e_2)\}$ vaguely and

$$(1 - \check{\eta})\mu_t = \mu_\eta \,|(K_1 \times K_2) \setminus \{(e_1, e_2)\}$$

which is (1). The uniqueness of the measure μ_η can be obtained. Let $\eta = \frac{1}{2}(\varepsilon_{(\chi,\rho)} + \varepsilon_{(\chi,\rho)^-}) \in Z$ for $(\chi, \rho) \in \widehat{K_1 \times K_2}$. Then by (1)

$$\int\limits_{(K_1 \times K_2) \setminus \{(e_1, e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu(x, y) = \mu_\eta \left((K_1 \times K_2) \setminus \{(e_1, e_2)\}\right) < \infty,$$

for each $(\chi, \rho) \in \widehat{K_1 \times K_2}$. □

The positive complex Radon measure μ on $(K_1 \times K_2) \setminus \{(e_1, e_2)\}$ is called the Levy measure Of (μ_t).

Lemma 7. *A continuous function $l : \widehat{K_1 \times K_2} \to \mathbb{R}$ such that $l(\mathbf{1}_1, \mathbf{1}_2) = 0$ is a homomorphism if and only if $l * \eta - l = 0$ for all $\eta \in Z$.*

Proof. Let $l(\chi_1, \chi_2) = l_1(\chi_1)\ l_2(\chi_2)$, where l_1 and l_2 are homomorphisms on $\widehat{K_i}$, $i = 1, 2$ described as in ([1], lemma 18.13). Let l is a homomorphism, then for $(\chi_1, \chi_2) \in \widehat{K_1 \times K_2}$

$$\begin{aligned} \int\limits_{\widehat{K_1 \times K_2}} l(\chi_1, \chi_2) d\eta(\chi_1, \chi_2) &= \int\limits_{\widehat{K_1} \times \widehat{K_2}} l_1(\chi_1) l_2(\chi_2) d\eta_1(\chi_1) d\eta_2(\chi_2) \\ &= \int\limits_{\widehat{K_1}} l_1(\chi_1) d\eta_1(\chi_1) \int\limits_{\widehat{K_2}} l_2(\chi_2) d\eta_2(\chi_2) \\ &= 0, \end{aligned}$$

where $\sigma_i, i = 1, 2$ are defined on $\widehat{K_i}$, $i = 1, 2$.Then

$$\begin{aligned} l * \eta(\chi_1, \chi_2) &= (l_1(\chi_1) *_1 d\eta_1(\chi_1))\,(l_2(\chi_2) *_2 d\eta_2(\chi_2)) \\ &= l_1(\chi_1) l_2(\chi_2) \\ &= l(\chi_1, \chi_2). \end{aligned}$$

Conversely, let $l * \eta - l = 0$, for (χ_1, χ_2) and $(\chi_{1^-}, \chi_{2^-}) \in \widehat{K_1 \times K_2}$

$$l\left((\chi_1, \chi_2) + (\chi_{1^-}, \chi_{2^-})\right) = l\left((\chi_1 + \chi_{1^-}, \chi_2 + \chi_{2^-})\right)$$

$$\begin{aligned} &= l_1(\chi_1 + \chi_{1-}) l_2(\chi_2 + \chi_{2-}) \\ &= l_1(\chi_1) l_2(\chi_2) + l_1(\chi_1) l_2(\chi_{2-}) + \\ &\quad + l_1(\chi_{1-}) l_2(\chi_2) + l_1(\chi_{1-}) l_2(\chi_{2-}), \end{aligned}$$

in particular, put $l_1(\chi_1)l_2(\chi_{2-}) + l_1(\chi_{1-})l_2(\chi_2) = 0$, then
$l((\chi_1, \chi_2) + (\chi_{1-}, \chi_{2-})) = l(\chi_1, \chi_2) + l(\chi_{1-}, \chi_{2-})$. □

Lemma 8. *Let $q : \widehat{K_1 \times K_2} \to \mathbb{R}$ be a continuous with $q(\alpha, \beta) = q(\alpha, \beta)^-$, $q(\mathbf{1}_1, \mathbf{1}_2) = 0$, then q is a quadratic form if and only if $q * \eta - q$ is a constant function for each $\eta \in Z$. Moreover, in the affirmative case q is nonnegative if and only if $q * \eta - q \geq 0$ for all $\eta \in Z$.*

Proof. Let

$$q(\alpha, \beta) = q(\alpha, \beta)^- = q_1(\overline{\alpha}) + q_2(\overline{\beta}) = q_1(\alpha) + q_2(\beta)$$

where $q_i : \widehat{K_i} \to \mathbb{R}$, $i = 1, 2$ are quadratic forms as in ([1], lemma 18.16), Then for $(\chi_1, \chi_2) \in \widehat{K_1 \times K_2}$

$$\begin{aligned} q * \eta(\chi_1, \chi_2) &= q_1 *_1 \eta_1(\chi_1) + q_2 *_2 \eta_2(\chi_2) \\ &= q_1(\chi_1) + \int\limits_{\widehat{K_1}} q_1(\chi_1) d\eta_1(\chi_1) + q_2(\chi_2) + \int\limits_{\widehat{K_2}} q_2(\chi_2) d\eta_2(\chi_2) \\ &= q_1(\chi_1) + q_2(\chi_2) + \int\limits_{\widehat{K_1}} q_1(\chi_1) d\eta_1(\chi_1) + \int\limits_{\widehat{K_2}} q_2(\chi_2) d\eta_2(\chi_2), \end{aligned}$$

then $q * \eta - q$ is a constant.

Conversely, let $q * \eta - q$ is a constant for all $\eta \in Z$, then we find for $(\chi_1, \chi_2), (\chi_{1-}, \chi_{2-}) \in \widehat{K_1 \times K_2}$ and $\eta_i = \frac{1}{2}(\varepsilon_{\chi_{i-}} + \varepsilon_{-\chi_{i-}})$, $i = 1, 2$, where

$$q * \eta(\chi_1, \chi_2) - q_1(\chi_1, \chi_2) = q_1 *_1 \eta_1(\chi_1) - q_1(\chi_1) + q_2 *_2 \eta_2(\chi_2) - q_2(\chi_2)$$

that

$$q_1 *_1 \eta_1(\chi_1) - q_1(\chi_1) + q_2 *_2 \eta_2(\chi_2) - q_2(\chi_2) = q_1 *_1 \eta_1(e_1) - q_1(e_1) + q_2 *_2 \eta_2(e_2) - q_2(e_2)$$

this equation depends on $q_i : \widehat{K_i} \to \mathbb{R}$, $i = 1, 2$, as constants, hence they are quadratic forms, then the assertion follows. □

Lemma 9. *Assume that μ is a positive measure on $K_1 \times K_2$ which is symmetric on $(K_1 \times K_2) \setminus \{(e_1, e_2)\}$ such that* $\int\limits_{(K_1\times K_2)\setminus\{(e_1,e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu(x,y) < \infty$ *, for each $\chi \in \widehat{K_1}$ and $\rho \in \widehat{K_2}$, Then the function $\psi_\mu : \widehat{K_1 \times K_2} \to \mathbb{R}$,*

$$\psi_\mu(\chi,\rho) = \int\limits_{(K_1\times K_2)\setminus\{(e_1,e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu(x,y)$$

is strongly negative definite.

Proof. From the above proposition, we have

$$\begin{aligned}
\psi_\mu(\chi,\rho) &= \lim_{t\to 0} \frac{1}{t} \left[1 - \exp(-t\psi(\chi,\rho))\right] \\
&= \lim_{t\to 0} \int\limits_{(K_1\times K_2)\setminus\{(e_1,e_2)\}} \left(1 - \overline{\chi(x)\rho(y)}\right) \frac{1}{t} d\mu_t \\
&= \int\limits_{(K_1\times K_2)\setminus\{(e_1,e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu.
\end{aligned}$$

which is the assertion. □

Theorem 10. *Let $\psi : \widehat{K_1 \times K_2} \to \mathbb{C}$ be a strongly negative definite function associated with convolution semigroup (μ_t) and Levy measure (symmetric) μ. Then ψ takes the form*

$$\psi(\chi,\rho) = c + il(\chi,\rho) + q(\chi,\rho) + \int\limits_{(K_1\times K_2)\setminus\{(e_1,e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu, \tag{2}$$

for $(\chi,\rho) \in (K_1 \times K_2)\hat{}$. Where, (c,l,q) are determined uniquely by (μ_t) : $c = \psi(\mathbf{1}_1, \mathbf{1}_2)$, $l = \mathrm{Im}\psi$ and

$$q(\chi,\rho) = \lim_{t\to\infty} \frac{\mathrm{Re}\psi(n(\chi,\rho))}{n^2} + \lim_{t\to\infty} \frac{\psi(n((\chi,\rho) * (\chi,\rho)^-))}{2n}. \tag{3}$$

Proof. Let $\psi \in SN(\widehat{K_1 \times K_2})$, since $c(\geq 0) \in SN(\widehat{K_1 \times K_2})$ then we can put $c = \psi(\mathbf{1}_1, \mathbf{1}_2)$. Assume that $\psi' := \psi - c$ which is still strongly negative definite function and has the same levy measure μ. Put $h := \psi' - \psi_\mu$ and for $\eta \in Z$ we have,

$$h * \eta - h = [\psi' - \psi_\mu] * \eta - [\psi' - \psi_\mu]$$

$$
\begin{aligned}
&= (\psi^{'} * \eta - \psi^{'}) - (\psi_\mu * \eta - \psi_\mu) \\
&= \widehat{\mu}_\eta - [(1-\check{\eta})\mu\hat{]}.
\end{aligned}
$$

Thus,

$$h * \eta - h = \mu_\eta(e_1, e_2) \tag{4}$$

since,

$$[(1-\check{\eta})\mu] = \mu_\eta \left|(K_1 \times K_2) \setminus \{(e_1, e_2)\}\right. .$$

From (4) we have,

$$\mathrm{Re}h * \eta - \mathrm{Re}h = \mu_\eta(e_1, e_2) \geq 0,$$

and

$$\mathrm{Im}h * \eta - \mathrm{Im}h = 0$$

from the above lemmas we get $\mathrm{Re}h = q$ which is nonnegative quadratic form and $\mathrm{Im}h = l$ is a homomorphism.

Since $\mathrm{Re}h + \mathrm{Im}h = \psi - c - \psi_\mu$ then

$$\psi(\chi, \rho) = c + il(\chi, \rho) + q(\chi, \rho) + \int\limits_{(K_1 \times K_2) \setminus \{(e_1, e_2)\}} (1 - \mathrm{Re}\chi(x)\rho(y))\, d\mu.$$

From the integral representation (2), let $\omega = (\chi, \rho) \in (K_1 \times K_2\hat{)}$

$$\frac{\mathrm{Re}\psi(n\omega)}{n^2} = \frac{\psi(\mathbf{1}_1, \mathbf{1}_2)}{n^2} + \frac{q(n\omega)}{n^2} + \int\limits_{(K_1 \times K_2) \setminus \{(e_1, e_2)\}} \frac{1}{n^2}\left[1 - (\mathrm{Re}\chi(x)\rho(y))^n\right] d\mu,$$

and

$$
\begin{aligned}
\frac{\psi(n(\omega * \omega^-))}{n} &= \frac{\psi(\mathbf{1}_1, \mathbf{1}_2)}{n} + \frac{q(n(\omega * \omega^-))}{n} \\
&\quad + \int\limits_{(K_1 \times K_2) \setminus \{(e_1, e_2)\}} \frac{1}{n}\left[1 - |\chi(x)\rho(y)|^{2n}\right] d\mu
\end{aligned}
$$

where

$$\lim \frac{q(n\omega)}{n^2} = q(\omega) - q(\omega * \omega^-)$$

and

$$q(n(\omega * \omega^-)) = nq(\omega * \omega^-)$$

which gives the formula for $q(\omega)$ provided the integrals tend to zero. Let $\chi(x) = r_1 \exp(i\theta_1)$ and $\rho(y) = r_2 \exp(i\theta_2)$, $0 < r_k \leq 1$ and $-\pi < \theta_k \leq \pi$, $k = 1, 2$.

$$\begin{aligned}
\frac{1}{n^2}\left[1 - (\mathrm{Re}\chi(x)\rho(y))^n\right] &= \frac{1 - (r_1 r_2)^n \cos n(\theta_1 + \theta_2)}{n^2} \\
&= \frac{2(r_1 r_2)^n \sin^2[\frac{n(\theta_1 + \theta_2)}{2}]}{n^2} + \frac{1 - (r_1 r_2)^n}{n^2}
\end{aligned}$$

since,

$$\frac{1 - (r_1 r_2)^n}{n^2} = \frac{1}{n^2}(1 - r_1 r_2)[1 + r_1 r_2 + ... + (r_1 r_2)^{n-1}]$$

so,

$$\begin{aligned}
\frac{1}{n^2}\left[1 - (\mathrm{Re}\chi(x)\rho(y))^n\right] &\leq a[2(r_1 r_2)\sin^2[\frac{\theta_1 + \theta_2}{2}] + (1 - r_1 r_2)] \\
&= a[1 - r_1 r_2 \cos(\theta_1 + \theta_2)] \\
&= a\left[1 - \mathrm{Re}\chi(x)\rho(y)\right], \; a > 0
\end{aligned}$$

then,

$$\lim_{t\to\infty} \int\limits_{(K_1 \times K_2)\setminus\{(e_1, e_2)\}} \frac{1}{n^2}\left[1 - (\mathrm{Re}\chi(x)\rho(y))^n\right] d\mu \to 0.$$

By the same,

$$\begin{aligned}
\frac{1}{n}\left[1 - |\chi(x)\rho(y)|^{2n}\right] &= \frac{1}{n}(1 - |\chi(x)\rho(y)|^2)(1 + |\chi(x)\rho(y)|^2 + ... \\
&\quad + |\chi(x)\rho(y)|^{2(n-1)}) \\
&\leq 1 - |\chi(x)\rho(y)|
\end{aligned}$$

but, $|\chi(x)\rho(y)| \leq 1$ so,

$$\lim_{t\to\infty} \int\limits_{(K_1 \times K_2)\setminus\{(e_1, e_2)\}} \frac{1}{n}\left[1 - |\chi(x)\rho(y)|^{2n}\right] d\mu \to 0$$

and the assertion

$$q(\omega) = \lim_{t\to\infty} \frac{\mathrm{Re}\psi(n(\omega))}{n^2} + \lim_{t\to\infty} \frac{\psi(n((\omega) * (\omega)^-))}{2n}$$

for $\omega = \omega = (\chi, \rho) \in (K_1 \times K_2)\hat{}$. $\square$

For more generalization, let K_1, K_2, ..., K_n are hypergroups which satisfying the property that, if $\mathbf{C} \subseteq \widehat{K}$ is compact then there exist a constant $M_c \geq 0$, a neighborhood U_c of e in K and a finite subset N_c of $\mathbf{C}$ such that for each $x \in U_c$

$$\sup\{1 - \mathrm{Re}\chi(x) : \chi \in \mathbf{C}\} \leq M_c \sup\{1 - \mathrm{Re}\chi(x) : \chi \in N_c\}. \tag{5}$$

Obviously each compact or discrete hypergroup satisfies the property (5). It is known that each locally compact abelian group satisfies property (5), see [11].

Lemma 11. *Let K_1, K_2, ..., K_n are hypergroups satisfying property (5) then $K_1 \times K_2 \times ... \times K_n$ satisfies property (5).*

Proof. Let $\chi_k(x_k) = r_k \exp(i\theta_k)$, $0 < r_1, r_2, ..., r_n \leq 1$ and $0 < \theta_1, \theta_2, ..., \theta_n \leq 2\pi$, $k = 1, 2, ..., n$. From the following

$$1 - r_1 r_2 ... r_n \leq (1 - r_1) + (1 - r_2) + ... + (1 - r_n),$$

$$1 - \cos(\theta_1 + \theta_2 + ... + \theta_n) \leq n[(1 - \cos\theta_1) + (1 - \cos\theta_2) + ... + (1 - \cos\theta_n)],$$

we get

$$\begin{aligned} 1 - r_1 ... r_n \cos(\theta_1 + \theta_2 + ... + \theta_n) \quad &\leq \quad (1 - r_1) + ... + (1 - r_n) \\ &\quad + n r_1 ... r_n [(1 - \cos\theta_1) + ... + (1 - \cos\theta_n)] \end{aligned} \tag{6}$$

$$\leq (1 - r_1) + ... + (1 - r_n) + n r_1 (1 - \cos\theta_1) + ... + n r_n (1 - \cos\theta_n)$$

$$\leq n[(1 - r_1 \cos\theta_1) + ... + (1 - r_n \cos\theta_n)],$$

for $\chi(x) = \prod_{k=1}^{n} \chi_k(x_k) \in (K_1 \times K_2 \times ... \times K_n)\hat{}$, where $\chi_k \in \widehat{K_k}$, $k = 1, 2, ..., n$. the form

$$\sup\{1 - \mathrm{Re}\chi(x)\} \leq M_c \sup\{1 - \mathrm{Re}\chi(x)\}$$

will be

$$\sup\{1 - \mathrm{Re} \prod_{k=1}^{n} \chi_k(r_k, \theta_k)\} \leq M_c \sup\{1 - \mathrm{Re} \sum_{k=1}^{n} \chi_k(r_k, \theta_k)\},$$

since $\chi_k(x_k) = r_k \exp(i\theta_k)$, then by (**??**) we can proof the assertion. $\square$

Example 12. (Jacobi Polynomial Hypergroups, see [7]) Let

$$E_j = \{(\alpha, \beta) : \alpha \geq \beta \geq -1 \text{ and either } \beta \geq -\frac{1}{2} \text{ or } \ \alpha + \beta \geq 0\}.$$

Let $R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$, where $\{P_n^{(\alpha,\beta)}\}$ are the Jacobi polynomials which are orthogonal on $[-1,1]$ with respect to the weight $(1-x^\alpha)(1+x\beta)$.Then if$(\alpha,\beta) \in E_j$ there is a hypergroup $J(\alpha,\beta)$ on I with character set

$$\Re^{(\alpha,\beta)} = \{R_n^{(\alpha,\beta)} : n \in \mathbb{N}_o\}.$$

$J(\alpha,\beta)$ is a continuous 1- variable hypergroup, Let $K = K_1 \times K_2$ a product hypergroup with, if $\phi \in \widehat{K}$ then $\phi(\mathbf{x}) =.\phi_1(x_1)\phi_2(x_2)$, for $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$, $\phi_1 \in \widehat{K_1}$ and $\phi_2 \in \widehat{K_2}$. Thus if $(\alpha_1,\beta_1;\alpha_2,\beta_2) \in E_{j_1} \times E_{j_2}$ then $J(\alpha,\beta) = J(\alpha_1,\beta_1) \times J(\alpha_2,\beta_2)$ is a product 2- variables continuous hypergroup.

Example 13. (Mixed Jacobi Hypergroups, see [6]) Consider the mixed Jacobi hypergroups $(\mathbb{R}_+ \times [-\pi,\pi], *)$, For $\alpha \geq 0$, $\lambda \in \mathbb{Z}$, and $\mu \in \mathbb{C}$ let $\phi_\mu^{(\alpha,\lambda)}$denote the Jacobi function associated with parameters α, λ and μ,the functions $\phi_{\mu,\lambda}^{(\alpha)}$ defined by

$$\phi_{\mu,\lambda}^{(\alpha)}(y,\theta) := e^{i\lambda\theta}(\cosh y)^\lambda \phi_\mu^{(\alpha,\lambda)}(y) \quad \text{for all } (y,\theta) \in K := \mathbb{R}_+ \times [-\pi,\pi]$$

satisfy the product formula

$$T_{(y,\theta)}^{(\alpha)} \phi_{\mu,\lambda}^{(\alpha)}(t,\tau) = \phi_{\mu,\lambda}^{(\alpha)}(y,\theta)\phi_{\mu,\lambda}^{(\alpha)}(t,\tau)$$

where $T_{(y,\theta)}^{(\alpha,\lambda)}$ is the generalized translation operator, for (y,θ), $(t,\tau) \in K$ and $f \in C_c$ the product convolution defined by

$$\varepsilon_{(y,\theta)} * \varepsilon_{(t,\tau)}(f) = T_{(y,\theta)}^{(\alpha)} f(t,\tau).$$

The hypergroup $(K,*)$ has an involution product $(y,\theta) \to (y,\theta)^- := (y,-\theta)$,thus

$$(y,\theta) = (t,\tau)^- \Leftrightarrow y = t \text{ and } \cos(\theta+1) = 1.$$

It follows that $(K,*)$ is not hermitian. The dual hypergroup take the form

$$\widehat{K} = \{(\lambda,\mu) \in \mathbb{Z} \times \mathbb{C} : \left\| \phi_{\mu,\lambda}^{(\alpha)} \right\|_\infty \leq 1\}.$$

And the integral representation of a strongly negative definite function f defined on $\widehat{K}$ is

$$\begin{aligned} f(\lambda,\mu) &= a + ib\lambda + \frac{c}{2(\alpha+1)}(\lambda^2 + \mu^2 + (\alpha+1)^2) \\ &+ \int\limits_{K\setminus\{(0,0)\}} (1 + \phi_{\mu,\lambda}^{(\alpha)}(y,\theta) + i\lambda\theta u(y,\theta))(\nu d(y,\theta)) \end{aligned}$$

for all $(\lambda,\mu) \in supp(\pi_K) \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is the unit character of K, $a,b,c,d \in \mathbb{R}$, $a,b,d \geq 0$, u is a function on the class

$$\begin{aligned} D_* \ &: \ = \{f \in \mathbb{C}^\infty(\mathbb{R}\times]-\pi,0[\cup]0,\pi[) \\ &: \ y \longmapsto f(y,\theta) \text{ is even with compact support, } \theta \longmapsto f(y,\theta) \text{ is} \\ & \quad\ 2\pi - \text{periodic}\} \end{aligned}$$

such that $0 < u < 1$ and $u = 1$ on a neighborhood of $(0,0)$, and $\nu \in M_+(K\backslash\{(0,0)\})$ with

$$\int\limits_{K\backslash\{(0,0)\}} \frac{y^2+\theta^2}{1+y^2+\theta^2}(\nu d(y,\theta)) < \infty.$$

For the details about K see [13].

References

[1] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Berlin-Heidelberg-New York, Springer (1975).

[2] C. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Theory of positive definite and related functions, Graduated texts in Math., **100**, Springer-Verlag, Berlin-Heidelberg-New-York (1984).

[3] W.R. Bloom, H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter, Berlin (1995).

[4] W.R. Bloom, H. Heyer, The Fourier transform for probability measures on hypergroups, *Rend. Mat.*, **2** (1982), 315-334.

[5] W.R. Bloom, H. Heyer, Convolution semigroups and resolvent families of measures on hypergroups, *Math. Z.*, **188** (1985), 449-474.

[6] W.R. Bloom, H. Heyer, Negative definite functions and convolution semigroups of probability measures on a commutative hypergroups, *Probability and Mathematical Statistics*, **16** (1996), 157-176.

[7] W.C. Contnet, A.L. Schwartz, *Continuous 2-Variable Polynomial Hypergroups*, In Applications of Hypergroups and Related Measure Algebras, American Mathematical Society (1995).

[8] C.F. Dunkl, The measure algebra of a localy compact hypergroup, *Trans. Amer. Math. Soc.*, **179** (1973), 331-348.

[9] R.I. Jewett, Spaces with an abstract convolution of measures, *Adv. in Math.*, **18** (1975), 1-101.

[10] R. Lasser, *Orthogonal Polynomials and Hypergroups: Contributions to Analysis and Probability Theory*, Technische University of Mtinchen, Institute for Mathematik (1981).

[11] R. Lasser, Convolution semigroups on hypergroups, *Pasfic Journal of Mathematics*, **127**, No. 2 (1987).

[12] R. Lasser, On the Levy-Hinin formula for commutative hypergroups, *Probability Measures on Groups*, Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach (1983), 298-308; *Lecture Notes in Math.*, **1064**, Berlin-Heidelberg-New York, Springer (1984).

[13] M. Sifi, Central limit theorem and infinitely divisible probabilities associated with partial drfferential operators, *J. Theoret. Probab.*, **8**, No. 3 (1995), 47-99.

[14] R. Spector, Mesures invariantes sur les hypergroupes, *Trans. Amer. Math. Soc.*, **239**, 147-165.