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CONVERGENCE THEOREMS FOR TWO VISCOSITY ITERATIVE ALGORITHMS FOR SOLVING EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract: In this paper, we introduce two viscosity iterative algorithms for finding the set of solution for equilibrium problems and fixed point problems in a Hilbert space. We show that the sequence converges strongly to a common element of the above two sets under some parameters controlling conditions. As applications, at the end of paper we utilize our results to study some convergence problem for strictly pseudocontractive mappings and finding the zeros of maximal monotone operators. Our results are generalizations and extensions of the results of Yao and Liou [Iterative Algorithms for Nonexpansive Mapping, *Fixed Point Theory and Applications*, Volume 2008, Article ID 384629, 10 pages.] and Su and Li [Strong convergence theorems on two iterative method for non-expansive mappings, *Applied Mathematics and Computation*, **181**, No. 1 (2006), 332-341.] and some recent results.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively and let C be a closed convex subset of H. A mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

We use $F(T)$ to denote the set of fixed points of T, that is,

$$F(T) = \{x \in C : Tx = x\}.$$

It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \tag{1.2}$$

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problem; see, for instance, see [2, 3, 4, 6, 7, 14].

Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F) \neq \emptyset$ and proved a strong convergence theorem. Motivated by the idea of Combettes and Hirstoaga [2], very recently, Takahashi and Takahashi [14] introduced a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Their results extend and improve the corresponding results announced by Combettes and Hirstoaga [2], Moudafi [5], Wittmann [15], and Tada and Takahashi [13].

On the other hand, Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H. Starting with an arbitrary $x_0 \in H$, defined a sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \ \ n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. In 2006, Su and Li [11] introduced the following two new iterative algorithms for a nonexpansive mapping T : for fixed $u \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(\beta_n u + (1-\beta_n)x_n), \tag{1.5}$$

$$y_{n+1} = \alpha_n(\beta_n u + (1-\beta_n)Ty_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i y_n, \tag{1.6}$$

respectively. Su and Li [11] proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to some fixed point of T under some assumptions. Consequently, Yao and Liou [17], introduced the following new iterative algorithms for a nonexpansive mapping T for fixed $u, v \in C$, let the sequences $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T(\delta_n v + (1-\delta_n)x_n), \tag{1.7}$$

$$x_{n+1} = \beta_n x_n + (1-\beta_n x_n)A_n(\alpha_n u + (1-\alpha_n)Tx_n), \tag{1.8}$$

where $A_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i$. They proved that the sequences $\{x_n\}$ converge strongly to a fixed point of T under some mild assumptions. Many authors studied the problem to finding a common element of the set of fixed points and the set of solutions of an equilibrium problem in the frame work of Hilbert spaces and Banach spaces; see, for instance, [4, 6, 7, 9, 12, 14, 18] and the references therein.

In this paper, motivated by the iterative schemes considered in [2, 3, 4, 6, 9, 12, 18], we introduce the following iterative process: Let F is bifunction of $C \times C$ into real numbers $\mathbb{R}$. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Suppose the sequences $\{r_n\}$ is a real sequence in $(0,\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$ and satisfy under some mild conditions. For fixed v and arbitrary given $x = x_0 \in C$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \ \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n v + (1-\delta_n)u_n) \end{cases} \tag{1.9}$$

and

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \ \forall x \in C, \\ y_n = \delta_n v + (1 - \delta_n)u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n A_n y_n, \end{cases} \tag{1.10}$$

where $A_n = \frac{1}{n+1}\Sigma_{i=0}^{n} T^i$ for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Then, we prove strong convergence theorems which is connected with [4, 9, 12, 17, 18] and some others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$.It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.6}$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1]

We need the following lemmas for proving our main results.

Lemma 2.1. (see Xu [16]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (see Osilike and Igbokwe [8]) *Let $(E, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad -\alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.\end{aligned}$$

Lemma 2.3. (see Blum and Oettli [1]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.4. (see Combettes and Hirstoaga [2]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. T_r *is single- valued;*

2. T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

3. $F(T_r) = EP(F)$;

4. $EP(F)$ *is closed and convex.*

Lemma 2.5. (see Suzuki [10]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$$

for all integers $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Main Theorems

In this section, we prove strong convergence theorems.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *is bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$. *Let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0, 1)$. *Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0, 1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$ *and* $\{r_n\}$ *is a real sequence in* $(0, \infty)$. *Suppose the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For fixed v *and arbitrary given* $x_0 \in C$, *the sequences* $\{x_n\}$ *and* $\{u_n\}$ *generated by* (1.9), *then* $\{x_n\}$ *converges strongly to*

$$z \in F(T) \cap EP(F),$$

where $z = P_{F(T)\cap EP(F)} f(z)$.

Proof. Let $p \in F(T) \cap EP(F)$ and $\alpha \in (0,1)$. Putting $u_n = T_{r_n} x_n$ for all $n \in \mathbb{N} \cup \{0\}$. For fixed $v \in C$ and let $y_n = \delta_n v + (1-\delta_n)u_n$ and $\sigma_n = \alpha_n(1-\alpha) + \gamma_n \delta_n$, we have

$$\|y_n - p\| \leq \delta_n \|v - p\| + (1-\delta_n)\|u_n - p\|, \tag{3.1}$$

it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(Ty_n - p)\| \\
&\leq \alpha_n(\alpha\|x_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| \\
&\quad + \gamma_n\|y_n - p\| \\
&\leq \alpha_n(\alpha\|x_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| \\
&\quad + \gamma_n(\delta_n\|v - p\| + (1-\delta_n)\|u_n - p\|) \\
&\leq (\alpha\alpha_n + \beta_n)\|x_n - p\| + \alpha_n\|f(p) - p\| + \gamma_n\delta_n\|v - p\| \\
&\quad + \gamma_n(1-\delta_n)\|x_n - p\| \\
&= (\alpha\alpha_n + \beta_n + \gamma_n(1-\delta_n))\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\quad + \gamma_n\delta_n\|v - p\|
\end{aligned}$$

$$\begin{aligned}
&= (\alpha\alpha_n + (1-\alpha_n) + \gamma_n(1-\delta_n))\|x_n - p\| \\
&\quad + \alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha} + \gamma_n\delta_n\|v - p\| \\
&\leq (1-\sigma_n)\|x_n - p\| + \sigma_n \max\{\frac{\|f(p) - p\|}{1-\alpha}, \|v - p\|\} \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}, \|v - p\|\}.
\end{aligned} \tag{3.2}$$

By induction on n, we obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\alpha}, \|v - p\|\}$$

for every $n \geq 0$ and $x_0 \in C$. So, $\{x_n\}$, $\{y_n\}$, $\{Ty_n\}$, and $\{u_n\}$ are bounded.

Next, to show that $\|x_{n+1} - x_n\| \to 0$, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|(\delta_{n+1}v + (1-\delta_{n+1})u_{n+1}) - (\delta_n v + (1-\delta_n)u_n)\| \\
&= \|(\delta_{n+1} - \delta_n)v + (1-\delta_{n+1})(u_{n+1} - u_n) \\
&\quad + (-\delta_{n+1} + \delta_n)u_n\| \\
&\leq |\delta_{n+1} - \delta_n|\|v\| + (1-\delta_{n+1})\|u_{n+1} - u_n\|
\end{aligned}$$

$$\begin{aligned} & +|\delta_{n+1}-\delta_n|\|u_n\| \\ \leq\ & |\delta_{n+1}-\delta_n|(\|v\|+\|u_n\|)+\|u_{n+1}-u_n\|. \end{aligned} \tag{3.3}$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we note that

$$F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0,\ \forall x \in C \tag{3.4}$$

and

$$F(u_{n+1}, x) + \frac{1}{r_{n+1}}\langle x - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \tag{3.5}$$

$\forall x \in C$. Putting $x = u_{n+1}$ in (3.4) and $x = u_n$ in (3.5), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2), we have

$$\begin{aligned} u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ \geq \langle u_{n+1} - u_n, u_{n+1} - u_n\rangle. \end{aligned}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have
$\|u_{n+1} - u_n\|^2$

$$\begin{aligned} \leq\ & \|u_{n+1} - u_n\|\|x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\| \\ \leq\ & \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| \ \leq\ & \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}\|r_{n+1} - r_n\|\|u_{n+1} - x_{n+1}\| \\ \leq\ & \|x_{n+1} - x_n\| + \frac{1}{c}\|r_{n+1} - r_n\|L, \end{aligned} \tag{3.6}$$

where $L = sup\|u_n - x_n\| : n \in \mathbb{N}$. Hence by (3.3) and (3.6), we have

$$\|y_{n+1} - y_n\| \ \leq\ |\delta_{n+1} - \delta_n|(\|v\| + \|u_n\|)$$

$$+\|x_{n+1}-x_n\|+\frac{1}{c}\|r_{n+1}-r_n\|L. \tag{3.7}$$

Set $x_{n+1}=\beta_n x_n+(1-\beta_n)z_n$. By (1.9), we see that $z_n=\frac{\alpha_n f(x_n)+\gamma_n T y_n}{1-\beta_n}$. Hence, for $\alpha\in(0,1)$ and (3.7), we have

$$\begin{aligned}
&\|z_{n+1}-z_n\|\\
&= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Ty_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n Ty_n}{1-\beta_n}\|\\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})f(x_n)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(Ty_{n+1}-Ty_n)+(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})Ty_n\|\\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Ty_n\|)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|\\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Ty_n\|)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\{|\delta_{n+1}-\delta_n|(\|v\|+\|u_n\|)+\|x_{n+1}-x_n\|\\
&\quad+\frac{1}{c}\|r_{n+1}-r_n\|L\}\\
&= \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Ty_n\|)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(|\delta_{n+1}-\delta_n|(\|v\|+\|u_n\|)+\frac{1}{c}\|r_{n+1}-r_n\|L)\\
&\quad+\|x_{n+1}-x_n\|-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Ty_n\|)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(|\delta_{n+1}-\delta_n|(\|v\|+\|u_n\|)+\frac{1}{c}\|r_{n+1}-r_n\|L)\\
&\quad-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|.
\end{aligned}$$

It follows from (i), (ii) and (iii), that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.8}$$

From Lemma 2.5 and (3.8), we obtain $\lim_{n\to\infty} \|z_n - x_n\| = 0$ and also

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0.$$

Next we show that $\|x_n - u_n\| \to 0$, as $n \to \infty$ for $p \in F(T) \cap EP(F)$ we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\ &\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2). \end{aligned} \tag{3.9}$$

Then by (3.9) become

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.10}$$

From (2.2) we have

$$\|y_n - p\|^2 = \delta_n\|v - p\|^2 + (1-\delta_n)\|u_n - p\|^2 - \delta_n(1-\delta_n)\|v - u_n\|^2,$$

it follow that

$$\|y_n - p\|^2 \le \delta_n\|v - p\|^2 + (1-\delta_n)\|u_n - p\|^2.$$

Therefore, from Lemma 2.2, (3.1) and (3.10), become

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(Ty_n - p)\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 \\ &\quad +\gamma_n(\delta_n\|v - p\|^2 + (1-\delta_n)\|u_n - p\|^2) \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\delta_n\|v - p\|^2 \\ &\quad +\gamma_n(1-\delta_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2) \\ &= \alpha_n\|f(x_n) - p\|^2 + (\beta_n + \gamma_n(1-\delta_n))\|x_n - p\|^2 \\ &\quad +\gamma_n\delta_n\|v - p\|^2 - \gamma_n(1-\delta_n)\|x_n - u_n\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 + \gamma_n\delta_n\|v - p\|^2 \\ &\quad -(1-\delta_n)\|x_n - u_n\|^2 \end{aligned} \tag{3.11}$$

and hence, from (3.11), we get
$(1-\delta_n)\|x_n - u_n\|^2$

$$\le \alpha_n\|f(x_n) - p\|^2 + \gamma_n\delta_n\|v - p\|^2$$

$$\begin{aligned} & +\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ = \ & \alpha_n \|f(x_n) - p\|^2 + \gamma_n \delta_n \|v - p\|^2 \\ & +(\|x_{n+1} - x_n\|)(\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned} \tag{3.12}$$

Since $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\delta_n \to 0$, imply that
$\|x_n - u_n\| \to 0$.
We show that $\|Ty_n - y_n\| \to 0$, since

$$\begin{aligned} \|x_n - Ty_n\| & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Ty_n\| \\ & = \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Ty_n - Ty_n\| \\ & \leq \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - Ty_n\| + \beta_n \|Ty_n - x_n\| \end{aligned} \tag{3.13}$$

and then
$(1 - \beta_n)\|Ty_n - x_n\|$

$$\leq \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - Ty_n\|. \tag{3.14}$$

Since $\|x_{n+1} - x_n\| \to 0$ and $\alpha_n \to 0$, we have $\|Ty_n - x_n\| \to 0$. Since $\delta_n \to 0$, then
$\|u_n - y_n\| = \|u_n - [\delta_n v + (1 - \delta_n)u_n]\| = \delta_n \|u_n - v\| \to 0$. Therefore,

$$\|Ty_n - y_n\| \leq \|Ty_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \to 0.$$

Next we show that $\limsup_{n\to\infty}\langle f(z) - z, Ty_n - z\rangle \leq 0$, where

$$z = P_{F(T)\cap EP(F)} f(z).$$

To show this, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, Ty_n - z\rangle = \lim_{i\to\infty}\langle f(z) - z, Ty_{n_i} - z\rangle.$$

Since $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converges weakly to y. We may assume without loss of generality that $y_{n_i} \rightharpoonup y$. Since $\|Ty_n - y_n\| \to 0$, we obtain $Ty_{n_i} \rightharpoonup y$. Then we can obtain $y \in F(T) \cap EP(F)$. In fact, let us fist show that $y \in F(T)$. Assume that $y \notin F(T)$. Since $y_{n_i} \rightharpoonup y$ and $Ty \neq y$, from Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty}\|y_{n_i} - y\| & < \liminf_{i\to\infty}\|y_{n_i} - T_y\| \\ & \leq \liminf_{i\to\infty}(\|y_{n_i} - Ty_{n_i}\| + \|Ty_{n_i} - T_y\|) \\ & \leq \liminf_{i\to\infty}\|y_{n_i} - y\|. \end{aligned}$$

This is a contradiction. Thus, we obtain $y \in F(T)$. Finally, we show that $y \in EP(F)$. By $u_n = T_{r_n} x_n$ and for all $x \in C$, we have

$$F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0. \tag{3.15}$$

Since $\|u_n - y_n\| \to 0$ and $y_{n_i} \rightharpoonup y$, then $u_{n_i} \rightharpoonup y$. By the same argument as that in the proof of [9, Theorem 3.1], we can show that $y \in EP(F)$. Therefore $y \in F(T) \cap EP(F)$.

Since $z = P_{F(T)\cap EP(F)} f(z)$ and by (2.4) we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \limsup_{n\to\infty}\langle f(z) - z, Ty_n - z\rangle \\
&= \limsup_{i\to\infty}\langle f(z) - z, Ty_{n_i} - z\rangle \\
&= \langle f(z) - z, y - z\rangle \leq 0.
\end{aligned}$$

We also have, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\langle f(z) - z, x_n - z\rangle \leq \epsilon$$

and $\gamma_n\delta_n\|v - z\|^2 \leq \epsilon$ for all $n \geq N$. Therefore,
$\|x_{n+1} - z\|^2$

$$\begin{aligned}
&= \langle x_{n+1} - z, x_{n+1} - z\rangle \\
&= \alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad +\gamma_n\langle Ty_n - z, x_{n+1} - z\rangle \\
&\leq \alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle + \frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad +\frac{\gamma_n}{2}(\|Ty_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \alpha_n\langle f(x_n) - f(z) + f(z) - z, x_{n+1} - z\rangle \\
&\quad +\frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\|y_n - z\|^2 + \|x_{n+1} - z\|^2)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +\frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\delta_n\|v - z\|^2 \\
&\quad +(1 - \delta_n)\|u_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +\frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n\delta_n}{2}\|v - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
= \;& \alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+\frac{\beta_n + \gamma_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n\delta_n}{2}\|v - z\|^2 \\
\leq \;& \alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+\frac{1-\alpha_n}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{\gamma_n\delta_n}{2}\|v - z\|^2 \qquad (3.16)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \;&\leq\; (1-\alpha_n)\|x_n - z\|^2 + 2\alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| \\
&\quad +2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \gamma_n\delta_n\|v - z\|^2 \\
&=\; (1-\alpha_n)\|x_n - z\|^2 + \lambda_n
\end{aligned}
$$

where $\lambda_n = 2\alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \gamma_n\delta_n\|v - z\|^2$. It is easily seen that $\lambda_n \to 0$ and by Lemma 2.1, we obtain

$$x_n \to z = P_{F(T)\cap EP(F)}f(z).$$

This completes the proof. □

Next we prove the strong convergence of the another iterative algorithm as follows.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F is bifunction of $C\times C$ into real numbers $\mathbb{R}$. Let $T : C \to C$ be a nonexpansive mapping such that $F(T)\cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:*

(i) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$; *and* $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\lim_{n\to\infty}\delta_n = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv). $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

For fixed v and arbitrary given $x_0 \in C$ and the sequences $\{x_n\}$ and $\{u_n\}$ generated interatively by

$$
\begin{cases}
F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \;\forall x \in C, \\
y_n = \delta_n v + (1-\delta_n)u_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n A_n y_n,
\end{cases} \qquad (3.17)
$$

where $A_n = \frac{1}{n+1}\Sigma_{i=0}^{n} T^i$. *Then* $\{x_n\}$ *converges strongly to* z *where*

$$z = P_{F(T)\cap EP(F)} f(z).$$

Proof. We show that $A_n : C \to C$ is nonexpansive. In fact, for any $x, y \in C$, we have that

$$\begin{aligned} \|A_n x - A_n y\| &= \|\frac{1}{n+1}\Sigma_{i=0}^{n} T^i x - \frac{1}{n+1}\Sigma_{i=0}^{n} T^i y\| \\ &\leq \frac{1}{n+1}\Sigma_{i=0}^{n}\|x - y\| = \|x - y\|. \end{aligned}$$

Therefore, from Theorem 3.1 the sequence $\{x_n\}$ generated in Theorem 3.2 converges strongly to
$z = P_{F(T)\cap EP(F)} f(z)$. □

4. Deduced Theorems

Using Theorem 3.1, we obtain the following theorems in Hilbert Spaces.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0, 1)$. *Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0, 1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. *Suppose the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For fixed v *and arbitrary given* $x_0 \in C$, *the sequences* $\{x_n\}$ *and* $\{u_n\}$ *generated interatively by*

$$\begin{cases} y_n = \delta_n v + (1 - \delta_n) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T y_n. \end{cases} \tag{4.1}$$

Then $\{x_n\}$ *converges strongly to* z *where* $z = P_{F(T)} f(z)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1 . So, from Theorem 3.1 the sequence $\{x_n\}$ generated in Theorem 4.1 converges strongly to $z = P_{F(T)} f(z)$. □

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For fixed v and arbitrary given $x_0 \in C$, the sequences $\{x_n\}$ and $\{u_n\}$ generated interatively by

$$\begin{cases} y_n = \delta_n v + (1 - \delta_n)u_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T y_n. \end{cases} \tag{4.2}$$

Then $\{x_n\}$ converges strongly to z where $z = P_{F(T)} f(z)$.

Proof. Put $f(x_n) = u$ for all $n \in \mathbb{N}$ in Theorem 4.1 . Thus, the sequence $\{x_n\}$ generated by 4.2 converges strongly to $z = P_{F(T)} f(z)$. □

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:*

(i) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For fixed v and arbitrary given $x_0 \in C$ and the sequences $\{x_n\}$ and $\{u_n\}$ generated interatively by

$$\begin{cases} y_n = \delta_n v + (1 - \delta_n)u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n (\frac{1}{n+1} \Sigma_{i=0}^{n} T^i) y_n, \end{cases} \tag{4.3}$$

Then $\{x_n\}$ converges strongly to z where $z = P_{F(T)} f(z)$.

5. Applications

In this section, we consider the problem of finding a zero of an accretive operator. Let E be a real Banach space and an operator $A \subset E \times E$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator A is said to satisfy the range condition of $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, I is the identity mapping on E, $R(I + \lambda A)$ is the range of $I + \lambda A$, and $\overline{D(A)}$ is the closure of $D(A)$. If A is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda A) \to D(A)$ by $J_\lambda = (I - \lambda A)^{-1}$, which is called the resolvent of A. We know that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}(0)$ for all $\lambda > 0$

The following three theorems are connected with the problem of obtaining of a common element of the sets of zeroes of a maximal monotone operator and an $\alpha-$inverse-strongly monotone operator.

Theorem 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $B : H \to 2^H$ be a maximal monotone operator such that $B^{-1}(0) \cap EP(F) \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, fixed $v \in C$ and*

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \quad \forall x \in H; \\ y_n = \delta_n v + (1 - \delta_n)u_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_r^B y_n, \end{cases} \tag{5.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$ and $\{r_n\}$ is a real sequence in $(0, \infty)$. Suppose the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in B^{-1}(0) \cap EP(F)$, where $z = P_{B^{-1}(0)EP(F)}x_0$.

Proof. Since J_r^B is nonexpansive. We have the following $F(J_r^B) = B^{-1}(0)$. Putting $T = J_r^B$ then by Theorem 3.1, we obtain the desired result. □

Theorem 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B : H \to 2^H$ be a maximal monotone operator such that $B^{-1}(0) \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, fixed $v \in C$ and*

$$\begin{cases} y_n = \delta_n v + (1-\delta_n)u_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_r^B y_n, \end{cases} \tag{5.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in B^{-1}(0)$, where $z = P_{B^{-1}(0)} x_0$.

Proof. Since J_r^B is nonexpansive. Putting $T = J_r^B$ then by Theorem 4.1, we obtain the desired result. □

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