International Journal of Pure and Applied Mathematics
Volume 72 No. 2 2011, 237-247

SOME RESULTS ON t-BEST APPROXIMATION IN FUZZY 2-NORMED LINEAR SPACES

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Abstract: The aim of this paper is to give the set of all t-best approximations on fuzzy 2-normed linear spaces and prove some theorems in the sense of Vaezpour and Karimi [13].

AMS Subject Classification: 46A30, 46S40, 46A70, 54A40
Key Words: 2-normed linear spaces, fuzzy 2-norms, t-best approximation, t-proximinal set, t-chebyshev set

1. Introduction

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by Katsaras in [11]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [10]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [12].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and

Received: August 10, 2011

Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [11], Felbin [6], and Bag and Samanta [1]. Many authors studied on fuzzy normed linear space [5]. The concept of 2-norm on a linear space has been introduced and developed by Gähler in [7,8] and Gunawan and Mashadi [9]. Recently, Vaezpour and Karimi [13], studied on the set of all t-best approximations on fuzzy normed linear spaces and proved several theorems pertaining to this set.

In this paper, we give the set of all t-best approximations on fuzzy 2-normed linear spaces and prove some theorems in the sense of Vaezpour and Karimi [13].

2. Preliminaries

Definition 1. Let X be a real linear space of dimension greater than one and let $\|\bullet,\bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions:

$2N_1$: $\|x,y\| = 0$ if and only if x and y are linearly dependent

$2N_2$: $\|x,y\| = \|y,x\|$

$2N_3$: $\|\alpha x,y\| = |\alpha|\|x,y\|$, for every $\alpha \in R$

$2N_4$: $\|x,y+z\| \leq \|x,y\| + \|x,z\|$

then the function $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet,\bullet\|)$ is called a 2-normed linear space.

Example 1. Let $X = R^3$ be a real linear space. Define $\|\bullet,\bullet\| : X \times X \longrightarrow R$ by $\|x,y\| = \max\{|x_1y_2 - x_2y_1|, |x_2y_3 - x_3y_2|, |x_3y_1 - x_1y_3|\}$, where $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3)$ are in R^3. Then $(X, \|\bullet,\bullet\|)$ is a 2-normed linear space.

Definition 2. Let X be a linear space over a real field F. A fuzzy subset N of $X \times X \times R$ is called a fuzzy 2-norm on X if the following conditions are satisfied for all $x,y,z \in X$.

$(2-N_1)$: For all $t \in R$ with $t \leq 0$, $N(x,y,t) = 0$,

$(2-N_2)$: For all $t \in R$ with $t > 0$, $N(x,y,t) = 1$ if and only if x,y are linearly dependent

$(2-N_3)$: $N(x,y,t)$ is invariant under any permutation of x,y

$(2-N_4)$: For all $t \in R$ with $t > 0$, $N(x,cy,t) = N(x,y,\frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(2-N_5)$: For all $s,t \in R$, $N(x, y+z, s+t) \geq \min\{N(x,y,s), N(x,z,t)\}$

$(2-N_6)$: $N(x,y,t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x,y,t) = 1$.

Then the pair (X, N) is called a fuzzy 2-normed linear space (briefly F-2-NLS).

Remark 1. From $(2-N_3)$, it follows that in F-2-NLS,

$(2-N_4)$: For all $t \in R$ with $t > 0$, $N(cx,y,t) = N(x,y,\frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(2-N_5)$: For all $s,t \in R$, $N(x+z,y,s+t) \geq \min\{N(x,y,s), N(z,y,t)\}$.

Example 2. Let $(X, \|\bullet,\bullet\|)$ be a 2-normed linear space. Define

$$N(x,y,t) = \begin{cases} \frac{t}{t+\|x,y\|}, & \text{if } \ t > 0, \quad t \in R, \quad x,y \in X \\ 0, & \text{if } \ t \leq 0, \quad t \in R, \quad x,y \in X. \end{cases}$$

Then (X, N) is a fuzzy 2-normed linear space.

Example 3. Let $(X, \|\bullet,\bullet\|)$ be a 2-normed linear space. Define

$$N(x,y,t) = \begin{cases} 0, & \text{if } \ t \leq \|x,y\|, \quad t \in R, \quad x,y \in X \\ 1, & \text{if } \ t > \|x,y\|, \quad t \in R, \quad x,y \in X. \end{cases}$$

Then (X, N) is a fuzzy 2-normed linear space.

Definition 3. A sequence $\{x_k\}$ in a fuzzy 2-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1, x_k - x, t) > 1 - r$, for all $k \geq n_0$.

Theorem 1. *In a fuzzy 2-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only* $\lim_{k\to\infty} N(x_1, x_k - x, t) = 1$, $\forall\, t > 0$.

3. Main Results

Definition 4. Let (X, N) be a fuzzy 2-normed linear space. The open ball $B(x,r,t)$ and the closed ball $B[x,r,t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x,r,t) = \{y \in X : N(x_1, x-y, t) > 1 - r\},$$

$$B[x,r,t] = \{y \in X : N(x_1, x-y, t) \geq 1 - r\}.$$

Definition 5. Let (X, N) be a fuzzy 2-normed linear space. A subset A of X is said to be open if there exists $r \in (0,1)$ such that $B(x,r,t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 6. Let (X, N) be a fuzzy 2-normed linear space. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$, i.e.

$$\lim_{k \longrightarrow \infty} N(x_1, x_k - x, t) = 1,$$

for all $t > 0$ implies that $x \in A$.

Definition 7. Let (X, N) be a fuzzy 2-normed linear space. A subset B of X is said to be closure of $A \subset B$ if for any $x \in B$, there exists a sequence $\{x_k\}$ in A such that $\lim_{k \longrightarrow \infty} N(x_1, x_k - x, t) = 1$, for all $t > 0$. We denote the set B by $\overline{A}$.

Definition 8. Let (X, N) be a fuzzy 2-normed linear space. A subset A of X is said to be compact if for any sequence $\{x_k\}$ in A has a sequence converging to an element of A.

Lemma 1. *If (X, N) be a fuzzy 2-normed linear space then:*

(i) the function $(x, y) \longrightarrow x + y$ is continuous;

(ii) the function $(\alpha, x) \longrightarrow \alpha x$ is continuous.

Proof. (i) If $x_k \longrightarrow x$ and $y_k \longrightarrow y$, then as $k \longrightarrow \infty$,

$$N(x_1, (x_k + y_k) - (x + y), t) \geq \min\{N(x_1, x_k - x, \frac{t}{2}), N(x_1, y_k - y, \frac{t}{2})\} \longrightarrow 1.$$

(ii) If $x_k \longrightarrow x$, $\alpha_k \longrightarrow \alpha$ and $\alpha_k \neq 0$ then

$$\begin{aligned} N(x_1, \alpha_k x_k - \alpha x, t) =& N(x_1, \alpha_k(x_k - x) + x(\alpha_k - \alpha), t) \\ &\geq \min\{N(x_1, \alpha_k(x_k - x), \frac{t}{2}), N(x_1, x(\alpha_k - \alpha), \frac{t}{2})\} \\ &= \min\{N(x_1, x_k - x, \frac{t}{2|\alpha_k|}), N(x_1, x, \frac{t}{2|\alpha_k - \alpha|})\} \\ &\longrightarrow 1 \text{ as } k \longrightarrow \infty. \end{aligned}$$

□

Definition 9. Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, and $A \neq \emptyset$. Let $d(A, x, t) = \sup\{N(x_1, x - y, t) : y \in A\}$, where $x \in X$, $t > 0$. An element $y_0 \in A$ is said to be a t-best approximation of x from A if $N(x_1, y_0 - x, t) = d(A, x, t)$.

Definition 10. Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, and $A \neq \emptyset$. For $x \in X$, $t > 0$, we shall denote the set of all elements of t-best approximation of x form A by $P_A^t(x)$; i.e.,

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t-best approximation in A then A is called a t-proximinal (respectively t-chebyshev) set.

Definition 11. Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, and $A \neq \emptyset$. For $t > 0$, A is said to be t-boundedly compact if for each $x \in X$ and $0 < r < 1$, $B[x, r, t] \cap A$ is a compact subset of X.

Theorem 2. *Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, and $A \neq \emptyset$ then:*

(i) $d(A + y, x + y, t) = d(A, x, t)$, for all $x, y \in X$ and $t > 0$,

(ii) $P_A^t(x + y) = P_A^t(x) + y$, for all $x, y \in X$ and $t > 0$,

(iii) $d(\alpha A, \alpha x, t) = d(A, x, \frac{t}{|\alpha|})$, for all $x \in X$, $t > 0$ and $\alpha \in R \backslash \{0\}$,

(iv) $P_{\alpha A}^{|\alpha| t}(\alpha x) = \alpha P_A^t(x)$, for all $x \in X$, $t > 0$ and $\alpha \in R \backslash \{0\}$,

(v) A is t-proximinal (respectively t-chebyshev) if and only if $A + y$ is t-proximinal (respectively t-chebyshev) for any given $y \in X$,

(vi) A is t-proximinal (respectively t-chebyshev) if and only if αA is $|\alpha| t$-proximinal (respectively $|\alpha| t$-chebyshev) for any given $\alpha \in R \backslash \{0\}$.

Proof. (i) For $x, y \in X$ and $t > 0$,

$$\begin{aligned} d(A + y, x + y, t) &= \sup\{N(x_1, (z + y) - (x + y), t) : z \in A\} \\ &= \sup\{N(x_1, z - x, t) : z \in A\} = d(A, x, t). \end{aligned}$$

(ii) On using (i), it follows that, $y_0 \in P_{A+y}^t(x + y)$ if and only if $y_0 \in A + y$ and $d(A + y, x + y, t) = N(x_1, x + y - y_0, t)$ if and only if $y_0 - y \in A$ and $d(A, x, t) = N(x_1, x - (y_0 - y), t)$ if and only if $y_0 - y \in P_A^t(x)$, i.e., $y_0 \in P_A^t(x) + y$.

(iii) We have

$$\begin{aligned} d(\alpha A, \alpha x, t) &= \sup\{N(x_1, \alpha x - \alpha z, t) : z \in A\} \\ &= \sup\{N(x_1, \alpha(x - z), t) : z \in A\} \\ &= \sup\{N(x_1, x - z, \frac{t}{|\alpha|}) : z \in A\} \end{aligned}$$

$$=d(A,x,\frac{t}{|\alpha|}).$$

(iv) On usaing (iii), it follows that $y_0 \in P_{\alpha A}^{|\alpha|t}(\alpha x)$ if and only if $y_0 \in \alpha A$ and $d(\alpha A, \alpha x, |\alpha|t) = N(x_1, \alpha x - y_0, |\alpha|t)$ if and only if $\frac{y_0}{\alpha} \in A$ and $N(x_1, x - \frac{y_0}{\alpha}, t) = d(A, x, t)$. However, this is equivalent to $\frac{y_0}{\alpha} \in P_A^t(x)$; i.e., $y_0 \in \alpha P_A^t(x)$.

(v) The proof of (v) is an immediate consequence of (ii).

(vi) The proof of (vi) follows from (iv). □

Corollary 3. *Let M be a nonempty subspace of a fuzzy 2-normed linear space X then:*

(i) $d(M, x+y, t) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $y \in M$,

(ii) $P_M^t(x+y) = P_M^t(x) + y$, for all $t > 0$, $x \in X$ and $y \in M$,

(iii) $d(M, \alpha x, |\alpha t|) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $\alpha \in R \backslash \{0\}$,

(iv) $P_M^{|\alpha|t}(\alpha x) = \alpha P_M^t(x)$, for all $t > 0$, $x \in X$ and $\alpha \in R \backslash \{0\}$.

Proof. The proof of (i) and (ii) follow from theorem 2(i) and 2(ii) and the fact that if M is a subspace and $y \in M$ then $M + y = M$.

The proof of (iii) and (iv) follow from theorem 2(iii) and 2(iv) and the fact that if M is a subspace and $\alpha \neq 0$ then $\alpha M = M$. □

Definition 12. For $x \in X$, $0 < r < 1$, $t > 0$,

$$S[x,r,t] = \{y \in X : N(x_1, x-y, t) = 1-r\} \quad \text{and} \quad e_A^t(x) = 1 - d(A,x,t).$$

Theorem 4. *Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, $x \in X \backslash \overline{A}$ and $t > 0$ then we have*

$$P_A^t(x) = A \cap B[x, e_A^t(x), t] = A \cap S[x, e_A^t(x), t]. \tag{1}$$

Proof. The inclusions;

$$P_A^t(x) \subseteq A \cap S[x, e_A^t(x), t] \subseteq A \cap B[x, e_A^t(x), t]. \tag{2}$$

are obvious by the definitions of $P_A^t(x)$ and $e_A^t(x)$.

Conversely, let $y \in A \cap B[x, e_A^t(x), t]$, then we have $y \in A$ and $N(x_1, y-x, t) \geq 1 - e_A^t(x) = d(A,x,t) \geq N(x_1, y-x, t)$. Therefore $y \in A$ and $N(x_1, y-x, t) = d(A,x,t)$, which implies that $y \in P_A^t(x)$. So, $A \cap B[x, e_A^t(x), t] \subset P_A^t(x)$. Hence by (2) we have (1) which completes the proof. □

Remark 2. Let (X,N) be a fuzzy 2-normed linear space, $A \subset X$ and $A \neq \emptyset$, $x \in X \backslash \overline{A}$ and $t > 0$ then we have

$$A \cap B(x, e_A^t(x), t) = \emptyset, \tag{3}$$

because, if $y_0 \in A \cap B(x, e_A^t(x), t)$ then $d(A,x,t) \geq N(x_1, x - y_0, t) > d(A,x,t)$ which is impossible.

Corollary 5. *Let (X,N) be a fuzzy 2-normed linear space, $A \subset X$ and $A \neq \emptyset$, $x \in X \backslash \overline{A}$ with $P_A^t(x) \neq \emptyset$ and $0 < r < 1$ such that,*

$$\emptyset \neq A \cap B[x,r,t] \subseteq S[x,r,t] \tag{4}$$

then we have $r = e_A^t(x)$, and we can write $A \cap B[x,r,t] = P_A^t(x)$.

Proof. If $r < e_A^t(x)$ then by the definition of $e_A^t(x)$ we have $A \cap B[x,r,t] = \emptyset$, which contradicts (4). If $r > e_A^t(x)$, since $P_A^t(x) \neq \emptyset$, then by (1) we have $\emptyset \neq P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x,r,t)$, which contradicts (4), and this completes the proof. □

Definition 13. Let (X,N) be a fuzzy 2-normed linear space, $0 < r < 1$ and $t > 0$. We shall say that a set $A \subset X$ supports the cell $B[x,r,t]$, or that A is a support set of the cell $B[x,r,t]$, if we have $d(A, B[x,r,t], t) = 1$ and $A \cap B[x,r,t] = \emptyset$.

Theorem 6. *Let (X,N) be a fuzzy 2-normed linear space, A a nonempty set in X, $x \in X \backslash \overline{A}$, $a_0 \in A$ and $t > 0$. We have $a_0 \in P_A^t(x)$ if and only if the set A supports the cell $B = B[x, 1 - N(x_1, a_0 - x, t), t]$.*

Proof. Assume that $a_0 \in P_A^t(x)$. Hence $N(x_1, a_0 - x, t) = d(A,x,t)$. Then by (3), we have $A \cap B[x, 1 - N(x_1, a_0 - x, t), t] = \emptyset$, on the other hand, since $a_0 \in A \cap B[x, 1 - N(x_1, a_0 - x, t), t]$, we have $d(A,B,t) = 1$. Consequently, the set A supports the cell B. Conversely, suppose $a_0 \notin P_A^t(x)$, hence $N(x_1, a_0 - x, t) < d(A,x,t)$, and let $0 < \varepsilon < 1$ such that $N(x_1, a_0 - x, t) < d(A,x,t) - \varepsilon$. Then there exists an $a \in A$ such that $N(x_1, a_0 - x, t) < d(A,x,t) - \varepsilon < N(x_1, a - x, t)$, hence $a \in B(x, 1 - N(x_1, a_0 - x, t), t)$. Consequently, A does not support the cell B. □

Remark 3. We recall that a set A in a topological space τ is said to be countably compact, if every countable open cover of A has a finite subcover, or, which is equivalent, if for every decreasing sequence $A_1 \supset A_2 \supset \dots$ of non-void closed subset of A we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$

Theorem 7. *Let (X, N) be a fuzzy 2-normed linear space, τ be an arbitrary topology on X and $t > 0$. If A is a nonempty set of X such that for $A \cap B[x, r, t]$ is $\tau-$countably compact, then A is t-proximinal.*

Proof. For all $n \in N$, $0 < 1 - d(A, x, t) + \frac{d(A,x,t)}{n+1} < 1$. put

$$A_n^t = A \cap B\left[x, 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1}, t\right] \quad (n = 1, 2, \dots).$$

Since for every $n \in N$, $d(A, x, t)\left(1 - \frac{1}{n+1}\right) < d(A, x, t)$, obviously $A_1^t \supset A_2^t \supset \dots$ and each $A_n^t \neq \emptyset$. Hence there exists $a_n^t \in A$ such that

$$d(A, x, t)\left(1 - \frac{1}{n+1}\right) < N(x_1, a_n^t - x, t).$$

It follows that $a_n^t \in A_n^t$. Now, since each A_n^t is $\tau-$countably compact and $\tau-$closed, we conclude that there exists an $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$. Then we have

$$d(A, x, t) \geq N(x_1, a_0 - x, t) \geq d(A, x, t)\left(1 - \frac{1}{n+1}\right) \quad (n = 1, 2, \dots),$$

hence $a_0 \in P_A^t(x)$ which completes the proof. □

Definition 14. Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$ and $A \neq \emptyset$. An element $y_0 \in A$ is said to be an F-best approximation of $x \in X$ from A if it is a t-best approximation of x from A, for every $t > 0$, i.e.,

$$y_0 \in \bigcap_{t \in (0,\infty)} P_A^t(x).$$

The set of all elements of F-best approximations of $x \in X$ from A is denoted by $FP_A(x)$, i.e.,

$$FP_A(x) = \bigcap_{t \in (0,\infty)} P_A^t(x).$$

If each $x \in X$ has at least (respectively exactly) one F-best approximation in A then A is called a F-proximinal (respectively F-chebyshev) set.

Example 4. Let $X = R^3$. Define $N : X \times X \times [0, \infty) \longrightarrow [0, 1]$ by

$$N(x_1, x_2, t) = \begin{cases} \left(\exp \frac{\|x_1, x_2\|_\infty}{t}\right)^{-1}, & \text{if } t > 0, \quad t \in R, \quad x_1, x_2 \in X, \\ 0, & \text{if } t \leq 0, \quad t \in R, \quad t \in R, \quad x_1, x_2 \in X, \end{cases}$$

where $\|x_1, x_2\|_\infty = \max\limits_{1\leq i\leq 2} \sum\limits_{j=1}^{3} |x_{ij}|$. Then (X, N) is a fuzzy 2-normed linear space. Let

$$A = \left\{(a, b, c) \in R^3 : a^2 + b^2 \leq 1, \quad 0 \leq c \leq a^2 + b^2\right\}$$

and $x_1 = (1, 0, 0)$, $x = (0, 0, 4)$ are in X. Let $a_0 = (0, -1, 1)$ and $a_1 = (0, 1, 1)$ are in A, Then for every $t > 0$,

$$N(x_1, a_0 - x, t) = N(x_1, (0, -1, 1) - (0, 0, 4), t) = \left(\exp\frac{4}{t}\right)^{-1},$$

$$N(x_1, a_1 - x, t) = N(x_1, (0, 1, 1) - (0, 0, 4), t) = \left(\exp\frac{4}{t}\right)^{-1}.$$

On the other hand

$$\begin{aligned} d(A, x, t) &= d(A, (0, 0, 4), t) = \sup\{N(x_1, u - (0, 0, 4), t) : u \in A\} \\ &= \sup\{N(x_1, (a, b, c) - (0, 0, 4), t) : a^2 + b^2 \leq 1, \quad 0 \leq c \leq a^2 + b^2\} \\ &= \sup\left\{\left(\exp\frac{\max(|x_{11}| + |x_{12}| + |x_{13}|, |x_{21}| + |x_{22}| + |x_{23} - 4|)}{t}\right)^{-1}\right\} \\ &= \left(\exp\frac{4}{t}\right)^{-1}. \end{aligned}$$

So, for every $t > 0$, $a_0 = (0, -1, 1)$ and $a_1 = (0, 1, 1)$ are t-best approximations of $(0, 0, 4)$ from A. Hence $a_0 = (0, -1, 1)$ and $a_1 = (0, 1, 1)$ are F-best approximations of $x = (0, 0, 4)$ from A. Therefore A is not an F-Chebyshev set.

Example 5. Let $X = R^3$. Define $N : X \times X \times R \longrightarrow [0, 1]$ by

$$N(x_1, x_2, t) = \begin{cases} \frac{t}{t + \|x_1, x_2\|}, & \text{if } t > 0, \quad t \in R, \quad x_1, x_2 \in X \\ 0, & \text{if } t \leq 0, \quad t \in R, \quad x_1, x_2 \in X, \end{cases}$$

where $\|x_1, x_2\|_\infty = \max\limits_{1\leq i\leq 2} \sum\limits_{j=1}^{3} |x_{ij}|$. Then (X, N) is a fuzzy 2-normed linear space. Let

$$A = \left\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 \leq 1\right\}.$$

Then, for every $a = (x, y, z) \in R^3$ where $x^2 + y^2 + z^2 > 1$, there exists a unique $a_0 = (x_0, y_0, z_0) \in A$ (especially in ∂A) which is an F-best approximation of a from A. So A is an F-proximinal set.

Remark 4. For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of A, and by $\mathcal{M}_A$ the set of all elements of the F-best approximation of the elements $x \in X$ from A. i.e.,

$$\mathcal{M}_A = \bigcup_{x \in X} FP_A(x).$$

Theorem 8. *Let (X, N) be a fuzzy 2-normed linear space, $A \subset X$, $A \neq \emptyset$, and A be a F-best proximinal set in X then $\partial A \subset \overline{\mathcal{M}_A}$.*

Proof. If $\partial A = \emptyset$, the proof is obvious. If $\partial A \neq \emptyset$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $\varepsilon' > \varepsilon$ and the cell $B(a_0, \varepsilon', \frac{t}{2})$ contains at least one element $x \in X \backslash A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, A is F-proximinal). Then we have,

$$\begin{aligned}
N(x_1, a_0 - \pi_A(x), t) &\geq \min\left\{N(x_1, a_0 - x, \frac{t}{2}), N(x_1, x - \pi_A(x), \frac{t}{2})\right\} \\
&= \min\left\{N(x_1, a_0 - x, \frac{t}{2}), N(x_1, A - x, \frac{t}{2})\right\} \\
&\geq \min\left\{N(x_1, a_0 - x, \frac{t}{2}), N(x_1, a_0 - x, \frac{t}{2})\right\} \\
&\geq \min\{\varepsilon', \varepsilon'\} = \varepsilon' \\
&> \varepsilon.
\end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \emptyset$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathcal{M}_A}$ which completes the proof. □

4. Acknowledgments

The author is grateful to the referee for careful reading of the article and suggested improvements.

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