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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HIGHER ORDER NONLINEAR DELAY IMPULSIVE DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract: In this paper, we investigated the sufficient conditions for asymptotic behavior of all solutions of higher order nonlinear delay impulsive differential equation of the form

$$\big(r(t)(x^{(2n-1)}(t))^{\alpha}\big)' + p(t)(x^{(2n-1)}(t))^{\alpha} + f(t, x(t-\delta)) = 0, \quad t \neq t_k,$$
$$x^{(i)}(t_k^+) = I_k^{(i)}(x^{(i)}(t_k)), \quad i = 0, 1, 2, \dots, 2n-1, \; k = 1, 2, 3 \dots .$$

Our results are extension of some known results.

Finally, we give an example to demonstrate our results.

AMS Subject Classification: 34A37, 34K25

Key Words: asymptotic behavior, higher order equation, impulsive differential equation

1. Introduction

In recent years the theory of impulsive differential equations emerging as an important area of research, since such equations have applications in control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of

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impulsive differential equation, see for example [1], [2]. The oscillation and asymptotic behavior of solutions of impulsive differential equations of various order are systematically studied by several authors, see [3], [4], [5], [6], [7], [8], [9]. In [6], the authors investigated the oscillation of higher order nonlinear functional impulsive differential equations. Recently, Liu [9] studied the asymptotic behavior of solutions of second order nonlinear impulsive differential equations.Motivated by the work of [6] and [9], we use impulsive inequality and the Riccati transformation to obtain the sufficient conditions for asymptotic behavior of all solutions of higher order nonlinear delay impulsive differential equations.Our results are extension of some known results.

Consider the impulsive differential equation

$$\big(r(t)(x^{(2n-1)}(t))^{\alpha}\big)' + p(t)(x^{(2n-1)}(t))^{\alpha} + f(t, x(t-\delta)) = 0, \quad t \neq t_k, \tag{1.1}$$

$$x^{(i)}(t_k^+) = I_k^{(i)}(x^{(i)}(t_k)), \quad i = 0, 1, 2, \dots, 2n-1, \ k = 1, 2, 3 \dots . \tag{1.2}$$

where $t \geq t_0, \alpha$ is the quotient of odd positive integers, $\delta > 0$, n is a natural number, $0 \leq t_0 < t_1 < \cdots < t_k \dots$ with $\lim\limits_{k\to\infty} t_k = +\infty$,

$$x^{(i)}(t_k) = \lim_{h\to 0^-} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k)}{h},$$

$$x^{(i)}(t_k^+) = \lim_{h\to 0^+} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k^+)}{h}.$$

A function $x : [t_0 - \delta, +\infty) \to \mathbb{R}$ is said be a solution of (1.1)-(1.2) if:

(i) $x(t)$ satisfies (1.1)on $[t_0, \infty) \setminus \{t_k \in N\}$;

(ii) $x^{(i)}(t)$ is continuous on $[t_0, \infty) \setminus \{t_k \in N\}$ and it is left continuous at t_k satisfies (1.2);

(iii) $x^{(i)}(t_0^+) = x_0^{(i)}$, $i = 0, 1, \dots 2n-1$, where $x^{(0)}(t) = x(t)$;

(iv) $x(t) = \phi(t)$ for $t \in [t_0 - \delta, t_0]$, where $\phi : [t_0 - \delta, t_0] \to \mathbb{R}$ has at most finite discontinuous points of the first kind and is left continuous at these points.

A solution of (1.1) and (1.2) is said to be non oscillatory if it is eventually positive or negative. Otherwise, it will be called oscillatory.

2. Main Results

In this paper, we assume that the following conditions hold:

(H1) f is continuous on $[t_0-\delta,+\infty)\times\mathbb{R}$, $xf(t,x)>0$ and $\frac{f(t,x)}{g(x)}\geq q(t)$ for $x\neq 0$, where $g(\gamma x)\geq \gamma g(x)$ for $\gamma>0$, $x'g'(x)>0$, and $q\geq 0, r'$ are continuous on $[t_0-\delta,+\infty)$, and for any t $\geq t_0$, $q(t)$ is not always equal to 0 in $[t,+\infty)$, $r(t)>0$.

(H2) $p(t), I_k^{(i)}(x)$ are continuous in $(-\infty,+\infty)$, and there exist positive numbers $a_k^{(i)}, b_k^{(i)}$ such that $a_k^{(i)}\leq \frac{I_k^{(i)}(x)}{x}\leq b_k^{(i)}, i=0,1,\dots,2n-1$.

(H3) $\lim\limits_{t\to\infty}\displaystyle\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^{(i)}}{b_k^{(i-1)}}ds=+\infty,$

(H4) $\lim\limits_{t\to\infty}\displaystyle\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^{(2n-1)}}{b_k^{(2n-2)}}\Big[\exp\Big(-\int_{t_0}^{s}\frac{r'(\sigma)+p(\sigma)}{\alpha r(\sigma)}d\sigma\Big)\Big]ds=+\infty,$

(H5) We have

$$\sum_{m=1}^{n-1}\prod_{k=m}^{n-1}\prod_{l=0}^{m-1}b_{j+k}^{(2n-2)}a_{j+l}^{(2n-1)}$$
$$\times\int_{t_{j+m-1}}^{t_{j+m}}\Big[\exp\Big(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\Big]du$$
$$+\prod_{k=0}^{n-1}a_{j+k}^{(2n-1)}\int_{t_{j+n-1}}^{t_{j+n}}\Big[\exp\Big(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\Big]du\to+\infty$$

as n $\to\infty$.

(H6) The following equality is valid

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{c_k}\exp\Big(\int_{t_0}^{s}\frac{p(\sigma)}{r(\sigma)}d\sigma\Big)q(s)ds=+\infty,$$

where

$$c_k=\begin{cases}(b_k^{(2n-1)})^{\alpha}, & t_k-\delta\neq t_j,\\ \frac{(b_k^{(2n-1)})^{\alpha}}{a_j^0}, & t_k-\delta=t_j.\end{cases}$$

In the following, we also assume that solutions to (1.1), (1.2) exist on $[t_0, +\infty)$.

Lemma 1. (see [1]) *Let the function $m \in PC^1(\mathbb{R}_+, \mathbb{R})$ satisfy the inequalities*

$$m'(t) \le u(t)m(t) + v(t), \quad t \ne t_k,$$
$$m(t_k^+) \le d_k m(t_k) + e_k, \quad k = 1, 2, \dots,$$

where $u, v \in PC(R_+, R)$ and $d_k \ge 0, e_k$ are constants, then

$$\begin{aligned} m(t) \le m(t_0) &\prod_{t_0<t_k<t} d_k \exp\Big(\int_{t_0}^{t} u(s)ds\Big) \\ &+ \sum_{t_0<t_k<t} \Big(\prod_{t_k<t_j<t} d_j \exp\Big(\int_{t_k}^{t} u(s)ds\Big)\Big) e_k \\ &+ \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\Big(\int_{s}^{t} u(\sigma)d\sigma\Big) v(s)ds, \quad t \ge t_0. \end{aligned} \tag{2.1}$$

Lemma 2. *Let $x(t)$ be a solution of* (1.1), *and conditions (H1)-(H3) be satisfied. Suppose that there exists some $T \ge t_0$ and $i \in \{1, 2, \dots, 2n-1\}$, such that $x^{(i)}(t) > 0$ (< 0), $x^{(i+1)}(t) \ge 0$ (≤ 0) for $t \ge T$. Then there exists some $T_1 \ge T$, such that $x^{(i-1)}(t) > 0$ (< 0), for $t \ge T_1$.*

Proof. Without loss of generality, let $T = t_0$, $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \ge 0$ for $t \ge T$. Assume that for any $t_k > T$, $x^{(i-1)}(t_k) < 0$. By $x^{(i+1)}(t) \ge 0$, $x^{(i)}(t) > 0$, $t \in (t_k, t_{k+1}]$, we have that $x^{(i)}(t)$ is monotonically nondecreasing on $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have $x^{(i)}(t) \ge x^{(i)}(t_1^+)$. In particular, $x^{(i)}(t_2) \ge x^{(i)}(t_1^+)$. Similarly, for $t \in (t_2, t_3]$, $x^{(i)}(t) \ge x^{(i)}(t_2^+) \ge a_2^{(i)} x^{(i)}(t_2) \ge a_2^{(i)} x^{(i)}(t_1^+)$. Therefore by induction, for $t > t_1$,

$$x^{(i)}(t) \ge \prod_{t_1<t_k<t} a_k^{(i)} x^{(i)}(t_1^+), \quad t \ne t_k.$$

From condition (H2), we have

$$x^{(i-1)}(t_k^+) \ge b_k^{(i-1)} x^{(i-1)}(t_k), \quad k = 2, 3, \dots.$$

Set $m(t) = -x^{(i-1)}(t)$. Then, for $t > t_1$,

$$m'(t) \le -\prod_{t_1<t_k<t} a_k^{(i)} x^{(i)}(t_1^+), t \ne t_k$$

$$m(t_k^+) \le b_k^{(i-1)} m(t_k), \quad k = 2, 3, ...$$

It follows from Lemma 1 that

$$\begin{aligned} m(t) &\le m(t_1^+) \prod_{t_1<t_k<t} b_k^{(i-1)} - x^{(i)}(t_1^+) \int_{t_1}^{t} \prod_{s<t_k<t} b_k^{(i-1)} \prod_{t<t_k<s} a_k^{(i)} ds \\ &= \prod_{t_1<t_k<t} b_k^{(i-1)} \Big[m(t_1^+) - x^{(i)}(t_1^+) \int_{t_1}^{t} \prod_{t_1<t_k<s} \frac{a_k^{(i)}}{b_k^{(i-1)}} ds \Big]. \end{aligned}$$

i. e. ,

$$x^{(i-1)}(t) \ge \prod_{t_1<t_k<t} b_k^{(i-1)} \Big[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+) \int_{t_1}^{t} \prod_{t_1<t_k<s} \frac{a_k^{(i)}}{b_k^{(i-1)}} ds \Big].$$

By condition (H3) and $a_k^{(i)} > 0$, $b_k^{(i-1)} > 0$, for all sufficiently large t, we have $x^{(i-1)}(t) > 0$, which is contrary to the assumption. Hence, there exists some j such that $t_j > T$ and $x^{(i-1)}(t_j) \ge 0$. Then

$$x^{(i-1)}(t_j^+) \ge a_j^{(i-1)} x^{(i-1)}(t_j) \ge 0.$$

Note that $x^{(i)}(t) > 0$ yields $x^{(i-1)}(t)$ being monotonically increasing on $(t_j, t_{j+1}]$. For $t \in (t_j, t_{j+1}]$, we have $x^{(i-1)}(t) > x^{(i-1)}(t_j^+) \ge 0$. Especially, $x^{(i-1)}(t_{j+1}) > x^{(i-1)}(t_j^+) > 0$. Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have $x^{(i-1)}(t) > x^{(i-1)}(t_{j+1}^+) \ge a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) > 0$. By induction, for $t \in (t_{j+m-1}, t_{j+m}]$, we have $x^{(i-1)}(t) > 0$. So for $t \ge t_{j+1}$, we have $x^{(i-1)}(t) > 0$. Summing up the above discussion, there exists some $T_1 \ge T$ such that $x^{(i-1)}(t) > 0$, $t \ge T_1$. The proof of the other case in this theorem is similar; so we omit it. $\square$

Lemma 3. *Let $x(t)$ be a solution of* (1.1) *and conditions (H1), (H2), (H3) be satisfied. Suppose that there exist an $i \in \{1, 2, \ldots, 2n\}$ and some $T \ge t_0$ such that $x(t) > 0$, $x^{(i)}(t) \le 0$, for $t \ge T$, and $x^{(i)}(t)$ is not always equal to 0 in $[t, +\infty)$. Then $x^{(i-1)}(t) > 0$ for all sufficiently large t.*

Proof. Without loss of generality, let $T = t_0$. We claim that $x^{(i-1)}(t_k) > 0$ for any $t_k \ge T$. If it is not true, then there exists some $t_j \ge T$, such that $x^{(i-1)}(t_j) \le 0$. Since $x^{(i)}(t) \le 0$, $x^{(i-1)}(t)$ is monotonically non-increasing in $(t_k, t_{k+1}]$ for $k \ge j$. Also because $x^{(i)}(t)$ is not always equal to 0 in $[t, +\infty)$, there exists some $t_l \ge t_j$ such that $x^{(i)}(t)$ is not always equal to 0 in $(t_l, t_{l+1}]$.

Without loss of generality, we can assume $l = j$, that is, $x^{(i)}(t)$ is not always equal to 0 in $(t_j, t_{j+1}]$. So we have

$$x^{(i-1)}(t_{j+1}) < x^{(i-1)}(t_j^+) \leq a_j^{(i-1)} x^{(i-1)}(t_j) \leq 0.$$

For $t \in (t_{j+1}, t_{j+2}]$, we have

$$x^{(i-1)}(t_{j+2}) < x^{(i-1)}(t_{j+1}^+) \leq a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) < 0.$$

By induction, for $t \in (t_{j+m}, t_{j+m+1}]$, we have $x^{(i-1)}(t) < 0$. So we have $x^{(i-1)}(t) < 0, x^{(i)}(t) \leq 0$, $t \in (t_{j+1}, +\infty)$. By Lemma 2, for all sufficiently large t, we have $x^{(i-2)}(t) < 0$. Similarly, we can conclude, using Lemma 2 repeatedly, that for all sufficiently large t, we have $x(t) < 0$. This is a contradiction to $x(t) > 0$ ($t \geq T$). Hence, we have $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. So we have $x^{(i-1)}(t) > 0$ for all sufficiently large t. The proof of Lemma 3 is complete. □

Lemma 4. *Let $x(t)$ be a solution of* (1.1) *and conditions (H1)-(H4) be satisfied. Suppose $T \geq t_0$, $x(t) > 0$ for $t \geq T$. Then there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T'$,*

$$\begin{aligned} x^{(i)}(t) > 0, &\quad i = 0, 1, \ldots, l; \\ (-1)^{i-1} x^{(i)}(t) > 0, &\quad i = l+1, \ldots, 2n-1. \end{aligned} \tag{2.2}$$

Proof. At first we prove that $x^{(2n-1)}(t_k) > 0$ for any $t_k \geq T$. If not, then there exists some j such that $t_j \geq T, x^{(2n-1)}(t_j) < 0$. So, $x^{(2n-1)}(t_j^+) = I_j^{(2n-1)}(x^{(2n-1)}(t_j)) \leq a_j^{(2n-1)} x^{(2n-1)}(t_j) < 0$. Let

$$x^{(2n-1)}(t_j) \exp\Big(\int_{t_0}^{t_j} \frac{r'(s) + p(s)}{\alpha r(s)} ds \Big) = \beta < 0.$$

From (1.1), it is clear that

$$\begin{aligned} &\Big(x^{(2n-1)}(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s) + p(s)}{\alpha r(s)} ds \Big) \Big)' \\ &\qquad = - \frac{f(t, x(t-\delta))}{\alpha r(t) (x^{(2n-1)}(t))^{\alpha-1}} \exp\Big(\int_{t_0}^{t} \frac{r'(s) + p(s)}{\alpha r(s)} ds \Big). \end{aligned}$$

Since α is the quotient of positive odd integers, $(x^{(2n-1)}(t))^{\alpha-1} > 0$, we obtain

$$\Big(x^{(2n-1)}(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s) + p(s)}{\alpha r(s)} ds \Big) \Big)' < 0.$$

Hence
$$x^{(2n-1)}(t)\exp\Big(\int_{t_0}^{t}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)$$
is decreasing on $(t_j,t_{j+1}]$. Therefore
$$\begin{aligned}x^{(2n-1)}(t_{j+1})\exp\Big(\int_{t_0}^{t_{j+1}}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\\ &\le x^{(2n-1)}(t_j^+)\exp\Big(\int_{t_0}^{t_j}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\\ &\le a_j^{(2n-1)}\beta<0\end{aligned}$$
and
$$x^{(2n-1)}(t_{j+2})\exp\Big(\int_{t_0}^{t_{j+2}}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\le a_{j+1}^{(2n-1)}a_j^{(2n-1)}\beta.$$
By induction, we obtain
$$x^{(2n-1)}(t_{j+n})\exp\Big(\int_{t_0}^{t_{j+n}}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\le\prod_{k=0}^{n-1}a_{j+k}^{(2n-1)}\beta,$$
while for $t\in(t_{j+n},t_{j+n+1}]$, we have
$$x^{(2n-1)}(t)\le\prod_{t_j\le t_k<t}a_k^{(2n-1)}\beta\exp\Big(-\int_{t_0}^{t}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)<0.$$

Also by Lemma 3, $x^{(2n-2)}(t)>0$ for large t. For convenience, we may assume that $x^{(2n-2)}(t)>0$ for $t\ge t_j$. By condition (H2) we have $x^{(2n-2)}(t_k^+)\le b_k^{(2n-2)}x^{(2n-2)}(t_k), k=j+1,j+2,\dots$. From Lemma 1
$$\begin{aligned}x^{(2n-2)}(t)\le\prod_{t_j<t_k<t}b_k^{(2n-2)}\Big[x^{(2n-2)}(t_j^+)-\beta\int_{t_j}^{t}\prod_{t_j<t_k<t}\frac{a_k^{(2n-1)}}{b_k^{(2n-2)}}\\ \times\exp\Big(-\int_{t_j}^{t}\frac{r'(v)+p(v)}{\alpha r(v)}dv\Big)ds\Big]\end{aligned}\tag{2.3}$$

Since $x^{(2n-2)}(t)>0(t\ge t_j)$, one can find that above inequality contradicts condition (H4) as $t\to\infty$. Hence $x^{(2n-1)}(t_k)>0$ for any $t_k\ge T$. So, we have $x^{(2n-1)}(t)>0$ for all sufficiently large t. Without loss of generality, let $x^{(2n-1)}(t)>0$ for $t\ge t_0$. So $x^{(2n-2)}(t)>0$ is monotonically nondecreasing on

$(t_k, t_{k+1}]$. If for any t_k, $x^{(2n-2)}(t_k) < 0$, then $x^{(2n-2)}(t) < 0 (t \geq t_0)$. If there exists some t_j such that $x^{(2n-2)}(t_j) \geq 0$, by that $x^{(2n-2)}(t)$ is monotonically increasing and $a_k^{(2n-2)} > 0$, $b_k^{(2n-2)} > 0$, we get $x^{(2n-2)}(t) > 0$ for $t > t_j$. So there exists some $T_1 \geq T$, such that one of the following statements hold

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) > 0, \quad \text{for } t \geq T_1 \tag{2.4}$$

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) < 0, \quad \text{for } t \geq T_1. \tag{2.5}$$

When (2.4) holds, Lemma 2 yields that $x^{(2n-3)}(t) > 0$ for all sufficiently large t. Using Lemma 1 repeatedly, for all sufficiently large t, we can conclude that

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) > 0, \ldots, x'(t) > 0, \quad x(t) > 0.$$

When (2.5) holds, by Lemma 3, we have $x^{(2n-3)}(t) > 0$, for all sufficiently large t. Hence, there exists some $T_2 \geq T_1$ such that

$$x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) > 0, \quad \text{for } t \geq T_2$$

$$x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) < 0, \quad \text{for } t \geq T_2$$

Repeating the discussion above, we obtain that there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$, such that for $t \geq T'$,

$$\begin{gathered} x^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l; \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, l+2, \ldots, 2n-1. \end{gathered}$$

The proof of Lemma 4 is complete. □

Lemma 5. *Let $x(t)$ be a solution of* (1.1) *and conditions (H1)-(H3), (H5) be satisfied. Suppose $T \geq t_0$, $x(t) > 0$ for $t \geq T$. Then there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T'$,*

$$\begin{gathered} x^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l; \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, \ldots, 2n-1. \end{gathered} \tag{2.6}$$

Proof. We first prove that $x^{(2n-1)}(t_k) > 0$ for any $t_k \geq T$. If not, then there exists some j such that $t_j \geq T, x^{(2n-1)}(t_j) < 0$. So, $x^{(2n-1)}(t_j^+) = I_j^{(2n-1)}(x^{(2n-1)}(t_j)) \leq a_j^{(2n-1)} x^{(2n-1)}(t_j) < 0$. From (1.1), it is clear that

$$\Big(x^{(2n-1)}(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s) + p(s)}{\alpha r(s)} ds\Big)\Big)'$$

$$= -\frac{f(t, x(t-\delta))}{\alpha r(t)(x^{(2n-1)}(t))^{\alpha-1}} \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big).$$

Since α is the quotient of positive odd integers, $(x^{(2n-1)}(t))^{\alpha-1} > 0$, we obtain

$$\Big(x^{(2n-1)}(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)\Big)' < 0.$$

Hence $x^{(2n-1)}(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)$ is decreasing on $(t_j, t_{j+1}]$.

$$\begin{aligned} x^{(2n-1)}(t_{j+1}) \exp\Big(\int_{t_0}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \\ \le x^{(2n-1)}(t_j^{+}) \exp\Big(\int_{t_0}^{t_j} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \end{aligned}$$

i. e. $x^{(2n-1)}(t_{j+1}) \le a_j^{(2n-1)} x^{(2n-1)}(t_j) \exp\Big(-\int_{t_j}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)$.
Similarly,

$$x^{(2n-1)}(t_{j+2}) \le a_{j+1}^{(2n-1)} a_j^{(2n-1)} x^{(2n-1)}(t_j) \exp\Big(-\int_{t_j}^{t_{j+2}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big).$$

By induction, we obtain

$$x^{(2n-1)}(t_{j+n}) \le \prod_{k=0}^{n-1} a_{j+k}^{(2n-1)} x^{(2n-1)}(t_j) \exp\Big(-\int_{t_j}^{t_{j+n}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)$$

Since $x^{(2n-1)}(t) \exp\Big(\int_{t_j}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)$ is decreasing in $(t_j, t_{j+i}]$, we have, for $t \in (t_j, t_{j+1}]$,

$$x^{(2n-1)}(t) \le a_j^{(2n-1)} x^{(2n-1)}(t_j) \exp\Big(-\int_{t_j}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) < 0,$$

Integrating the above inequality from s to t, we have

$$\begin{aligned} x^{(2n-2)}(t) \le x^{(2n-2)}(s) + a_j^{(2n-1)} x^{(2n-1)}(t_j) \\ \times \int_{s}^{t} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du. \end{aligned}$$

Let $t \to t_{j+1}$, $s \to t_j^{+}$, we have

$$x^{(2n-2)}(t_{j+1}) \le x^{(2n-2)}(t_j^+) + a_j^{(2n-1)} x^{(2n-1)}(t_j) \times \int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du.$$

Since $x^{(2n-1)}(t) < 0$ then by Lemma $3, x^{(2n-2)}(t) > 0$ for large t. For convenience, we may assume that $x^{(2n-2)}(t) > 0$ for $t \ge t_j$. By condition (H2) we have $x^{(2n-2)}(t_k^+) \le b_k^{(2n-2)} x^{(2n-2)}(t_k), k = j+1, j+2, \ldots$. Therefore

$$x^{(2n-2)}(t_{j+1}) \le b_j^{(2n-2)} x^{(2n-2)}(t_j) + a_j^{(2n-1)} x^{(2n-1)}(t_j) \times \int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du.$$

and $x^{(2n-2)}(t_{j+2})$

$$\begin{aligned}
&\le x^{(2n-2)}(t_{j+1}^+) + a_{j+1}^{(2n-1)} x^{(2n-1)}(t_{j+1}) \\
&\qquad \times \int_{t_{j+1}}^{t_{j+2}} \exp\Big(-\int_{t_{j+1}}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du \\
&\le b_{j+1}^{(2n-2)} b_j^{(2n-2)} x^{(2n-2)}(t_j) \\
&+ b_{j+1}^{(2n-2)} a_j^{(2n-1)} x^{(2n-1)}(t_j) \int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du \\
&+ a_{j+1}^{(2n-1)} a_j^{(2n-1)} x^{(2n-1)}(t_j) \int_{t_{j+1}}^{t_{j+2}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
x^{(2n-2)}(t_{j+n}) \le \prod_{k=0}^{n-1} b_{j+k}^{(2n-2)} x^{(2n-2)}(t_j) \Big[\sum_{m=1}^{n-1} \prod_{k=m}^{n-1} \prod_{l=0}^{m-1} b_{j+k}^{(2n-2)} a_{j+l}^{(2n-1)} \\
\times \int_{t_{j+m-1}}^{t_{j+m}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du + \prod_{k=0}^{n-1} a_{j+k}^{(2n-2)} \\
\times \int_{t_{j+n-1}}^{t_{j+n}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) du \Big] x^{(2n-1)}(t_j).
\end{aligned}$$

Since $x^{(2n-2)}(t) > 0 (t \ge t_j)$, one can find that above inequality contradicts condition (H5) as $t \to \infty$. Hence $x^{(2n-1)}(t_k) > 0$ for any $t_k \ge T$ and $x^{(2n-1)}(t) > 0$ for all $t \in (t_k, t_{k+1}]$. Remaining proof is the same as that of in Lemma 4. The proof of Lemma 5 is complete. ☐

Using Lemma 4, we have the following Theorem.

Theorem 6. *If the conditions* (H1)-(H4)*and (H6) are satisfied, then every solution* $x(t)$ *of* (1.1)- (1.2) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let $x(t)$ be a solution of (1.1)-(1.2), and suppose by contradiction that $\liminf_{t\to\infty} |x(t)| > 0$. So $x(t)$ is nonoscillatory. Without loss of generality, we may assume that $x(t) > 0$ on $(t_0, +\infty)$. By Lemma 4 and (1.1), there exists $T' \geq t_0$ such that , for $t \geq T'$, we have $x^{(2n-1)}(t) > 0, x'(t) > 0, x(t) > 0$. We use a Riccati transformation of the form

$$V(t) = \frac{r(t)(x^{(2n-1)}(t))^\alpha}{g(x(t-\delta))}. \tag{2.7}$$

Differentiating $V(t)$, we obtain

$$\begin{aligned} V'(t) &= \frac{-p(t)(x^{(2n-1)}(t))^\alpha - f(t, x(t-\delta))}{g(x(t-\delta))} - \frac{x'(t-\delta)g'(x(t-\delta))}{r(t)(x^{(2n-1)}(t))^\alpha} V^2(t) \\ &\leq -p(t)\frac{V(t)}{r(t)} - q(t). \end{aligned}$$

From (2.7) and (H1), it is clear that

$$\begin{aligned} V(t_k^+) &= \frac{r(t_k^+)(x^{(2n-1)}(t_k^+))^\alpha}{g(x(t_k^+ - \delta))} \\ &\leq \begin{cases} \frac{r(t_k)(x^{(2n-1)}(t_k))^\alpha (b_k^{(2n-1)})^\alpha}{g(x(t_k - \delta))} = (b_k^{(2n-1)})^\alpha V(t_k) \\ \frac{r(t_k)(x^{(2n-1)}(t_k))^\alpha (b_k^{(2n-1)})^\alpha}{g(x(t_j^+))} \leq \frac{(b_k^{(2n-1)})^\alpha}{a_j^{(0)}} V(t_k) \end{cases} \\ &= c_k V(t_k) \end{aligned}$$

where c_k 's are defined in (H6). Applying Lemma 1, we have

$$\begin{aligned} V(t) \leq V(t_0) &\prod_{t_0 < t_k < t} c_k \exp\Big(- \int_{t_0}^t \frac{p(s)}{r(s)} ds\Big) \\ &\quad - \int_{t_0}^t \prod_{s < t_k < t} c_k \exp\Big(- \int_s^t \frac{p(\sigma)}{r(\sigma)} d\sigma\Big) q(s) ds \\ = &\prod_{t_0 < t_k < t} c_k \exp\Big(- \int_{t_0}^t \frac{p(s)}{r(s)} ds\Big) \end{aligned}$$

$$\times\Big[V(t_0)-\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{c_k}\exp\Big(\int_{t_0}^{s}\frac{p(\sigma)}{r(\sigma)}d\sigma\Big)q(s)ds\Big].$$

By (H6), the above inequality is impossible. The proof is complete. □

Using Lemma 5, we have the following Theorem.

Theorem 7. *If the conditions (H1)-(H3) and (H5)-(H6) are satisfied, then every solution $x(t)$ of* (1.1)- (1.2) *satisfies* $\liminf_{t\to\infty}|x(t)|=0$.

Proof. The proof is similar to the proof of Theorem 6. □

3. Example

Consider

$$\Big(t(x^{(2n-1)}(t))^3\Big)'-(x^{(2n-1)}(t))^3+\frac{1}{t^2}x(t-\frac{1}{3})=0,\quad t\neq k,\ t\geq\frac{1}{2},$$
$$x^{(i)}(k^+)=\frac{k}{k+1}x^{(i)}(k),\ x(k^+)=x(k),\quad i=1,2,\ldots,(2n-1),\ k\in N.$$

Comparing with (1.1), (1.2), we see that $r(t)=t$, $\alpha=3$, $p(t)=-1$, $q(t)=\frac{1}{t^2}$, $\delta=\frac{1}{3}$, $t_0=\frac{1}{2}$, $t_k=k$, $t_{k+1}-t_k>\frac{1}{3}$, $a_k^{(0)}=b_k^{(0)}=1$, $a_k^{(i)}=b_k^{(i)}=\frac{k}{k+1}$, $i=1,2,\ldots,(2n-1)$. It is obvious that the conditions (H1) and (H2) are satisfied.

When $i=1$, $a_k^{(1)}=\frac{k}{k+1}$, $b_k^{(0)}=1$ and when $i>1$, $a_k^{(i)}=b_k^{(i-1)}=\frac{k}{k+1}$.

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^{(1)}}{b_k^{(0)}}ds=\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{k}{k+1}ds=\infty,$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^{(i)}}{b_k^{(i-1)}}ds=\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}(1)ds=\frac{1}{2}+1+1+\cdots=\infty.$$

From the above (H3) holds.

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^{(2n-1)}}{b_k^{(2n-2)}}\Big[\exp\Big(-\int_{t_0}^{s}\frac{r'(\sigma)+p(\sigma)}{\alpha r(\sigma)}d\sigma\Big)\Big]ds=\lim_{t\to\infty}\int_{t_0}^{t}ds=\infty.$$

This shows that (H4) holds.

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{c_k}\exp\Big(\int_{t_0}^{s}\frac{p(\sigma)}{r(\sigma)}d\sigma\Big)q(s)ds$$

$$= \lim_{t\to\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} (b_k^{(2n-1)})^{-3} \exp\Big(\int_{t_0}^{s} \frac{-1}{\sigma} d\sigma \Big) \frac{1}{s^2} ds$$

$$= \frac{1}{2} \lim_{t\to\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \Big(\frac{k+1}{k} \Big)^3 \frac{1}{s^3} ds = \infty.$$

So, (H6) holds. Clearly, all the conditions of Theorem 6 are satisfied. Therefore, every solution $x(t)$ satisfies $\lim_{t\to\infty} \inf |x(t)| = 0$.

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