MATCHING NUMBER AND EDGE COVERING NUMBER
ON KRONECKER PRODUCT OF $C_n$

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph $G_1$ and $G_2$ has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. In this paper, let $G$ is a simple graph with order $m$, we prove that

$$\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m\lfloor \frac{n}{2} \rfloor\} \text{ and } \beta'(C_n \otimes G) = \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}.$$

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let $G_1$ and $G_2$ be graphs. The Kronecker product of graph $G_1$ and $G_2$, denote by $G_1 \otimes G_2$, be the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters. A subset of the edge set $E$ of $G$ is said to be matching or an independent edge set of $G$, if no two distinct edges in $M$ have a common vertex. A matching $M$ is maximum matching in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality
of maximum matching of \( G \) is called the matching number of \( G \), denoted by \( \alpha'(G) \).

An edge of graph \( G \) is said to cover the two vertices incident with it, and an edge cover of a graph \( G \) is a set of edges covering all the vertices of \( G \). The minimum cardinality of an edge cover of a graph \( G \) is called the edge covering number of \( G \), denoted by \( \beta'(G) \).

By definitions of matching number, edge covering number, clearly that
\[
\alpha'(C_n) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \beta'(C_n) = \left\lceil \frac{n}{2} \right\rceil.
\]

In [1], there are some properties about Kronecker product of graph. We recall here.

**Proposition 1.** Let \( H = G_1 \otimes G_2 = (V(H), E(H)) \) then:

(i) \( n(V(H)) = n(V(G_1))n(V(G_2)) \);

(ii) \( n(E(H)) = 2n(E(G_1))n(E(G_2)) \);

(iii) for every \((u, v) \in V(H)\), \( d_H((u, v)) = d_{G_1}(u)d_{G_2}(v) \).

Note that for any graph \( G \), we have \( G_1 \otimes G_2 \cong G_2 \otimes G_1 \)

**Theorem 2.** Let \( G_1 \) and \( G_2 \) be connected graphs, The graph \( H = G_1 \otimes G_2 \) is connected if and only if \( G_1 \) or \( G_2 \) contains an odd cycle.

**Theorem 3.** Let \( G_1 \) and \( G_2 \) be connected graphs with no odd cycle then \( G_1 \otimes G_2 \) has exactly two connected components.

Next we get that general form of graph of Kronecker Product of \( C_n \) and a simple graph.

**Proposition 4.** Let \( G \) be connected graph order \( m \), the graph of \( C_n \otimes G \) is
\[
\bigcup_{i=1}^{n-1} H_i \cup H_n
\]
where \( V(H_i) = W_i \cup W_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \); \( W_i = \{(i, 1), (i, 2), \ldots, (i, m)\} \);
\( E(H_i) = \{(i, u)(i + 1, v)/uv \in E(G)\} \) and \( V(H_n) = W_n \cup W_{n+1}; E(H_n) = \{(n, u)/uv \in E(G)\} \) Moreover, if \( G \) has no odd cycle then for each \( H_1 \) and \( H_n \) has exactly two connected components isomorphic to \( G \).

**Example.**

2. Matching Number of the Graph of \( C_n \otimes G \)

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 7 that show character of matching for each \( H_i \).
**Definition 5.** Given a matching $M$, an $M$-alternating path is a path that alternates between edges in $M$ and edges not in $M$. An $M$-alternating path whose endpoints are unsaturated by $M$ is an $M$-augmenting path.

**Theorem 6.** A matching $M$ in a graph $G$ is a maximum matching in $G$ if and only if $G$ has no $M$-augmenting path.

Next, we giving Lemma 7 which show character of matching for each $H_i$.

**Lemma 7.** Let $C_n \otimes G = ( \bigcup_{i=1}^{n-1} H_i ) \cup H_n$. For each $H_i$ and $H_n$, then $\alpha'(H_i) = \alpha'(H_n) = 2\alpha'(G)$.

*Proof.* Suppose $G$ has no odd cycle, by proposition 1.4 we get $H_i=2G$. So $\alpha'(H_i) = 2\alpha'(G)$. If $G$ has odd cycle, for each $H_i$, vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} H_i = C_n \otimes (G - \tau)$ when $\tau$ is an edge in odd cycle, $M$ be the maximum matching of
G. We get \( \overline{H_i} = 2(G - \overline{e}) \) then
\[
\alpha'(\overline{H_i}) = 2\alpha'(G - \overline{e}) = \begin{cases} 
2[\alpha'(G) - 1], & \text{if } \overline{e} \text{ is in } M, \\
2\alpha'(G), & \text{otherwise.}
\end{cases}
\]

When we add \( \overline{e} \) comeback, we get \( \alpha'(H_i) = \alpha'(\overline{H_i}) + 1 \). Hence \( \alpha'(H_i) = 2\alpha'G \). Similarly, \( \alpha'(H_n) = 2\alpha'G \). \( \square \)

Next, we establish Theorem 8 for a matching number of \( C_n \otimes G \)

**Theorem 8.** Let \( G \) be connected graph order \( m \), then
\[ \alpha'(C_n \otimes G) = \max\{na'(G), m\left\lfloor \frac{n}{2} \right\rfloor\}. \]

**Proof.** Let \( V(C_n) = \{u_i, i = 1, 2, ..., n\} \), \( V(G) = \{v_j, j = 1, 2, ..., m\} \), \( S_i = \{(u_i, v_j) \in V(C_n \otimes G) : j = 1, 2, ..., m\} \), \( i = 1, 2, ..., n \) and since \( \alpha'(C_n) = \left\lfloor \frac{n}{2} \right\rfloor \).

Let \( \alpha'(G) = k \), assume that the maximum matching of \( C_n, G \) be
\[
M_1 = \{u_1u_2, u_3u_4, ..., u_{2\left\lfloor \frac{n}{2} \right\rfloor - 1}u_{2\left\lfloor \frac{n}{2} \right\rfloor}\},
\]
\[
M_2 = \{v_jv_{j+1}/j = 1, 3, ..., 2k - 1\},
\]
respectively.

By Lemma 2.2 we have \( \alpha'(H_i) = 2\alpha'(G) \). Since \( C_n \otimes G \) is \( \bigcup_{i=1}^{n-1} H_i \) union \( H_n \) which have matching in \( H_1, H_3, ..., H_{2\left\lfloor \frac{n}{2} \right\rfloor - 1} \), then \( \alpha'(C_n \otimes G) \geq na'(G) \).

By definition of matching, we get another matching of \( C_n \otimes G \) be set of edges such that incident with vertices in \( S_i \) and \( S_i+1, i = 1, 3, ..., 2\left\lfloor \frac{n}{2} \right\rfloor - 1 \). So \( \alpha'(C_n \otimes G) \geq m\left\lfloor \frac{n}{2} \right\rfloor \).

Hence \( \alpha'(C_n \otimes G) \geq \max\{na'(G), m\left\lfloor \frac{n}{2} \right\rfloor\} \).

If \( na'(G) > m\left\lfloor \frac{n}{2} \right\rfloor \), suppose that \( \alpha'(C_n \otimes G) > na'(G) \), then there exist a matching \( M \) is a augmenting path. That is not true because each vertices in \( C_n \otimes G \) always incident with edges in
\[
M = \bigcup_{i=1,3,2\left\lfloor \frac{n}{2} \right\rfloor - 1}^{n-1} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, ..., 2k - 1\}
\]
\[ \bigcup \bigcup_{i=1,3,2\left\lfloor \frac{n}{2} \right\rfloor - 1}^{n-1} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, ..., 2k\} \]

and another edges which are not in \( M \):
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Figure 2: The Matching $M$ when $n\alpha'(G) > m\lfloor n/2 \rfloor$ and $n$ is odd

$N = \bigcup_{i=2,4,2\lfloor n/2 \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, \ldots, 2k - 1\} \cup \bigcup_{i=2,4,2\lfloor n/2 \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, \ldots, 2k\} \cup \{\,(u_1, v_j)(u_n, v_{j+1})/j = 1, 3, \ldots, 2k - 1\} \cup \{\,(u_1, v_j)(u_n, v_{j-1})/j = 2, 4, \ldots, 2k\}$,

so the endpoints of $M$ are saturated by $M$.

If $n\alpha'(G) < m\lfloor n/2 \rfloor$, suppose that $\alpha'(C_n \otimes G) > m\lfloor n/2 \rfloor$, it is not true because every $S_i$ have $|S_i| = m$.

Hence $\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m\lfloor n/2 \rfloor\}$. \hfill \Box

3. Edge Covering number of the graph of $C_n \otimes G$

We begin this section by giving Lemma 9 that shows a relation of matching number and edge covering number and Lemma 10 that show character of edge cover number for each $H_i$.

Lemma 9. Let $G$ be a simple graph with order $n$. Then $\alpha'(G) + \beta'(G) = n$
Figure 3: The Matching $M$ when $na'(G) < m\lfloor \frac{n}{2} \rfloor$ and $n$ is odd

**Lemma 10.** Let $C_n \otimes G = \bigcup_{i=1}^{n-1} H_i \cup H_n$. For each $H_i$ and $H_n$ then

$$\beta'(H_i) = \beta'(H_n) = 2\beta'(G)$$

**Proof.** Suppose $G$ has no odd cycle, by proposition 1.4, we get $H_i=2G$. So $\beta'(H_i) = 2\beta'(G)$.

If $G$ has odd cycle, for each $(u_{i+1}, v) \in W_i$, $(u_i, v) \in W_{i+1}$ in $V(H_i)$ and $(u_n, v) \in W_n$ in $V(H_n)$ have $d_{H_i}((u_i, v)) = d_H(u_{i+1}, v) = d_G(v) = d_{H_n}((u_n, v)) = d_{H_n}(u_1, v))$. Let $\bigcup_{i=1}^{n-1} H_i = C_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, $C$ be the minimum edge covering set of $G$. We get $\overline{H_i} = 2(G - \overline{e})$ then

$$\beta(\overline{H_i}) = \begin{cases} 2[\beta(G) + 2], & \text{if } \overline{e} = xy \in C \text{ with } d(x) > 1 \text{ and } d(y) > 1, \\ 2[\beta(G) - 1], & \text{if } \overline{e} = xy \in C \text{ with } d(x) \geq 1 \text{ or } d(y) \geq 1, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$
When we add $\overline{e}$ comeback, in the case $\beta^*(G - \overline{e}) = \beta^*(G) - 1$, we get $\beta^*(H_i) = \beta^*(\overline{H_i}) + 1$. And in the case $\beta^*(G - \overline{e}) = \beta^*(G) + 2$, we get $\overline{e} = xy \in C$ of $G$ replace edges $ux, yv$ (edge cover of $G - \overline{e}$), so $\beta^*(G - \overline{e}) = \beta^*(G) - 2$.

Hence $\beta^*(H_i) = 2\beta^*(G)$. Similarly, $\beta^*(H_n) = 2\beta^*(G)$.

Next, we establish Theorem 11 for a minimum edge covering number of $C_n \otimes G$.

**Theorem 11.** Let $G$ be connected graph order $m$, then $\beta^*(C_n \otimes G) = \min\{n\beta^*(G), m\lceil\frac{n}{2}\rceil\}$

**Proof.** Let $V(C_n) = \{u_i, i = 1, 2, \ldots, n\}, V(G) = \{v_j, j = 1, 2, \ldots, m\}, S_i = \{(u_i, v_j) \in V(C_n \otimes G) : j = 1, 2, \ldots, m\}, i = 1, 2, \ldots, n$ and since $\beta^*(C_n) = \lceil\frac{n}{2}\rceil$.

Let $\beta^*(G) = k$, assume that the maximum matching of $G$ be $M_2$, and minimum edge covering set of $C_n, G$ be

$$C_1 = \begin{cases} \{u_1u_2, u_3u_4, \ldots, u_{n-1}u_n\} & \text{where } n \text{ is even,} \\ \{u_1u_2, u_3u_4, \ldots, u_{n-2}u_{n-1}, u_nu_1\} & \text{where } n \text{ is odd,} \end{cases}$$

$$C_2 = M_2 \cup \{v_jv/j = 2k+1, 2k+2, \ldots, m \text{ and } v \text{ is endvertex of matching in } M_2\},$$

respectively.
By Lemma 3.2 we have $\beta'(H_i) = 2\beta'(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which have edge cover in $H_1, H_3, \ldots, H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, then $\beta'(C_n \otimes G) \leq n\beta'(G)$.

Since definition of edge cover, we get another edge cover of $C_n \otimes G$ be set of edges, such that incident with vertices in $S_i$ and $S_{i+1}$, $i = 1, 3, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 1$. So $\beta'(C_n \otimes G) \leq m\left\lceil \frac{n}{2} \right\rceil$.

Hence $\beta'(C_n \otimes G) \leq \min\{n\beta'(G), m\left\lceil \frac{n}{2} \right\rceil\}$.

If $n\beta'(G) < m\left\lceil \frac{n}{2} \right\rceil$, suppose that $\beta'(C_n \otimes G) < n\beta'(G)$, then there exist edges $xy$ in edge covering of each $H_1, H_3, \ldots, H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, which is endvertex $x$ and $y$ incident with another edges in edge covering of each $H_1, H_3, \ldots, H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, it not impossible.

If $n\beta'(G) > m\left\lceil \frac{n}{2} \right\rceil$, suppose that $\beta'(C_n \otimes G) > m\left\lceil \frac{n}{2} \right\rceil$, that is not true because every $S_i$ have $|S_i| = m$.

Hence $\beta'(C_n \otimes G) = \min\{n\beta'(G), m\left\lceil \frac{n}{2} \right\rceil\}$. \(\square\)

By Theorem 2.3 and Lemma 3.1, we can also show that:

$$\alpha'(C_n \otimes G) + \beta'(C_n \otimes G) = mn,$$
\[
\max\{n\alpha'(G), m\left\lfloor \frac{n}{2} \right\rfloor\} + \beta'(C_n \otimes G) = mn,
\]

\[
\beta'(C_n \otimes G) = mn - \max\{n\alpha'(G), m\left\lfloor \frac{n}{2} \right\rfloor\}
\]
\[
= mn + \min\{-n\alpha'(G), -m\left\lfloor \frac{n}{2} \right\rfloor\}
\]
\[
= \min\{n(m - \alpha'(G)), m(n - \left\lfloor \frac{n}{2} \right\rfloor)\}
\]
\[
= \min\{n\beta'(G)), m\left\lceil \frac{n}{2} \right\rceil\}.
\]

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\section*{References}


