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OSCILLATION CRITERIA FOR THE SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH IMPULSE EFFECT

M.B. Dimitrova
Department of Mathematics
Technical University of Sliven
8800, Sliven, BULGARIA

Abstract: Necessary and sufficient conditions are found for oscillation of all solutions of a class of nonlinear delay differential equations of second order with impulse effects at fixed moments.

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1. Introduction

In the recent two decades the number of investigations of the oscillatory and nonoscillatory behavior of the solutions of functional differential equations is constantly growing. Greater part of the works on this subject are given in [14]. The oscillatory and asymptotic properties of various classes of functional differential equations are systematically studied in the monographs [7], [10] and [11].

The oscillation theory of the impulsive differential equations in recent years is also developed. See, for example the monographs [1],[2] and [13].

Needless to say that today, the mixture between the functional differential equations and the impulsive differential equations, which we call impulsive differential equations with deviating argument (IDEDA), is very attractive for explorations. Indeed, the impulsive part of IDEDA reflects on the disconti-

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nuities of first kind and can be used to model mathematically the short-time disturbances of some processes in the nature, whereas the differential part with its deviating argument is very convenient instrument for simulation of the dependance of the same processes on their history. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnology, industrial robotics, biology, electrical engineering, etc. The pioneer work, devoted to the investigation of the oscillatory properties of the solutions of impulsive differential equations with a deviating argument was [8]. Among numerous papers, published on IDEDA, we choose to refer to [3]-[6] and [9]. In spite of wide possibilities for their application, the theory of these equations is developing rather slowly because of difficulties of technical and theoretical character related to their study, especially the equations of second, or higher order.

In the present paper necessary and sufficient conditions for oscillation of all solutions of a class of nonlinear delay differential equations of second order are considered, where the first derivative (the velocity of the process) is subject to impulse perturbations at fixed moments.

2. Preliminary Notes

Consider the nonlinear delay differential equation with impulse effects on the first derivative at fixed moments

$$\begin{aligned} &\big(r(t)y'(t)\big)' + f(t, y(t), y(t-h)) = 0, \quad t \neq \tau_k,\ k \in N, \\ &\Delta\big(r(\tau_k)y'(\tau_k)\big) + g_k(y(\tau_k), y(\tau_k - h)) = 0, \quad \Delta y(\tau_k) = 0. \end{aligned} \tag{1}$$

with initial condition

$$\begin{aligned} y(t) &= \varphi(t), \quad t \in [-h, 0), \\ y(0) &= y_0, \qquad y'(0) = y_0', \end{aligned} \tag{2}$$

where h is a positive constant and $\varphi \in C^2([-h,0), R)$.

The points $\tau_k \in (0, +\infty),\ k \in N$ are the moments of impulsive effect (let us call them jump points), where the respective function reveals its discontinuities of first kind as jumps. In order to manifest these jumps of the unknown function $y(t)$, or the jumps of its first derivative, we use the notation

$$\Delta y(\tau_k) = y(\tau_k + 0) - y(\tau_k - 0), \qquad \Delta y'(\tau_k) = y'(\tau_k + 0) - y'(\tau_k - 0),$$

$$\Delta\big(r(\tau_k)y'(\tau_k)\big) = r(\tau_k + 0)y'(\tau_k + 0) - r(\tau_k - 0)y'(\tau_k - 0),$$

where

$$
\begin{aligned}
y(\tau_k-0) &= \lim_{\varepsilon\to 0} y(\tau_k-\varepsilon),\\
y'(\tau_k-0) &= y'(t)|_{t=\tau_k-} = \lim_{\varepsilon\to 0} y\frac{y(\tau_k-\varepsilon)-y(\tau_k-0)}{-\varepsilon},\\
y(\tau_k+0) &= \lim_{\varepsilon\to 0} y(\tau_k+\varepsilon),\\
y'(\tau_k+0) &= y'(t)|_{t=\tau_k+} = \lim_{\varepsilon\to 0} \frac{y(\tau_k+\varepsilon)-y(\tau_k+0)}{\varepsilon}.
\end{aligned}
$$

Denote by $P_\tau C(R,R)$ the set of all piecewise continuous on $(\tau_k,\tau_{k+1}],\ k\in N$ functions $u\colon R\to R$, which at the points τ_k are continuous from the left, i.e. $u(\tau_k-0) = \lim\limits_{t\to\tau_k-0} u(t) = u(\tau_k)$, for which there exists a sequence of reals $\{u(\tau_k+0)\}$, such that $u(\tau_k+0) = \lim\limits_{t\to\tau_k+0} u(t)$, and which at the jump points $\tau_k,\ k\in N$ may have discontinuities of first kind, that we characterize as down-jumps when $\Delta u(\tau_k)<0$, or as up-jumps when $\Delta u(\tau_k)>0,\ k\in N$.

Denote by $P_\tau C^k(R,R)$ the set of all functions $u\colon R\to R$, for which $\dfrac{d^k u}{dt^k}\in P_\tau C(R,R)$.

Introduce the following hypotheses, where $R_+=(0,+\infty)$ and $\overline{R}_+=[0,+\infty)$:

H1. $0<\tau_1<\tau_2<\ldots<\tau_k<\ldots,\ \lim\limits_{k\to+\infty}\tau_k=+\infty,\ \max\{\tau_{k+1}-\tau_k\}<+\infty,\ k\in N$.

H2. $f\in C(\overline{R}_+\times R^2,R)$; $f(t,u,v)$ is a nondecreasing function with respect to u and v for each fixed $t\in\overline{R}_+$ and $uf(t,u,v)>0$ for $uv>0,\ t\in\overline{R}_+$.

H3. $g_k\in C(R^2,R)$; $ug_k(u,v)>0$ for $uv>0,\ k\in N,\ g_k(u,v)$ are nondecreasing functions with respect to u and v, $k\in N$.

H4. $r\in P_\tau C(\overline{R}_+,R_+),\ \ r(\tau_k+0)>0,\ k\in N$ and $\lim\limits_{t\to+\infty}R(t)=+\infty$, where

$$R(t)=\int_0^t \frac{ds}{r(s)}.$$

Introduce the notations $\tau_{kh}=\tau_k+h,\ k\in N$. We should note that in general, we have $\{\tau_{kh}|k\in N\}\cap\{\tau_k|k\in N\}\neq\emptyset$. Let $\{\theta_k|k\in N\}=\{\tau_{kh}|k\in N\}\cup\{\tau_k|k\in N\}$ and $0<\theta_1<\theta_2<\ldots$. Obviously, $\theta_1=\tau_1$.

Definition 1. A *solution* of the equation (1) with initial conditions (2) will be called such a function $y\in P_\tau C^1([-h,+\infty),R)$, for which the following conditions are fulfilled:

1. If $-h \leq t < 0$, then $y(t) = \varphi(t)$.

2. If $t = 0$, then $y(0) = y_0$ and $y'(0) = y'_0$.

3. If $0 < t \leq \theta_1 = \tau_1$, then the solution y coincides with the solution of the equation without impulses

$$\big(r(t)y'(t)\big)' + f(t, y(t), y(t-h)) = 0 \tag{3}$$

with initial conditions (2).

4. If $\theta_k < t \leq \theta_{k+1}$, $k \in N$ and $\theta_k \in \{\tau_k | k \in N\} \setminus \{\tau_{kh} | k \in N\}$, then the function y coincides with the solution of the equation (3) with initial conditions

$$\begin{aligned} y(\theta_k + 0) =& y(\theta_k), \\ y'(\theta_k + 0) =& \frac{1}{r(\theta_k + 0)}\Big[r(\theta_k)y'(\theta_k) - g_k(y(\theta_k), y(\theta_k - h))\Big]. \end{aligned} \tag{4}$$

5. If $\theta_k < t \leq \theta_{k+1}$, $k \in N$ and $\theta_k \in \{\tau_{kh} | k \in N\} \setminus \{\tau_k | k \in N\}$, then the solution of the problem (1), (2) coincides with the solution of the equation

$$\big(r(t)y'(t+0)\big)' + f(t, y(t), y(t-h)) = 0 \tag{5}$$

with initial conditions

$$y(\theta_k + 0) = y(\theta_k), \qquad y'(\theta_k + 0) = y'(\theta_k).$$

6. If $\theta_k < t \leq \theta_{k+1}$, $k \in N$ and $\theta_k \in \{\tau_{kh} | k \in N\} \cap \{\tau_k | k \in N\}$, the solution of the problem under consideration coincides with the solution of the equation (5) with initial conditions (4).

Definition 2. The solution $y(t)$ of the equation (1) is said to be *eventually positive*, if there exist a number $a \in R_+$, such that $y(t) > 0$, $t \geq a$.

Analogously, the solution $y(t)$ of the equation (1) is said to be *eventually negative*, if there exist a number $a \in R_+$, such that $y(t) < 0$, $t \geq a$.

Definition 3. The solution $y(t)$ of the equation (1) is said to be *oscillatory*, if there exist a number $b \in R_+$, such that $\{t\colon\ y(t) > 0,\ t > a\} \neq \emptyset$ and $\{t\colon\ y(t) < 0,\ t > a\} \neq \emptyset$, for each $a > b$. Otherwise, the solution $y(t)$ is called *nonoscillatory.*

Definition 4. The function $f(t, u, v)$ (resp. $g_k(u, v)$) is said to be *strictly superlinear*, if there exists a constant $\beta > 1$ such that $\dfrac{|f(t, u, v)|}{|u|^\beta}$ $\left(\text{resp.} \dfrac{|g_k(u, v)|}{|u|^\beta}\right)$ is nondecreasing function with respect to $|u|$ and $|v|$ for each fixed $t \geq 0$ ($k \in N$).

Definition 5. The function $f(t,u,v)$ (resp. $g_k(u,v)$) is said to be *strictly sublinear*, if there exists a constant β $(0<\beta<1)$ such that the function $\dfrac{|f(t,u,v)|}{|u|^\beta}$ $\left(\text{resp.}\dfrac{|g_k(u,v)|}{|u|^\beta}\right)$ is nonincreasing with respect to $|u|$ and $|v|$ for each fixed $t\geq 0$ $(k\in N)$.

3. Main Results

Our first result consider the neutral terms of the equation (1) as strictly sublinear.

Theorem 1. *Let we suppose that:*

1. *The hypotheses H1-H4 are satisfied.*

2. *The functions $f(t,u,v)$ and $g_k(u,v)$, $(k\in N)$, are strictly sublinear with one and the same constant β $(0<\beta<1)$.*

Then the following assertions are equivalent:

(a) *For each nonzero constant c, we have*

$$\int\limits_0^\infty \big|f(s,cR(s),cR(s-h))\big|ds+\sum_{k=1}^\infty \big|g_k(cR(\tau_k),cR(\tau_k-h))\big|=+\infty. \tag{6}$$

(b) *Every solution of the equation* (1) *is oscillatory.*

Proof. $(b)\Longrightarrow(a)$: Suppose that the equation (1) is oscillatory, but (a) is not satisfied. Without loss of generality we may assume, that there exist a constant $c>0$ and a point $T>h$ such that

$$\int\limits_T^\infty f(s,cR(s),cR(s-h))ds+\sum_{\tau_k\geq T} g_k(cR(\tau_k),cR(\tau_k-h))\leq\frac{c}{2}. \tag{7}$$

Consider the set

$$M=\Big\{y\in C([T-h,+\infty),R):\ y(t)=0,\ t\in[T-h,T]\ \text{ and }\ \frac{c}{2}\big[R(t)-R(T)\big]\leq y(t)\leq c\big[R(t)-R(T)\big],\quad t>T\Big\}.$$

Define the operator $S\colon\ M \to C([T-h,+\infty),R)$, by the formula

$$(Sy)(t)=\begin{cases}0, & T-h\le t\le T,\\ \displaystyle\int_T^t \frac{1}{r(p)}\Big[\frac{c}{2}+\int_p^\infty f(s,y(s),y(s-h))ds & \\ \displaystyle\qquad +\sum_{p\le\tau_k} g_k(y(\tau_k),y(\tau_k-h))\Big]dp, & t>T.\end{cases} \tag{8}$$

Let $y\in M$. It follows from the definition of the set M, the hypotheses H2-H4 and from (7), that

$$(Sy)(t)\ge \frac{c}{2}\big[R(t)-R(T)\big], \qquad t\ge T.$$

From the inequality $y(t)\le c\big[R(t)-R(T)\big]$, it follows, that $y(t)\le cR(t)$. In view of the last inequality, as well as in view of the hypotheses H2 and H3, we arrive at

$$\int_p^\infty f(s,y(s),y(s-h))ds\le \int_T^\infty f(s,cR(s),cR(s-h))ds, \qquad p\ge T \tag{9}$$

and

$$\sum_{\tau_k\ge p} g_k(y(\tau_k),y(\tau_k-h))\le \sum_{\tau_k\ge T} g_k(cR(\tau_k),cR(\tau_k-h)), \qquad p\ge T. \tag{10}$$

It follows from (7)-(10), that $(Sy)(t)\le c\big[R(t)-R(T)\big]$, $t\ge T$. Therefore, $SM\subseteq M$.

Let us define now the functions $u_n\colon\ [T-h,+\infty)\to R,$ by the recursive relation

$$u_n(t)=(Su_{n-1})(t), \qquad n\in N,$$

where

$$u_0(t)=\begin{cases}0, & t\in[T-h,T),\\ \frac{c}{2}\big[R(t)-R(T)\big], & t\ge T.\end{cases}$$

Direct verification shows the validity of the inequalities

$$\frac{c}{2}\big[R(t)-R(T)\big]\le u_{n-1}(t)\le u_n(t)\le c\big[R(t)-R(T)\big], \qquad t\ge T.$$

Therefore, for $t \geq T - h$ there exists the limit $\lim\limits_{n \to +\infty} u_n(t) = u(t)$. Now, the Lebesque dominated convergence theorem implies $u \in M$ and $u(t) = (Su)(t)$, whence $u(t)$ is a solution of the equation (1).

It follows from the hypothesis H4 that $\lim\limits_{t \to \infty} R(t) = \infty$. Finally, we have the inequality $\frac{c}{2}\big[R(t) - R(T)\big] \leq u(t)$ for $t \geq T$, which implies $\lim\limits_{t \to \infty} u(t) = \infty$. Moreover, $u(t)$ becomes nonoscillatory. So, if (6) does not hold true, then the equation (1) has a nonoscillatory solution. This contradicts to our assumption, that (1) is oscillatory.

$(a) \Longrightarrow (b)$: Suppose, that (6) is fulfilled, but the equation (1) still has a nonoscillatory solution y. Since the negative of a solution of (1) is again a solution of (1), it suffices to prove the theorem considering this solution as an eventually positive function. So, suppose that $y(t) > 0$ for $t \geq t_0 \geq 0$. It is clear that $y(t-h) > 0$ for $t \geq t_1 = t_0 + h$. Then, (1) and the hypotheses H2 and H3 imply

$$\big(r(t)y'(t)\big)' < 0 \qquad \text{and} \qquad \Delta\big(r(\tau_k)y'(\tau_k)\big) < 0 \quad \text{for} \quad t \geq t_1.$$

Therefore, the function $r(t)y'(t)$ is a decreasing function for $t \geq t_1$.

Let $r(t)y'(t) < 0$ for $t \geq t_1$. Since $r(t)y'(t)$ is a decreasing function for $t \geq t_1$, it follows that $r(t)y'(t) \leq r(t_1)y'(t_1) = c < 0$, $t \geq t_1$. If we integrate the last inequality from t_1 to t, $(t > t_1)$, we obtain

$$y(t) \leq y(t_1) + c\int_{t_1}^{t} \frac{ds}{r(s)}.$$

Taking into account the hypothesis H4, we conclude from the above inequality as $t \to +\infty$, that $\lim\limits_{t \to +\infty} y(t) = -\infty$. But, this contradicts the assumption that y is a positive solution of the equation (1).

Let $r(t)y'(t) \geq 0$ for $t \geq t_1$. From the fact that $r(t)y'(t)$ is a decreasing function in $[t_1, +\infty)$, it follows that there exists the finite limit

$$\lim_{t \to +\infty} r(t)y'(t) = A, \qquad A \in [0, +\infty).$$

If we integrate (1) from t to $+\infty$, $(t \geq t_1)$, we arrive at

$$y'(t) \geq \frac{1}{r(t)}\left[\int_{t}^{\infty} f(s, y(s), y(s-h))ds + \sum_{\tau_k \geq t} g_k(y(\tau_k), y(\tau_k - h))\right]. \tag{11}$$

Let $t_2 > t_1$ be such that $R(t) - R(t_1) \geq \frac{1}{2}R(t)$ for $t \geq t_2$. Next, if we integrate (11) from t_1 to t $(t \geq t_2)$, we obtain

$$\begin{aligned}
y(t) \geq & y(t) - y(t_1) \\
\geq & \int_{t_1}^{t} \frac{1}{r(s)} \left[\int_{s}^{\infty} f(p, y(p), y(p-h))dp + \sum_{s \leq \tau_k} g_k(y(\tau_k), y(\tau_k - h)) \right] ds \\
\geq & \int_{t_1}^{t} \frac{1}{r(s)} \left[\int_{t}^{\infty} f(p, y(p), y(p-h))dp + \sum_{t \leq \tau_k} g_k(y(\tau_k), y(\tau_k - h)) \right] ds \\
\geq & \frac{1}{2}R(t) \left[\int_{t}^{\infty} f(p, y(p), y(p-h))dp + \sum_{t \leq \tau_k} g_k(y(\tau_k), y(\tau_k - h)) \right].
\end{aligned} \tag{12}$$

Since there exists a constant $c > 0$, such that $y(t) \leq cR(t)$ for $t \geq t_2$, it follows from condition 2 of the theorem that

$$\begin{aligned}
f(t, y(t), y(t-h)) = & \frac{f(t, y(t), y(t-h))}{y^{\beta}(t)} y^{\beta}(t) \\
\geq & \frac{f(t, cR(t), cR(t-h))}{c^{\beta}} \left[\frac{y(t)}{R(t)} \right]^{\beta}
\end{aligned} \tag{13}$$

and

$$g_k(y(\tau_k), y(\tau_k - h)) \geq \frac{g_k(cR(\tau_k), cR(\tau_k - h))}{c^{\beta}} \left[\frac{y(\tau_k)}{R(\tau_k)} \right]^{\beta}, \qquad \tau_k \geq t_2. \tag{14}$$

Now, (12), (13) and (14) yield

$$\begin{aligned}
\frac{y(t)}{R(t)} \geq \frac{1}{2c^{\beta}} \Bigg[& \int_{t}^{\infty} f(s, cR(s), cR(s-h)) \left[\frac{y(s)}{R(s)} \right]^{\beta} ds \\
& + \sum_{t \leq \tau_k} g_k(cR(\tau_k), cR(\tau_k - h)) \left[\frac{y(\tau_k)}{R(\tau_k)} \right]^{\beta} \Bigg] \equiv U(t).
\end{aligned}$$

Moreover, the last inequality implies

$$U'(t) \leq -\frac{1}{2c^{\beta}} f(t, cR(t), cR(t-h)) U^{\beta}(t), \quad t \neq \tau_k,$$

$$\Delta U(\tau_k) = -\frac{1}{2c^\beta} g_k(cR(\tau_k), cR(\tau_k - h))U^\beta(\tau_k).$$

Therefore, from the fact that $U(t)$ is a decreasing function, and from the inequality $a^{1-\beta} - b^{1-\beta} \leq (1-\beta)b^{-\beta}(a-b)$ for $0 < a < b$, $0 < \beta < 1$, there follows

$$\begin{aligned} \big[U^{1-\beta}(t)\big]' &= (1-\beta)U^{-\beta}(t)U'(t) \\ &\leq -\frac{1}{2}(1-\beta)c^{-\beta}f(t, cR(t), cR(t-h)), \quad t \neq \tau_k, \\ \Delta\big[U^{1-\beta}(\tau_k)\big] &= U^{1-\beta}(\tau_k + 0) - U^{1-\beta}(\tau_k) \leq (1-\beta)U^{-\beta}(\tau_k)\Delta U(\tau_k) \\ &\leq -\frac{1}{2}(1-\beta)c^{-\beta}g_k(cR(\tau_k), cR(\tau_k - h)). \end{aligned} \tag{15}$$

Further, if we integrate (15) from t_2 to t ($t \geq t_2$), we obtain

$$U^{1-\beta}(t) - U^{1-\beta}(t_2) \leq -\tfrac{1}{2}(1-\beta)c^{-\beta}\left[\int_{t_2}^{t} f(s, cR(s), cR(s-h))ds + \sum_{t_2 \leq \tau_k < t} g_k(cR(\tau_k), cR(\tau_k - h))\right].$$

Passing here to the limit as $t \to +\infty$ $(\lim_{t\to+\infty} U(t) = 0)$ implies the contradiction

$$\int_{t_2}^{\infty} f(s, cR(s), cR(s-h))ds + \sum_{t_2 \leq \tau_k} g_k(cR(\tau_k), cR(\tau_k - h)) < +\infty.$$

The proof is complete.

The next result consider the neutral terms of the equation (1) as strictly superlinear.

Theorem 2. *Let we suppose that:*

1. *The hypotheses H1-H4 are satisfied.*

2. *The functions f and g_k $(k \in N)$ are strictly superlinear with one and the same constant $\beta > 1$.*

Then the following assertions are equivalent:

(a) *For each constant* $c \neq 0$ *we have*
$\int_0^\infty R(s)\big|f(s,c,c)\big|ds + \sum_{k=1}^{\infty} R(\tau_k)\big|g_k(c,c)\big| = +\infty.$

(b) *Every solution of the equation* (1) *is oscillatory.*

Proof. $(b) \Longrightarrow (a)$: Let us suppose that the equation (1) is oscillatory, but the assertion (a) is not satisfied, i.e.,

$$\int_0^\infty R(s)\big|f(s,c,c)\big|ds + \sum_{k=1}^\infty R(\tau_k)\big|g_k(c,c)\big| < +\infty$$

for some constant $c \neq 0$. Without loss of generality we may assume, that $c > 0$. Then, there exists a point $T \geq h$ such that

$$\int_T^\infty R(s)f(s,c,c)ds + \sum_{T\leq\tau_k} R(\tau_k)g_k(c,c) \leq \frac{c}{2}. \tag{16}$$

Introduce the set

$$M = \Big\{y \in C([T-h,+\infty),R) : y(t) = \frac{c}{2},\ t \in [T-h,T) \text{ and } \frac{c}{2} \leq y(t) \leq c,\ t \geq T\Big\}.$$

Define the operator $S\colon\ M \to C([T-h,+\infty),R)$, by the formula

$$(Sy)(t) = \begin{cases} \dfrac{c}{2}, & T-h \leq t < T, \\ \dfrac{c}{2} + \displaystyle\int_T^t \frac{1}{r(s)}\Big[\int_s^\infty f(p,y(p),y(p-h))dp \\ \qquad + \sum_{s\leq\tau_k} g_k(y(\tau_k),y(\tau_k-h))\Big]ds, & t \geq T. \end{cases}$$

Let $y \in M$. Then $(Sy)(t) \geq \dfrac{c}{2}$ for $t \geq T-h$. Moreover, it follows from (16) and the definition of the operator S, that $(Sy)(t) \leq c$. Therefore, $SM \subseteq M$. Analogously to the proof of Theorem 1 we obtain that the operator S has a fixed point $u \in M$, i.e., $u(t) = (Su)(t)$, $t \geq T-h$. It can be verified directly that u is a solution of the equation (1), such that $\dfrac{c}{2} \leq u(t) \leq c$ for $t \geq T$, i.e., $u(t)$ is a nonoscillatory solution of the equation (1). This contradicts to our assumption, that (1) is oscillatory.

$(a) \Longrightarrow (b)$: Suppose, that (a) is fulfilled, but the equation (1) still has a nonoscillatory solution y. Since the negative of a solution of (1) is again a solution of (1), it suffices to prove the theorem considering this solution as an eventually positive function. So, suppose that $y(t) > 0$ for $t \geq t_0 \geq 0$. Clearly, $y(t-h) > 0$ for $t \geq t_1 = t_0 + h$. Then, (1) and the hypotheses H2 and H3 imply that $\big(r(t)y'(t)\big)' < 0$ and $\Delta\big(r(\tau_k)y'(\tau_k)\big) < 0$ for $t \geq t_1$. Therefore, the function $r(t)y'(t)$ is a decreasing function for $t \geq t_1$. Analogously to the proof of Theorem 1 we arrive at the inequality $r(t)y'(t) > 0$ for $t \geq t_1$, i.e., y is an increasing function for $t \geq t_1$ whence

$$y(t) \geq c = y(t_0) > 0, \qquad y(t-h) \geq c > 0, \qquad t \geq t_1. \tag{17}$$

As in the proof of Theorem 1 we derive (11). It follows from (17) and condition 2 of Theorem 2, that

$$f(t, y(t), y(t-h)) = \frac{f(t, y(t), y(t-h))}{y^\beta(t)} y^\beta(t) \geq \frac{f(t, c, c)}{c^\beta} y^\beta(t), \quad t \geq t_1 \tag{18}$$

and

$$g_k(y(\tau_k), y(\tau_k - h)) \geq \frac{g_k(c, c)}{c^\beta} y^\beta(\tau_k), \quad \tau_k \geq t_1. \tag{19}$$

On the other hand, (11), (18) and (19) yield

$$y'(t) \geq \frac{1}{r(t)} \left[\int_t^\infty f(s, c, c) ds + \sum_{t \leq \tau_k} g_k(c, c) \right] \frac{y^\beta(t)}{c^\beta}, \tag{20}$$

since $y(t)$ is an increasing function for $t \geq t_1$.

If we divide (20) by $y^\beta(t)$ and integrate from t_1 to $+\infty$, we arrive at

$$c^{-\beta} \int_{t_1}^\infty \frac{1}{r(t)} \left[\int_t^\infty f(s, c, c) ds + \sum_{t \leq \tau_k} g_k(c, c) \right] dt \leq \frac{y^{1-\beta}(t_1)}{\beta - 1} < +\infty.$$

The last inequality implies

$$\int_0^\infty R(s) f(s, c, c) ds + \sum_{k=1}^\infty R(\tau_k) g_k(c, c) < +\infty,$$

which contradicts to our assumption. The proof is complete.

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