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METHOD OF AVERAGING FOR OPTIMAL CONTROL PROBLEMS WITH IMPULSIVE EFFECTS

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Abstract: In this paper are presented some results connected to the applications of the averaging method for solving of optimal control problems, where the models are systems of differential equations with impulsive effects. We suppose additional control in the impulses. Four schemes for averaging are presented.

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1. Introduction

In works devoted to the application of the averaging method in optimal control, are developed two approaches.

The first approach use necessary conditions for optimality, the original optimal control problem is reduced to a boundarry value problem, which can be solved using the method of averaging.

In the second method the equations of a moving object can be averaged directly, and then one solves the optimal control problem for the reduced system.

In this work we consider numerical-asymptotic methods for solving optimal control problems, which are based on the second approach. Briefly, this approach includes the following:

1. To the nonautonomous optimal control problem, using various averaging schemes for differential inclusions (in our case differential inclusions with impulses) we assign autonomous optimal control problem.

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2. So obtained simpler optimal control problem can be solved by using numerical methods.

This approach considerably reduces the calculations and makes the solution of the problem easier

Moreover, this approach extends the class of optimal control problems which are solvable by averaging methods.

We suppose that optimal control problem can be solved by numerical methods. The numerical methods usually have iterative character.In order to improve the functional, at each step the control is changed. For every step it is necessary to check whether the chosen control belongs to the given admissible set. In some cases a suitable way to do these checks is the theory of the support functions.

Below we propose and justify two averaging schemes. These schemes are suitable when support functions are used to check the belonging of chosen control to the given admissible set.

2. Preliminary Notes

In the domain $Q = D \times [0, L] \subset R^n \times R$ we consider the following impulsive systems of differential inclusions

$$\dot{x} \in F^1(t, x, \varepsilon),\ x(0) = x_0,\ t \neq \varepsilon\tau_i^1, \tag{1p}$$

$$\Delta x \mid_{t=\varepsilon\tau_i^{\mathbf{1}}} \in \varepsilon I_i^1(x) \tag{2p}$$

$$\dot{y} \in F^2(t, y, \varepsilon),\ y(0) = x_0,\ t \neq \varepsilon\tau_i^2, \tag{3p}$$

$$\Delta y \mid_{t=\varepsilon\tau_i^{\mathbf{2}}} \in \varepsilon I_i^2(x), \tag{4p}$$

where $t \in [0, L],\ L = const,\ x,\ y \in D \subset R^n,\ F^l(t, x, \varepsilon),\ (l = 1,\ 2)$ are multi-valued functions defined in $(n + 2)$-dimensional Euclidean space with compact and convex values, i.e. $F^l : Q \times R^{'} \longrightarrow conv(R^n)$. $conv(R^n)$ denote the spaces of the compact and convex subsets of R^n, $I_i^l : D \longrightarrow conv(R^n)$, τ_i^1 and τ_i^2 are fixed moments of the impulses $((i = 1, 2, ..., [k/\varepsilon]$, k is a natural number), Δx and Δy are the jumps of the solutions $x(t)$ and $y(t)$ of the differential inclusions and ε is a small parameter.

For the systems (1p), (2p) and (3p), (4p) we suppose the following condition

$$\lim_{\varepsilon\to 0} h \left[\frac{1}{\Delta t} \left(\int_t^{t+\Delta t} F^1(t, x, \varepsilon) dt + \varepsilon \sum_{t<\varepsilon\tau_i^{\mathbf{1}}<t+\Delta t} I_i^1(x) \right), \right.$$

$$\frac{1}{\Delta t}\left(\int_{t}^{t+\Delta t} F^2(t,x,\varepsilon)dt + \varepsilon \sum_{t<\varepsilon\tau_i^{\mathbf{2}}<t+\Delta t} I_i^2(x)\right)\Big] = 0 \tag{5p}$$

holds.

Here $h[A,B]$ denotes the Hausdorf's distance between two sets A and B; The integrals are Auman's integrals.

The condition ($5p$) is called "condition for integral continuity".

We need from the following theorem:

Theorem 1. (see [23]) *Let in the domain Q the following conditions be fulfilled:*

1) The multifunctions $F^1(t,x,\varepsilon)$ and $F^2(t,y,\varepsilon)$ are continuous with respect to t, Lipschitz' continuous w.r. to x with constant λ, and uniformly bounded, i.e.

$$\| F^1(t,x,\varepsilon) \| \leq M, \ \| F^2(t,y,\varepsilon) \| \leq M \ (M = const).$$

2) The multifunctions $I_i^1(x)$ and $I_i^2(x)$ are Lipschitz' continuous with constant λ, they are uniformly bounded, i.e.

$$(\| I_i^1(x) \| \leq M, \ \| I_i^2(x) \| \leq M, \ M = const)$$

and

$$x + I_i^l(x) \subset D.$$

3) The condition for integral continuity (5p) hold uniformly w.r. to (t,x).

4) The inequality $| \tau_i^1 - \tau_i^2 | < \eta = const$ hold.

5) For every $x^0 \in int\, D$ and for every $\varepsilon \in (0, \varepsilon_1]$, ($\varepsilon_1 = const$), there exists a constant $\rho > 0$, such that the ρ-neighbourhoods of solutions of the impulsive systems belong to the domain D for every $t \in [0, L]$.

Then for every $\xi > 0$ there exists such $\varepsilon^0 = \varepsilon^0(\xi) > 0$, that for $\varepsilon \in (0, \varepsilon^0]$ the following statements (affirmations) are truth:

1) For every solution $y(t) = y(t;\varepsilon)$ of (3p), (4p) there exists a solution $x(t) = x(t;\varepsilon)$ of (1p), (2p), such that

$$\| x(t) - y(t) \| < \xi. \tag{6p}$$

2) For every solution $x(t) = x(t;\varepsilon)$ of (1p), (2p) there exists a solution $y(t) = y(t;\varepsilon)$ of (3p), (4p)satisfying (which satisfy)(6p).

Corollary 1. *Let the condition of the Theorem 1 be fulfilled. Let the impulsive systems (1p)-(4p) be 2π-periodical systems with respect to t, i.e. $F^l(t,x,\varepsilon) = F^l(t+2\pi,x,\varepsilon)$ and there exists natural number p, such that $\tau^l_{i+p} = \tau^l_i + 2\pi$ and $I^l_{i+p}(x) = I^l_i(x)$, $(l = 1,\ 2)$.*

Than there exist constants $\varepsilon^0 > 0$ and $C > 0$, such that for every $\varepsilon \in (0,\varepsilon^0)$ the estimation (6p) has the following form

$$\| x(t) - y(t) \| < C\varepsilon. \tag{7p}$$

Let us consider the following systems in the standard form

$$\dot{x} \in \varepsilon X(t,x),\ t \neq \tau^1_i,\ x(0) = x_0,\ t \in [0, L\varepsilon^{-1}], \tag{8p}$$

$$\Delta x\ |_{t=\tau_i} \in \varepsilon I^1_i(x) \tag{9p}$$

and

$$\dot{y} \in \varepsilon Y(t,x),\ t \neq \tau_i,\ y(0) = x_0,\ t \in [0, L\varepsilon^{-1}], \tag{10p}$$

$$\Delta y\ |_{t=\tau_i} \in \varepsilon I^2_i(x). \tag{11p}$$

We assume that the following condition is true:

$$\lim_{T\to\infty} h \left[\frac{1}{T} \left(\int_t^{t+T} X(t,x)dt + \sum_{t<\tau_i<t+T} I^1_i(x) \right), \right. \tag{12p}$$

$$\left. \frac{1}{T} \left(\int_t^{t+T} Y(t,y)dt + \sum_{t<\tau_i<t+T} I^2_i(y) \right) \right] = 0.$$

The condition (12p) can be considered as a generalization of the Bogolyubov-Mitropolski's condition in the case for impulsive differential inclusions.

We will show that the condition for averaging (12p) follows from the condition (5p).

Changing the variable $s = \varepsilon t$, $(t = \frac{s}{\varepsilon})$ from the systems (8p)÷(11p) we obtain the systems

$$\frac{dx}{ds} \in X\left(\frac{s}{\varepsilon}, x\right) = X(s,x,\varepsilon),\ s \neq \varepsilon\tau^1_i, \tag{13p}$$

$$\Delta x\ |_{s=\varepsilon\tau^1_i} \in \varepsilon I^1_i(x), \tag{14p}$$

$$\frac{dy}{ds} \in Y\left(\frac{s}{\varepsilon}, y\right) = Y(s, y, \varepsilon),\ \ s \neq \varepsilon\tau_i^1, \tag{15p}$$

$$\Delta y\mid_{s=\varepsilon\tau_i^{2}} \in \varepsilon I_i^2(x). \tag{16p}$$

For the systems (13p)-(16p) the condition (5p) has the form:

$$\lim_{\varepsilon\to 0} h\left[\frac{1}{\Delta s}\left(\int_s^{s+\Delta s} X\left(\frac{s}{\varepsilon}, x\right) ds + \varepsilon \sum_{s<\varepsilon\tau_i^{1}<s+\Delta s} I_i^1(x)\right),\right.$$

$$\left.\frac{1}{\Delta s}\left(\int_s^{s+\Delta s} Y\left(\frac{s}{\varepsilon}, x\right) ds + \varepsilon \sum_{s<\varepsilon\tau_i^{2}<s+\Delta s} I_i^2(x)\right)\right] = 0.$$

We will show that from the above condition it follows the condition (12p). Indeed

$$\lim_{\varepsilon\to 0} h\left[\frac{1}{\Delta s}\left(\int_s^{s+\Delta s} X\left(\frac{s}{\varepsilon}, x\right) ds + \varepsilon \sum_{s<\varepsilon\tau_i^{1}<s+\Delta s} I_i^1(x)\right),\right.$$

$$\left.\frac{1}{\Delta s}\left(\int_s^{s+\Delta s} Y\left(\frac{s}{\varepsilon}, x\right) ds + \varepsilon \sum_{s<\varepsilon\tau_i^{2}<s+\Delta s} I_i^2(x)\right)\right] = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \lim_{\varepsilon\to 0} h\left[\frac{1}{\Delta s}\left(\varepsilon\int_{\frac{s}{\varepsilon}}^{\frac{s+\Delta s}{\varepsilon}} X\left(\frac{s}{\varepsilon}, x\right) d\frac{s}{\varepsilon} + \varepsilon \sum_{s<\varepsilon\tau_i^{1}<s+\Delta s} I_i^1(x)\right),\right.$$

$$\left.\frac{1}{\Delta s}\left(\varepsilon\int_{\frac{s}{\varepsilon}}^{\frac{s+\Delta s}{\varepsilon}} Y\left(\frac{s}{\varepsilon}, x\right) d\frac{s}{\varepsilon} + \varepsilon \sum_{s<\varepsilon\tau_i^{2}<s+\Delta s} I_i^2(x)\right)\right] = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \lim_{\varepsilon\to 0} h\left[\frac{1}{\frac{\Delta s}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s}{\varepsilon}+\frac{\Delta s}{\varepsilon}} X\left(\frac{s}{\varepsilon}, x\right) d\frac{s}{\varepsilon} + \frac{1}{\frac{\Delta s}{\varepsilon}} \sum_{\frac{s}{\varepsilon}<\tau_i^{1}<\frac{s}{\varepsilon}+\frac{\Delta s}{\varepsilon}} I_i^1(x),\right.$$

$$\left.\frac{1}{\frac{\Delta s}{\varepsilon}}\varepsilon\int_{\frac{s}{\varepsilon}}^{\frac{s}{\varepsilon}+\frac{\Delta s}{\varepsilon}} Y\left(\frac{s}{\varepsilon}, x\right) d\frac{s}{\varepsilon} + \frac{1}{\frac{\Delta s}{\varepsilon}} \sum_{\frac{s}{\varepsilon}<\tau_i^{2}<\frac{s}{\varepsilon}+\frac{\Delta s}{\varepsilon}} I_i^2(x)\right] = 0 \Longrightarrow$$

$$\Longrightarrow \lim_{T\to\infty} h\left[\frac{1}{T}\int_t^{t+T} X(t,x)dt + \frac{1}{T}\sum_{t<\tau_i^1<t+T} I_i^1(x),\right.$$

$$\left.\frac{1}{T}\int_t^{t+T} Y(t,x)dt + \frac{1}{T}\sum_{t<\tau_i^2<t+T} I_i^2(x)\right] = 0,$$

where $T = \dfrac{\Delta s}{\varepsilon}$ and $(\varepsilon \to 0) \Longrightarrow (T \to \infty)$ ($T \to \infty$ because of $\varepsilon \to 0$).

The proven relation between conditions (5p) and (12p) show that Theorem 1 can be considered as a generalization of Bogolyubov's Theorem in the case for impulsive differential inclusions.

Corollary 2. *If the following limits exist:*

$$\overline{X}(x) = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} X(s,x)\,ds,$$

$$\overline{X}(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{t\leq\tau_i<t+T} I_i^1(x),$$

then the the system (8p), (9p) can be writen as a differential inclusion without impulses in the following form:

$$\dot{x} \in \varepsilon[\overline{X}(x) + \overline{I}(x)],\;\; x(0) = x_0$$

3. Statement of the Problem

In the domain $Q = \{t \geq 0,\; x \in D \subset R^n,\; u \in U \subset R^m,\; w_i \in W \subset R^{m_i}\}$ we consider the following impulsive control system:

$$\dot{x} = \varepsilon[f(t,x) + A(x)\varphi(t,u)],\; x(0) = x_0,\; t \neq t_i, \tag{1}$$

$$\Delta x|_{t=t_i} = \varepsilon I_i(x, w_i), \tag{2}$$

where x is n-dimensional phase vector; $\varepsilon > 0$ is a small parameter; $f(t,x)$ and $\varphi(t,n)$ are 2π-periodical functions with respect to $t \in [0, L\varepsilon^{-1}]$ ($L = const$), $A(x)$ is $n \times m$ matrix; $u \in U \in comp(R^m)$ is the control; Δx is the jump of the solution of the system at the given moments $t = t_i$; $I_i(x, w_i)$ are the functions giving the values of the jumps (impulses) ($i = 1,\ 2,\ ...,\ k$);

$w_i \in W \in comp(R^{m_i})$ is the control by impulses, $comp(R^m)$ $(comp(R^{m_i}))$ is the space of compact subsets of R^m (R^{m_i}).

Everywhere below we suppose that all functions are uniformly bounded, measurable with respect to t and Lipschitz continuous w.r. to x and w.r. to u.

Now, let us consider the following optimal control problem:

We have to find admissible controls $u(t), w_i$ of the system (1), (2), which minimize (at the moment $T = L\varepsilon^{-1}$) the given functional in the following form:

$$J[u, w_i] = \Phi(x(T)) \tag{3}$$

So, we consider the following optimal control problem:

$$\dot{x} = \varepsilon[f(t, x) + A(x)\varphi(t, u)], \ t \neq t_i, \ x(0) = x_0, \tag{1}$$

$$\Delta x|_{t=t_i} = \varepsilon I_i(x, w_i), \tag{2}$$

$$J[u, w_i] = \Phi(x(T)) \to \min. \tag{3}$$

4. Main Result

We use the method of averaging to solve the optimal control problem (1)-(3).

We propose and justify two schemes for total averaging. As we said above these schemes are suitable when support functions are used for verifying whether the chosen control belongs to the given admissible set.

We suppose that in the domain Q the following conditions are true:

(C_1) the functions $f(t, x)$ and $\varphi(t, u)$ are measurable and 2π-periodical functions with respect to t, Lipshitz's continuous (LC) with respect to x and with respect to u with a constant λ; and they are uniformly bounded with a constant M.

(C_2) the matrix-function $A(x)$ is uniformly bounded with a constant M and LC with a constant λ.

(C_3) the functions $I_i(x, w_i)$ are uniformly bounded with a constant M, Lipschitz's continuous with respect to x with a constant λ, $I_i(x, w_i)$ are continuous with respect to w_i and $I_i(x, W) \in convR^n$.

$(C4)$ the function $\Phi(x)$ is Lipschitz continuous with a constant λ.

$(C5)$ for all admissible controls $\{v, w_i\}$ the trajectories of the system (4),(5) with some their ρ - neighbourhood belong to the domain D.

Schemes for Total Averaging for Impulsive Optimal Control Problems

First Scheme for Total Averaging

To the problem (1) - (3), we assign the following averaged problem

$$\dot{y} = \varepsilon[\bar{f}(y) + A(y)v], \;\; t \neq t_i, \;\; y(0) = x_0 \tag{4}$$

$$\Delta y \mid_{t=t_i} = \varepsilon I_i(y, z_i) \tag{5}$$

$$J_1[v, w_i] = \Phi(y(T)) \to min \tag{6}$$

where

$$\bar{f}(y) = \frac{1}{2\pi}\int_0^{2\pi} f(t,y)dt \tag{7'}$$

$$v \in V = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t,U)dt \tag{7''}$$

The integral of multivalued map in (7") is Auman's integral.

In the case when $f(t,x)$ and $\varphi(t,u)$ are 2π-periodical functions we have (see [21])

$\frac{1}{2\pi}\int_0^{2\pi} f(t,y)dt$ instead of $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^{T} f(t,y)dt$ and

we take $\frac{1}{2\pi}\int_0^{2\pi} \varphi(t,U)dt$ instead of $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^{T} \varphi(t,U)dt$

The next theorem justifies this averaging scheme, i. e we shall prove, that the solution of the problem (1) - (3) is arbitrary close to the solution of (4) - (6) when ε is sufficiently small and vice-versa.

Let J^* be the optimal value of functional (3) of the original problem (1) - (3).

Let J_1^* be the optimal control value of the functional (6) of the averaged problem (4) - (6).

Let $J[v^*, z_i^*]$ be the value of the functional (3) of the original problem (1)–(3), where $\{v^*, z_i^*\}$ are the optimal controls of the problem (4) - (6).

Theorem 2. *Let the conditions (C1) - (C5) hold. Then there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following estimations are valid:*

$$| J^* - J_1^* | \leq \varepsilon C \tag{8}$$

$$J[v^*, z_1^*] - J^* \leq \varepsilon C \tag{9}$$

Proof. From the conditions (C1) - (C5) it follows that the attainable sets of the system (1), (2) and (4), (5) are compact in D. Hence (see [18]) the systems have optimal solutions $x(t)$ and $y(t)$ with optimal values of the functionals $J^*[u^*(t), w_i^*]$ and $J_1^*[v_i^*, z_i^*]$ respectively.

Let us write the original system (1), (2) and the averaged system (4), (5) as differential inclusions:

$$\dot{x} \in [f(t,x) + A(x)\varphi(t,U)], \;\; t \neq t_i, \;\; x(0) = x_0, \;\; t \in [0, L\varepsilon^{-1}] \tag{10}$$

$$\Delta x \,|_{t=t_i} \in I_i(x, W) \tag{11}$$

and

$$\dot{y} \in [\overline{f}(y) + A(y)V], \;\; t \neq t_i, \;\; y(0) = x_0, \;\; t \in [0, L\varepsilon^{-1}] \tag{12}$$

$$\Delta y \,|_{t=t_i} \in \varepsilon I_i(y, W) \tag{13}$$

From the conditions (7'), (7'') it follows that the system (12), (13) is the averaged system of the system (10), (11). Hence according to Theorem 1 and Corollary 1 there exist constants $\varepsilon \in (0, \sigma]$ and $C > 0$, such that the following estimation is true

$$h[X(T), Y(T)] < C\varepsilon \tag{14}$$

Here $X(T)$ and $Y(T)$ are the attainable sets of the differential inclusion (10), (11) and (12), (13) respectively or they are the attainable sets of the systems (1), (2) and (4), (5) respectively.

Now let us proof the estimation (8) For J^* and J_1^* we have

$$J^* = \min_{x \in X(T)} \Phi(x), \;\; J_1^* = \min_{y \in Y(T)} \Phi(Y) \tag{15}$$

Let us estimate $| J^* - J_1^* |$. To do that we estimate $| \min_{x \in X(T)} \Phi(x) - \min_{y \in Y(T)} \Phi(y) |$

Let $x^*(t)$ be the optimal solution of the system (1), (2) (or (10), (11)) corresponding to the optimal control $\{u^*(t), w_i^*\}$.

Let $y^*(t)$ be the optimal solution of the system (4), (5) (or (12)(13)) corresponding to the optimal control $\{v^*(t), z_i^*\}$.

Then $\Phi(x^*) = \min_{x \in X(T)} \Phi(x)$ and $\Phi(y^*) = \min_{y \in X(T)} \Phi(y)$

According to (14) for $x^* \in X(T)$ there exists $\overline{y} \in Y(T)$ such that the following estimation $\|x^* - \overline{y}\| \leq C\varepsilon$ holds.

Similarly according to (14) for $y^* \in Y(T)$ there exists $\overline{x} \in X(T)$ such that the estimation $\|y^* - \overline{x}\| \leq C\varepsilon$ holds.

Since the function $\Phi(.)$ is a Lipcshitz function with constant λ, we will have:

$$\mid J^* - J_1(\overline{y}) \mid = \mid \Phi(x^*) - \Phi(\overline{y}) \mid \leq \lambda\|x^* - \overline{y}\| \leq \lambda\varepsilon C \tag{16}$$

and

$$\mid J_1^* - J(\overline{x}) \mid = \mid \Phi(y^*) - \Phi(\overline{x}) \mid \leq \lambda\|y^* - \overline{x}\| \leq \lambda\varepsilon C \tag{17}$$

We two possibilities:

$a)$ $J^* \leq J_1^*$ and $b)$ $J_1^* \leq J^*$.

If inequality $a)$ holds, then according to (16), we have:

$a)$ $J^* \leq J_1^* \Rightarrow J^* \leq J_1^* \leq J_1(\overline{y}) \Rightarrow J_1 - J^* \leq J_1(\overline{y}) - J^* \leq \lambda\varepsilon C.$

If inequality $b)$ holds, then according to (17), we have:

$b)$ $J_1^* \leq J^* \Rightarrow J_1^* \leq J^* \leq J(\overline{x}) \Rightarrow J^* - J_1^* \leq J(\overline{x}) - J_1^* \leq \lambda\varepsilon C$

From $a)$ and $b)$ substituting C with λC, we obtain the estimation (8).

Now, let us prove the estimation (9).

Let us write the system (1), (2) by the optimal control v^*, z_i^* of the averaged system (4), (5). So, we obtain the system

$$\dot{\tilde{x}} = [f(t, \tilde{x}) + A(\tilde{x})\varphi(t, v^*(t))], \ \ t \neq t_i, \ \ \tilde{x}(0) = x(0) \tag{18}$$

$$\Delta\tilde{x} \mid_{t=t_i} = \varepsilon I_i(\tilde{x}, z_i^*) \qquad (19))$$

($u(t)$ is replaced by $v^*(t)$ and w_i is replaced by z_i^*)

Now, we consider the system (4), (5) with the optimal control v^*, z_i^*, We obtain:

$$\dot{y}^* = \varepsilon[\overline{f}(y^*) + A(y^*)v^*(t)], \ \ t \neq t_i, \ \ y_0 = x_0 \tag{20}$$

$$\Delta y^* \mid_{t=t_i} = \varepsilon I(y^*, z_i^*) \tag{21}$$

It is obvious that the system (20), (21) is the averaged system of the system (18), (19). So according to Cogollary 1 there exists a constant $\varepsilon_0 > 0$ and there exists a constant $C_1 > 0$, that if $\varepsilon \in (0, \varepsilon_0]$ the following estimation is valid:

$$\|\tilde{x}(t) - y^*(t)\| \leq C_1\varepsilon, \ \ C_1 = const, \ \ t \in [0, L\varepsilon^0]$$

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$$\mid J_1^* - J[v^*(t), z_i^*] \mid = \mid \Phi(y^*(T)) - \Phi(\tilde{x}(T)) \mid \leq \lambda \mid y^*(T) - \tilde{x}(T) \mid \leq \lambda C_1\varepsilon \tag{22}$$

Now let us estimate $J[v^*(t), z_i^*] - J^*$

Using (8) and (22) we get

$$J[v^*, z_i^*] - J^* =\mid J^* - J[v^* - z_i^*] \mid\leq\mid J^* - J_1^* \mid + \mid J_1^* - J[v^* - z_i^*] \mid\leq C\varepsilon + \lambda C_1\varepsilon$$

Substituting C with $C + \lambda C_1$ we obtain the estimation (9), which complete the proof. □

In the first scheme for total averaging, the equation (4) does not depend on time, but the impulsive effects make the system nonautonomous

In the next scheme for total averaging the system is autonomous.

Let us consider the following scheme for total averaging:

Second Scheme for Total Averaging

To the problem (1) - (2), we assign the following averaged problem

$$\frac{dy}{d\tau} = \overline{f}(y) + A(y)v + q, \;\; y(0) = x_0, \tag{23}$$

where $\tau = \varepsilon t$, $\tau \in [0, L]$, $\overline{f}(y)$, and v are obtained by (7') and (7") and

$$q \in I_0(y) = \frac{1}{2\pi} \sum_{0\leq t_i\leq 2\pi} I_i(y, W) \tag{24}$$

Let us consider the following optimal control problem:

We have to determine the admissible controls $\{u(t), w_i\}$ of the system (1), (2) which minimize at the moment $T = L\varepsilon^{-1}$, the functional

$$J[u, w_i] = \Phi(x(T)) \tag{25}$$

We can establish correspondence between the controls $\{u(t), w_i\}$ of the problem (1), (2) and the controls $\{v(\tau), q(\tau)\}$ of the problem (23), (24) in the following sense (see [21]):

To the control $q(\tau)$ we assign the following step-function $q^0(t)$:

$$q^0(t) = \Big\{q^{0i} \mid q^{0i} = \frac{1}{2\pi} \int_{2\pi i}^{2\pi(i+1)} q(\varepsilon s)\, ds, \quad t \in [2\pi i, 2\pi(i + 1)], \quad i = 0, 1, 2, ...k\Big\} \tag{26}$$

It is clear, that there exist $w_i \in W$, such that

$$q^{0i} = \frac{1}{2\pi} \sum_{2\pi i\leq t<2\pi(i+1)} I_i(y, w_i). \tag{27}$$

So obtained control w_i corresponds to the control $q(\tau)$. The correspondence between controls $v(\tau)$ and $u(t)$ for $t \in [2\pi i, 2\pi(i+1)]$ is given by the following relation:

$$\int_{2\pi i}^{2\pi(i+1)} v(\varepsilon t)\, dt = \int_{2\pi i}^{2\pi(i+1)} \varphi(t, u(t))\, dt \tag{28}$$

The relations (26) — (28) allow us to establish correspondence between every control $\{q(\tau), v(\tau)\}$ of the averaged system (23), (24) and the control $\{w_i, u(t)\}$ of the original system (1), (2) and vice versa.

In the first scheme the functional, which we minimize has the form:

$$J_1[v, w_i] = \Phi(y(T)), \tag{29}$$

and in the second schemes has the form:

$$J_2[v, q] = \Phi(y(L)), \tag{30}$$

The next theorem justifies the second scheme for total averaging.

Let us introduce the following notations:

Let J^* be the optimal values of the functional (25) of the original problem (1), (2);

Let J_2^* be the optimal values of the functional (30) of the averaged problem (23), (24).

Let $J[u'(t), w_i']$ be the value of the functional (25) of the problem (1)–(3), where $u'(t)$ and w_i' are obtained by the relations (26) — (28) using the optimal control $v^*(\tau)$, $q^*(\tau)$ of the averaged system (23), (24).

Theorem 3. *Let in the domain* $Q = \{t \geq 0,\ x \in D \subset R^n,\ u \in U,\ w_i \in W\}$ *the conditions (C1) – (C4) hold and moreover:*

5) for all attainable controls $v(\tau)$, $q(\tau)$ *the trajectories of the system (23) belong to the domain* D *with their* ρ*-neighbourhood;*

Then there exists a constant C *and there exists a constant* $\varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0]$ *the following estimations:*

$$|J^* - J_2^*| < C\varepsilon, \tag{31}$$

$$J[u^1(t), w_i^1] - J^* < C\varepsilon. \tag{32}$$

hold.

The constant C *does not depend on* ε*.*

Proof. Let us write the original equation (1), (2) as a differential inclusion:

$$\dot{x} \in \varepsilon[f(t,x) + A(x)\varphi(t,U)], \quad t \neq t_i \tag{33}$$

$$\Delta x|_{t=t_i} \in I_i(x, W_i) \tag{34}$$

According to Corollary 2 of Theorem 1 to the inclusions (33), (34) we can assign the following differential inclusion:

$$\dot{y} \in \varepsilon[\bar{f}(y) + A(y)V + I_0(y)] \tag{35}$$

It is clear, that the differential inclusion (35) corresponds to the controlable system (23).

Since $f(t,x)$ and $\varphi(t,u)$ are 2π-periodical functions, then there exist a constant C_1 and a constant ε_0, such that the following estimation is valid:

$$h(X(t), Y(t)) \leq C_1\varepsilon, \ \ \forall\varepsilon \in (0, \varepsilon_0] \tag{36}$$

where $X(t)$ and $Y(t)$ are the attainable sets of the systems (33),(34)((1),(2)) and (35)((23),(24)), respectively.

We have $J^* = \min\limits_{x\in X(T)} \Phi(x)$ and $J_2^* = \min\limits_{y\in Y(T)} \Phi(y)$

Let us estimate the difference

$$|J^* - J_2^*| = |\min_{x\in X(T)} \Phi(x) - \min_{y\in Y(L)} \Phi(x)|$$

Let $x^*(t)$ and u^*, w_i^* be the optimal solution and optimal control of the system (33),(34). Let $y^*(t)$ and v^*, q^* be the optimal solution and optimal control of the system (35).

So we have $\Phi(x^*) = \min\limits_{x\in X(T)} \Phi(x)$, $\Phi(y^*) = \min\limits_{y\in X(L)} \Phi(y)$.

Since $h(X(t), Y(t)) \leq C_1\varepsilon$, then for $x^* \in X(T)$ there exists $\overline{y} \in Y(L)$, such that $|x^* - \overline{y}| \leq C_1\,\varepsilon$, and for $y^* \in Y(L)$ there exists $\bar{x} \in X(T)$ such that $|y^* - \bar{x}| \leq C_1\varepsilon$.

Since $\Phi(x)$ is Lipschitz function (with constant λ) we have:

$$|J^* - J_2(\bar{y})| = |\Phi(x^*) - \Phi(\bar{y})| \leq \lambda|x^* - \bar{y}| \leq \lambda\,C_1\varepsilon \tag{m1}$$

$$|J_2^* - J(\bar{x})| = |\Phi(y^*) - \Phi(\bar{x})| \leq \lambda|y^* - \bar{x}| \leq \lambda\,C_1\varepsilon \tag{m2}$$

Obvious that $J^* \leq J(\bar{x})$ and $J_2^* \leq J_2(\bar{y})$

There are two possibilities $J^* \leq J_2^*$ or $J_2^* \leq J^*$.

a) If $J^* \le J_2^*$, then $J^* \le J_2^* \le J_2^*(\bar{y})$. According to $(m1)$ we have

$$|J_2^* - J^*| \le |J_2(\bar{y}) - J^*| \le \lambda C_1 \varepsilon$$

b) If $J_2^* \le J^*$, then according to $(m2)$ we have

$$J_2^* \le J^* \le J(\bar{x}) \implies |J^* - J_2^*| \le |J(\bar{x}) - J_2^*| \le \lambda C_1 \varepsilon$$

Substituting $C = \lambda C_1$ we obtain the inequality (31).

Now let us prove the inequality (32).

First we write the equation (23) with the optimal control $v^*(t),\ q^*(t)$:

$$\begin{gathered} \dot{y}^* = \varepsilon[\bar{f}(y^*) + A(y^*)v^*(t) + q^*(t)] \\ y^*(0) = x_0 \end{gathered} \tag{37}$$

Using (26)–(28) we construct control $u^1(t),\ w_i^1$ of the original system and help-function $q^0(t)$, which correspond to the optimal controls $v^*(t),\ q^*(t)$ of the averaged system.

Consider the equation:

$$\begin{gathered} \dot{y}^1 = \varepsilon[f(y^1) + A(y^1)\varphi(t, u^1(t)) + q^0(t)] \\ y^1(0) = x_0 \end{gathered} \tag{38}$$

From the construction (26)–(28) we have:

$$\int_{2\pi i}^{2\pi(i+1)} q^0(t)\, dt = \int_{2\pi i}^{2\pi(i+1)} q^*(t)\, dt,$$

$$\int_{2\pi i}^{2\pi(i+1)} v^*(t)\, dt = \int_{2\pi i}^{2\pi(i+1)} \varphi(t, u^1(t))\, dt$$

Then the equation (37) is the partially averaged equation of the equation (38).

Hence, there exists a constant C_1 and there exists a constant ε_1, such that $\forall \varepsilon \in (0, \varepsilon_1]$ $\forall t \in [0, L\varepsilon^{-1}]$ the inequality

$$\|y^*(t) - y^1(t)\| \le C_1 \varepsilon \tag{39}$$

hold.

We consider the following differential equation with impulses:

$$\dot{x}^2 = \varepsilon\left[f(t, x^2) + A(x^2)\varphi(t, u^1))\right] \tag{40}$$

$$\Delta x^2|_{t=t_i} = \varepsilon I_i(x^2(t_i), w_i^1) \tag{41}$$

The equation (38) is the partially averaged equation of the equation (40), (41).

Hence there exist constants C_2 and $\varepsilon_2 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_2]$ and $\forall t \in [0, L\varepsilon^{-1}]$ the following inequality is fulfilled:

$$\|x^2(t) - y^1(t)\| \le C_2\varepsilon \tag{42}$$

From (39) and (42) it follows that

$$\|y^*(t) - x^2(t)\| \le C\varepsilon \tag{43}$$

From (43) and from the fact, that $\Phi(\cdot)$ is Lipschitz function, we obtain

$$|J[u^1(t), w_i^1] - J_2^*| = |\Phi(x^2(T)) - \Phi(y^*(T))| < \lambda C\varepsilon \tag{44}$$

From (31) and (44) it follows that

$$J[u^1(t), w_i^1] - J^* = |J[u^1(t), w_i^1] - J^*| = |J[u^1(t), w_i^1] - J_2^* + J_2^* - J^*| < \lambda\, C\varepsilon + C\varepsilon,$$

therefore (32) is true. □

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