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OSCILLATION CRITERIA FOR EVEN ORDER DYNAMIC EQUATIONS ON TIME–SCALES

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Abstract: Some new criteria for the oscillation of even order linear dynamic equations on time-scales of the form

$$x^{\Delta^n}(t) + q(t)x(t) = 0$$

are established.

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1. Introduction

This paper is concerned with the oscillatory behavior of all solutions of the even order dynamic equation

$$x^{\Delta^n}(t) + q(t)x(t) = 0 \tag{1.1}$$

on an arbitrary time-scale$\mathbb{T} \subseteq \mathbb{R}$ with $\sup \mathbb{T} = \infty$, where $q : \mathbb{T} \to \mathbb{R}^+ = (0, \infty)$ is real-valued rd-continuous function.

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We recall that a solution x of equation (1.1) is said to be nonoscillatory if there exists a $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$; otherwise, it is said to be oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been an increasing interest in studying the oscillatory behavior of first and second order dynamic equations on time-scales, see [1–4]. With respect to dynamic equations on time-scales, it is fairly new topic and for general basic ideas and background, we refer to [1]. In fact, there are very few results regarding the oscillation of equation (1.1). Therefore, the purpose of this paper is to establish some new criteria for the oscillation of equation (1.1). We may note that the results of this paper are new for the case $\mathbb{T} = \mathbf{Z}$.

2. Main Results

We shall employ the following well-known lemma.

Lemma 2.1. *Let $x(t) \in C^m_{rd}([t_0, \infty), \mathbb{R}^+)$. If $x^{\Delta^m}(t)$ is of constant sign on $[t_0, \infty)$ and not identically zero on $[t_1, \infty)$ for any $t_1 \geq t_0$, then there exist a $t_x \geq t_1$ and an integer ℓ, $0 \leq \ell \leq m$ with $m + \ell$ even for $x^{\Delta^m}(t) \geq 0$ or $m + \ell$ odd for $x^{\Delta^m}(t) \leq 0$ such that*

$$\ell > 0 \quad \textit{implies} \quad x^{\Delta^k}(t) > 0 \quad \textit{for} \quad t \geq t_x, \quad k \in \{1, 2, \cdots, \ell - 1\} \tag{2.1}$$

and

$$\ell \leq m - 1 \quad \textit{implies} \quad (-1)^{\ell+k} x^{\Delta^k}(t) > 0 \quad \textit{for} \quad t \geq t_x, \quad k \in \{\ell, \ell + 1, \cdots, m - 1\}. \tag{2.2}$$

It will be convenient to employ the Taylor monomials (see [1, Sec. 1.6]) $\{h_n(t,s)\}_{n=0}^{\infty}$ which are defined recursively as follows

$$h_0(t,s) \; = \; 1, \quad h_{n+1}(t,s) \; = \; \int_s^t h_n(u,s)\Delta u, \quad t, s \in \mathbb{T} \quad \text{and} \quad n \geq 0.$$

It follows that $h_1(t,s) = t - s$ for any time-scale, but simple formulas in general do not hold for $n \geq 2$.

Now we have the following oscillation result for equation (1.1).

Theorem 2.1. *Let*

$$\int^{\infty} \int_s^{\infty} q(u)\Delta u \Delta s \; = \; \infty. \tag{2.3}$$

If there exists a positive non-decreasing delta-differentiable function η such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\eta(s)q(s) - h_{n-1}^{-1}(s,t_0)\eta^{\Delta}(s)\right] \Delta s \;=\; \infty, \tag{2.4}$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) on $[t_0, \infty)_{\mathbb{T}}$. It suffices to discuss the case x is eventually positive (as $-x$ also solves (1.1) if x does), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. Now, we see that $x^{\Delta^n}(t) \leq 0$ for all $t \geq t_1$, where $x^{\Delta^n}(t)$ is not identically zero for all large t. Using Lemma 2.1, there exists an integer $\ell \in \{1, 3, \cdots, n-1\}$ such that (2.1) and (2.2) hold for all $t \geq t_1$. Clearly, we see that $x^{\Delta^{n-1}}(t) > 0$ and $x^{\Delta}(t) > 0$ for all $t \geq t_1$. Thus there exists a constant $c > 0$ such that

$$x(t) \;\geq\; c \quad \text{for} \quad t \geq t_1. \tag{2.5}$$

We claim that $\ell = n-1$. To this end, we assume that $x^{\Delta^{n-2}}(t) < 0$ for $t \geq t_1$. Integrating equation (1.1) from t to $u \geq t \geq t_1$ and letting $u \to \infty$, we get

$$x^{\Delta^{n-1}}(t) \;\geq\; \int_t^{\infty} q(s)x(s)\Delta s.$$

Using (2.5) in the above inequality, we have

$$x^{\Delta^{n-1}}(t) \;\geq\; c\int_t^{\infty} q(s)\Delta s.$$

Integrating this inequality from t_1 to t, we find

$$-x^{\Delta^{n-2}}(t) + x^{\Delta^{n-2}}(t_1) \;\leq\; -c\int_{t_1}^{t}\int_s^{\infty} q(u)\Delta u\Delta s,$$

or,

$$0 \;<\; -x^{\Delta^{n-2}}(t) \;\leq\; -c\int_{t_1}^{t}\int_s^{\infty} q(u)\Delta u\Delta s \quad\to\quad -\infty,$$

a contradiction. Next, we claim that

$$x^{\Delta}(t) \;\geq\; h_{n-2}(t,t_1)x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_1 \tag{2.6}$$

and

$$x(t) \;\geq\; h_{n-1}(t,t_1)x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_1. \tag{2.7}$$

Since $\ell = n-1$, we have

$$x^{\Delta^n}(t) \;\leq\; 0 \quad \text{and} \quad x^{\Delta^i}(t) \;>\; 0, \quad i = 0, 1, \cdots, n-1 \quad \text{and} \quad t \geq t_1. \tag{2.8}$$

Now,

$$x^{\Delta^{n-2}}(t) - x^{\Delta^{n-2}}(t_1) \;=\; \int_{t_1}^{t} x^{\Delta^{n-1}}(s)\Delta s.$$

Using the fact that $x^{\Delta^{n-1}}(t)$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$, we get

$$x^{\Delta^{n-2}}(t) \;\geq\; h_1(t, t_1) x^{\Delta^{n-1}}(t).$$

Integrating this inequality $(n-3)$–times from t_1 to t we obtain (2.6). Next, integrating (2.6) from t_1 to t, we obtain (2.7). This proves the claim. Now let

$$w := \;\eta \frac{x^{\Delta^{n-1}}}{x} \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}.$$

Then on $[t_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} w^{\Delta} &= \left(\frac{\eta}{x}\right)^{\Delta} \left(x^{\Delta^{n-1}}\right)^{\sigma} + \frac{\eta}{x}\left(x^{\Delta^n}\right) \\ &= -\,\eta q + \left(x^{\Delta^{n-1}}\right)^{\sigma} \left(\frac{\eta^{\Delta} x - \eta x^{\Delta}}{x\; x^{\sigma}}\right) && (2.9) \\ &\leq -\,\eta q + \eta^{\Delta} \left(\frac{x^{\Delta^{n-1}}}{x}\right)^{\sigma} \\ &\leq -\,\eta q + \eta^{\Delta} \left(\frac{x^{\Delta^{n-1}}}{x}\right) \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}. && (2.10) \end{aligned}$$

Using (2.8) in (2.10), we find

$$w^{\Delta}(t) \;\leq\; -\,\eta(t)q(t) + h_{n-1}^{-1}(t, t_1)\eta^{\Delta}(t) \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}. \tag{2.11}$$

Integrating (2.11) from $t_2 > t_1$ to t, we obtain

$$0 \;<\; w(t) \;\leq\; w(t_2) - \int_{t_2}^{t} \left[\eta(s)q(s) - h_{n-1}^{-1}(s, t_1)\eta^{\Delta}(s)\right] \Delta s,$$

which gives a contradiction using (2.4). This completes the proof. □

Next, we have the following interesting result for the oscillation of equation (1.1).

Theorem 2.2. *Let condition (2.3) hold. If there exists a positive nondecreasing delta-differentiable function η such that for $t_1 \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\eta(s)q(s) - h_{n-2}^{-1}(s,t_0) \left(\frac{\eta^{\Delta}(s)}{2\sqrt{\eta(s)}} \right)^2 \right] \Delta s \;=\; \infty, \tag{2.12}$$

then equation (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_1 \ge t_0$. Proceeding as in the proof of Theorem 2.1, we obtain (2.6) – (2.8) for $t \ge t_1$. We define the function w as in the proof of Theorem 2.1 and obtain (2.9). Thus,

$$w^{\Delta} \;\le\; -\eta q + \left(\frac{\eta^{\Delta}}{\eta^{\sigma}} \right) w^{\sigma} - \left(\frac{\eta}{\eta^{\sigma}} \right) w^{\sigma} \frac{x^{\Delta}}{x}, \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}.$$

Using (2.6) in this inequality, we find

$$\begin{aligned} w^{\Delta} \;&\le\; -\eta q + \left(\frac{\eta^{\Delta}}{\eta^{\sigma}} \right) w^{\sigma} - h_{n-2}(t,t_1) \left(\frac{\eta}{\eta^{\sigma}} \right) w^{\sigma} \frac{x^{\Delta^{n-1}}}{x} \\ &=\; -\eta q + \left(\frac{\eta^{\Delta}}{\eta^{\sigma}} \right) w^{\sigma} - h_{n-2}(t,t_1) \frac{w^{\sigma}}{\eta^{\sigma}} w \\ &\le\; -\eta q + \left(\frac{\eta^{\Delta}}{\eta^{\sigma}} \right) w^{\sigma} - \frac{h_{n-2}(t,t_1)}{\eta^{\sigma}} (w^{\sigma})^2 \\ &=\; -\eta q - \left(\sqrt{\frac{h_{n-2}(t,t_1)}{\eta^{\sigma}}} w^{\sigma} - \frac{\eta^{\Delta}}{2\sqrt{\eta^{\sigma} h_{n-2}(t,t_1)}} \right)^2 + \frac{(\eta^{\Delta})^2}{4\eta^{\sigma}} h_{n-2}^{-1}(t,t_1) \\ &\le\; -\eta q + \left(\frac{\eta^{\Delta}}{2\sqrt{\eta}} \right)^2 h_{n-2}^{-1}(t,t_1) \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}. \end{aligned}$$

Integrating this inequality from $t_2 > t_1$ to t, we have

$$w(t) \;\le\; w(t_2) - \int_{t_2}^{t} \left[\eta(s)q(s) - \left(\frac{\eta^{\Delta}(s)}{2\sqrt{\eta(s)}} \right)^2 h_{n-2}^{-1}(s,t_1) \right] \Delta s. \tag{2.13}$$

Taking $\limsup$ of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (2.12). This completes the proof. □

As an example, we let $\mathbb{T} = \mathbf{Z}$. In this case equation (1.1) takes the form

$$\Delta^n x(t) + q(t)x(t) \;=\; 0 \tag{2.14}$$

and Theorems 2.1 and 2.2 reduce to the following new results for equation (2.14).

Theorem 2.3. *Let*

$$\sum_{s=t_0}^{\infty}\sum_{u=s}^{\infty} q(u) \;=\; \infty. \tag{2.15}$$

If there exists a positive nondecreasing sequence $\{\eta(t)\}$ *such that for every* $t_1 \geq t_0 \in \mathbb{N}_0$,

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t} \left[\eta(s)q(s) - \left((s-t_0)^{(n-1)}\right)^{-1}(\Delta\eta(s))\right] \;=\; \infty, \tag{2.16}$$

then equation (2.15) is oscillatory.

Theorem 2.4. *Let condition (2.14) hold. If there exists a positive nondecreasing sequence* $\{h(t)\}$ *such that for* $t_1 \geq t_0$,

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t} \left[\eta(s)q(s) - \left(\frac{\Delta\eta(s)}{2\sqrt{\eta(s)}}\right)^2 \left((s-t_0)^{(n-2)}\right)^{-1}\right] \;=\; \infty, \tag{2.17}$$

then equation (2.14) is oscillatory.

3. Remarks

1. The results of this paper can be extended to more general equation

$$x^{\Delta^n}(t) + f(t, x(g(t))) \;=\; 0,$$

where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and $xf(t,x) > 0$ for $x \neq 0$ and all $t \geq t_0$, $g : \mathbb{T} \to \mathbb{T}$ is an rd-continuous function, $g(t) \leq t$ and $\lim_{t\to\infty} g(t) = \infty$. The details are left to the reader.

2. As an example, we have reformulated the obtained results for the discrete case when $\mathbb{T} = \mathbf{Z}$. We may employ other types of time–scales, e.g., $\mathbb{T} = \mathbb{R}, \mathbb{T} = h\mathbf{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2, \cdots$ etc., see [1]. The details are left to the reader.

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