International Journal of Pure and Applied Mathematics
Volume 73 No. 1 2011, 111-128

ON THE NONOSCILLATION AND OSCILLATION OF THE SOLUTIONS OF A FIRST ORDER NEUTRAL NONCONSTANT DELAY IMPULSIVE DIFFERENTIAL EQUATIONS WITH VARIABLE OR OSCILLATING COEFFICIENTS

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Abstract: In this paper we consider first order neutral impulsive differential equation with variable coefficients. The asymptotic behavior of a nonoscillatory solution for such equations is investigating and sufficient conditions for oscillation of all the solutions are found.

AMS Subject Classification: 34A37, 34K11, 34K40, 34K45
Key Words: oscillation, impulsive effect, neutral differential equation

1. Introduction

The Neutral Impulsive Differential Equations (NIDE) are part of the Impulsive Differential Equations with Deviating Arguments (IDEDA). Among the numerous publications concerning the oscillation properties of IDEDA - with delayed or advanced arguments, we choose to refer to [1], [2], [7], [8], [9], [13] and [14]. NIDE are characterized with neutral argument in which the highest-order derivative of the unknown function appears in the equation both with and without delay. Moreover, the unknown function in them, may have discontinuities of first kind at points, which we call jump points. Such equations can be used to model processes, that occur in the theory of optimal control, industrial robotics, biotechnologies, etc. Some results on the oscillation theory

Received: September 22, 2011
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of this type of equations can be found in [3], [4] and [6].

As concern the behavior of the solutions of the differential equations with deviating arguments, but without impulses, we choose to refer to [5], [10], [11] and [12].

The authors investigated neutral delay impulsive differential equations with constant coefficients and found there necessary and sufficient conditions for existence of eventually positive solutions in [3] and established oscillation criteria in [4], as well. In the present paper we study the asymptotic behavior of the eventually non-oscillatory solutions of (E_1) and obtain oscillation criteria when the coefficients are variable and the delays are nonconstant.

2. Preliminary Notes

The object of investigation in the present work is the first order impulsive differential equation with variable coefficients and nonconstant neutral delay argument of the form

$$\frac{d}{dt}\{y(t) - c(t)y(h(t))\} + p(t)y(\sigma(t)) = 0,\ t \neq \tau_k\ k \in N \qquad (E_1)$$

$$\Delta\{y(\tau_k) - c_{\tau_k}y(h(\tau_k))\} + p_{\tau_k}y(\sigma(\tau_k)) = 0,\ k \in N$$

as well as the corresponding to it inequalities

$$\frac{d}{dt}\{y(t) - c(t)y(h(t))\} + p(t)y(\sigma(t)) \leq 0,\ t \neq \tau_k,\ k \in N \qquad (N_{1,\leq})$$

$$\Delta\{y(\tau_k) - c_{\tau_k}y(h(\tau_k))\} + p_{\tau_k}y(\sigma(\tau_k)) \leq 0,\ k \in N$$

and

$$\frac{d}{dt}\{y(t) - c(t)y(h(t))\} + p(t)y(\sigma(t)) \geq 0,\ t \neq \tau_k,\ k \in N \qquad (N_{1,\geq})$$

$$\Delta\{y(\tau_k) - c_{\tau_k}y(h(\tau_k))\} + p_{\tau_k}y(\sigma(\tau_k)) \geq 0,\ k \in N.$$

The points $\tau_k \in (0, +\infty),\ k \in N$ are the moments of impulsive effect (let us call them jump points), where the unknown function reveals its discontinuities of first kind as jumps. In order to manifest these jumps of the unknown function $y(t)$, we use the notation

$$\Delta\{y(\tau_k) - c_{\tau_k}y(h(\tau_k))\} = \Delta y(\tau_k) - c_{\tau_k}\Delta y(h(\tau_k)),\ \ \Delta y(\tau_k) = y(\tau_k+0) - y(\tau_k-0).$$

Denote by $P_\tau C(R, R)$ the set of all functions $u\colon R \to R$, which satisfy the following conditions:

(i) u is piecewise continuous on $(\tau_k, \tau_{k+1}],\ k \in N$,

(ii) u is continuous from the left at the points τ_k, i.e.

$$u(\tau_k - 0) = \lim_{t \to \tau_k - 0} u(t) = u(\tau_k),$$

(iii) there exists a sequence of reals $\{u(\tau_k + 0)\}_{k=1}^{\infty}$, such that

$$u(\tau_k + 0) = \lim_{t \to \tau_k + 0} u(t),$$

(iv) u may have discontinuities of first kind at the jump points $\tau_k,\ k \in N$, that we qualify as down-jumps when $\Delta u(\tau_k) < 0$, or as up-jumps when $\Delta u(\tau_k) > 0,\ k \in N$.

Introduce the following hypotheses, where $R^+ = (0, +\infty)$ and $R_0^+ = [0, +\infty)$:

($\mathbf{H_1}$) $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots,\ \lim\limits_{k \to +\infty} \tau_k = +\infty,\ \max\left\{\tau_{k+1} - \tau_k\right\} < +\infty,\ k \in N$;

($\mathbf{H_2}$) $h,\ \sigma \in C^1(R^+, R^+),\ h'(t) > 0,\ \sigma'(t) > 0,\ \sigma(t) < t,\ h(t) < t$ and $\lim\limits_{t \to +\infty} h(t) = +\infty,\ \lim\limits_{t \to +\infty} \sigma(t) = +\infty$;

($\mathbf{H_3}$) $c \in C(R, (0, c_0]),\ c_0 \in (0, 1),\ c_{\tau_k} = c(\tau_k)$;

($\mathbf{H_4}$) $p \in P_\tau C(R, R),\ p(t)$ is not identically zero on any positive half-line, $\sum\limits_{k=1}^{+\infty} p_{\tau_k}^2 \neq 0$;

($\mathbf{H_5}$) $p \in P_\tau C(R, R^+),\ p_{\tau_k} \in R_0^+,\ k \in N,\ \int\limits_0^{+\infty} p(s)ds + \sum\limits_{k=1}^{+\infty} p_{\tau_k} = +\infty$.

Let $\rho(t) = \min\limits_{t \in R^+} \{\sigma(t), h(t)\}$. We say that a real valued function $y(t)$ is a *solution* of the equation (E_1), if there exists a number $T_0 \in R$ such that $y \in P_\tau C([\rho(T_0), +\infty), R)$, the function $z(t) = y(t) - c(t)y(h(t))$ is continuously differentiable for $t \geq T_0,\ t \neq \tau_k,\ k \in N$ and $y(t)$ satisfies (E_1) for all $t \geq T_0$.

Without further mentioning we will assume throughout this paper, that every solution $y(t)$ of equation (E_1) that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some ray of the form $[T_y, +\infty)$ and for each $T \geq T_y$ it is fulfilled $\sup\left\{|y(t)|:\ t \geq T\right\} > 0$. Such a solution is called a *regular solution* of (E_1).

We say that a real valued function u defined on an interval $[a, +\infty)$ has some property *eventually,* if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A regular solution $y(t)$ of equation (E_1) is said to be *nonoscillatory,* if there exists a number $t_0 \geq 0$ such that $y(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called *oscillatory*. Also, note that a *nonoscillatory* solution is called *eventually positive* (*eventually negative*), if the constant sign that determines its *nonoscillation* is positive (negative). Equation (E_1) is called *oscillatory* , if all its solutions are oscillatory.

Moreover, in this article, when we write a functional relation (or inequality), we will mean that it holds for all sufficiently large values of the argument.

In order to assist our investigations on the oscillation of the equation (E_1), we shall consider in the next section the delay impulsive differential equation with variable coefficients of the form

$$z'(t) + q(t)z(s(t)) = 0, \; t \neq \tau_k \qquad (E_2)$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) = 0, \; k \in N$$

and the corresponding to it inequalities

$$z'(t) + q(t)z(s(t)) \leq 0, \; t \neq \tau_k \qquad (N_{2,\leq})$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) \leq 0, \; k \in N$$

and

$$z'(t) + q(t)z(s(t)) \geq 0, \; t \neq \tau_k \qquad (N_{2,\geq})$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) \geq 0, \; k \in N,$$

under the hypotheses:

H$_2$ $*$ $\quad s \in C^1(R^+, R^+), \; s'(t) > 0, \; \lim_{t \to +\infty} s(t) = +\infty, \;$ and $s(t) < t$.

H$_3$ $*$ $\quad q \in P_\tau C(R^+, R^+), \; q_{\tau_k} \in R, \; 1 > q_{\tau_k} \geq 0, \; k \in N$.

3. Some Useful Lemmas

Consider $y(t)$ as a solution of equation (E_1) and set the auxiliary function

$$z(t) = y(t) - c(t)y(h(t)), \qquad \Delta z(\tau_k) = \Delta y(\tau_k) - c_{\tau_k}\Delta y(h(\tau_k)), \quad c_{\tau_k} = c(\tau_k), \; k \in N. \qquad (*)$$

We introduce two lemmas, which investigate the asymptotic behavior of the function $z(t)$, when $y(t)$ is a non-oscillatory solution of (E_1). The first one is formulated and proved for an eventually positive solution $y(t)$ of the equation (E_1).

Lemma 1. *Assume that the hypotheses* $(H_1)-(H_5)$ *are satisfied and* $y(t)$ *be an eventually positive solution of* (E_1). *Then* $z(t)$ *is a decreasing eventually positive function of* t *with not strict down-jumps and* $\lim_{t\to+\infty} z(t) = 0$ *with* $\lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.

Proof. Let $y(t)$ be an eventually positive solution of the equation (E_1), i.e. $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$ such that $y(t) > 0$ for $t \geq \rho(T_0)$. Then, from (E_1) and utilizing $(*)$ we have

$$z'(t) = -p(t)y(\sigma(t)), \qquad t \neq \tau_k,\ k \in N,\ t \geq T_0, \tag{1}$$

$$\Delta z(\tau_k) = -p_{\tau_k} y(\sigma(\tau_k)), \qquad k \in N,\ \tau_k \geq T_0.$$

From (1), in view of (H_5), it follows that $z(t)$ is an eventually decreasing function of t $(z'(t) < 0)$ with not strict down-jumps $(\Delta z(\tau_k) \leq 0)$ for $t \in [T_0, +\infty)$.

Assume $z(t) < 0$ eventually. Then, for some $t_1 \geq T_0$ there exists $\delta_\nu > 0$ such that $z(t) \leq -\delta_\nu$, for every $t \geq t_1$, $t \neq \tau_k$, i.e. $y(t) - c(t)y(h(t)) \leq -\delta_\nu$, $t \neq \tau_k$, $t \geq t_1$. In the meantime, for the same $\delta_\nu > 0$, there will be such a position ν in the sequence of the impulsive moments $\{\tau_k\}$, whereafter $z(\tau_k) \leq -\delta_\nu$, for every $\tau_k \geq \tau_\nu$ where $k \geq \nu$, $k \in N, \nu \in N$. Hence, $y(\tau_k) - c_{\tau_k} y(h(\tau_k)) \leq -\delta_\nu$, $\tau_k \geq \tau_\nu$, $k \geq \nu$. Denote $t_\nu = \max\{t_1, \tau_\nu\}$. Using (H_3), we can combine the last two inequalities as

$$y(t) \leq -\delta_\nu + c(t)y(h(t)) \leq -\delta_\nu + c_0 y(h(t)), \qquad t \geq t_\nu.$$

By iterations, from the last inequality we get

$$y(t) \leq -\delta_\nu(1 + c_0 + c_0^2 + ... + c_0^{n-1}) + c_0^n y(h^n(t)), \qquad t \geq t_\nu, \tag{2}$$

In view of (H_3), the inequality (2) implies

$$y(t) \leq -\frac{\delta_\nu}{1-c}, \qquad t \geq t_\nu.$$

This is a contradiction. Hence, our assumption, that eventually $z(t) < 0$, is impossible.

Assume $z(t) \equiv 0$. Then, from (1), it follows $p(t)y(\sigma(t)) \equiv 0$ and $p(\tau_k)y(\sigma(\tau_k)) = 0$. But $y(t)$ is an eventually positive function, so we should have $p(t) \equiv 0$, $p_{\tau_k} = 0$, $k \in N$, which contradicts (H_4).

Thus, $z(t) \geq 0$ eventually. Moreover, in view of (1) and (H_5), we conclude that there exists $\lim\limits_{t\to+\infty} z(t)$ and it is a finite positive number or zero. Observe, that the last fact implies $\lim\limits_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.

Assume $\lim\limits_{t\to+\infty} z(t) = L,\ L > 0$. Then, if we integrate (E_1) from T_0 to t, we obtain

$$\int_{T_0}^{t} z'(s)ds + \int_{T_0}^{t} p(s)y(\sigma(s))ds = 0,$$

or

$$z(t) - z(T_0) - \sum_{T_0<\tau_k<t} \Delta z(\tau_k) + \int_{T_0}^{t} p(s)y(\sigma(s))ds = 0,$$

i.e.

$$z(t) = z(T_0) + \sum_{T_0<\tau_k<t} \Delta z(\tau_k) - \int_{T_0}^{t} p(s)y(\sigma(s))ds. \tag{3}$$

But $\Delta z(\tau_k) = -p_{\tau_k} y(\sigma(\tau_k))$ and from (3) we get

$$z(t) = z(T_0) - \sum_{T_0<\tau_k<t} p_{\tau_k} y(\sigma(\tau_k)) - \int_{T_0}^{t} p(s)y(\sigma(s))ds. \tag{4}$$

Note, that $L < z(t) < y(t)$, i.e. $y(t)$ is bounded from below. Then (4) reduces to

$$z(t) \leq z(T_0) - L\left(\sum_{T_0<\tau_k<t} p_{\tau_k} + \int_{T_0}^{t} p(s)ds\right),$$

which together with (H_5) implies $\lim\limits_{t\to+\infty} z(t) = -\infty$ and contradicts our assumption. Therefore, $\lim\limits_{t\to+\infty} z(t) = 0$. The proof is complete.

The second lemma is only formulated for an eventually negative solution $y(t)$ of the equation (E_1), but the proof is carried out respectively to the proof of Lemma 1.

Lemma 2. *Assume that the hypotheses $(H_1)-(H_5)$ are satisfied. Let $y(t)$ be an eventually negative solution of (E_1). Then $z(t)$ is an increasing eventually negative function of t with not strict up-jumps and $\lim\limits_{t\to+\infty} z(t) = 0$ with $\lim\limits_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.*

Our aim into the next lemma is to establish appropriate sufficient condition under which the equation (E_2) is oscillatory. To this end, we introduce the following result.

Lemma 3. *Assume the hypotheses* $(H_1), (H_2*), (H_3*)$ *are satisfied. Suppose also that:*

$$\frac{1}{e}\limsup_{t\to\infty}\prod_{s(t)<\tau_k<t}(1-q_{\tau_k}) < \liminf_{t\to\infty}\int\limits_{s(t)}^{t} q(r)dr, \qquad k\in N.$$

Then:

(a) the equation (E_2) *is oscillatory;*

(b) the inequality $(N_{2,\le})$ *has no eventually positive solutions;*

(c) the inequality $(N_{2,\ge})$ *has no eventually negative solutions.*

Proof. Since the proofs of (a),(b) and (c) can be carried out by similar arguments, it suffices to prove only the case (a). To this end, we assume for the sake of contradiction, that equation (E_2) has a nonoscillatory solution. Since the negative of a solution of (E_2) is again a solution of (E_2), it suffices to prove the lemma considering this solution as an eventually positive function. So, suppose that $z(t)$ is a solution of (E_2) and there exists a number $t_0>0$, such that $z(t)>0$, for every $t\ge s(t_0)$. Then, it follows from (E_2) that $z'(t)=-q(t)z(s(t))<0$ and $\Delta z(\tau_k)=-q_{\tau_k}z(s(\tau_k))\le 0$, for every $t,\ \tau_k\ge s(t_0)$, $k\in N$, i.e. $z(t)$ is a decreasing function with not strict down-jumps.

Now, we can rearrange (E_2), dividing by $z(t)$, in order to obtain

$$\frac{z'(t)}{z(t)} = -q(t)\frac{z(s(t))}{z(t)} < -q(t), \qquad t\ne\tau_k,\ k\in N, \tag{5}$$

$$\Delta z(\tau_k) = -q_{\tau_k}z(s(\tau_k)) < -q_{\tau_k}z(\tau_k), \qquad k\in N.$$

It follows from the condition of the lemma, that there exist a constant $L>0$ and a number $t_1\ge t_0$, such that

$$\frac{\int\limits_{s(t)}^{t} q(r)dr}{m} \ge L > \frac{1}{e}, \qquad t\ge t_1, \tag{6}$$

where we denote

$$m=\limsup_{t\to\infty}\prod_{s(t)<\tau_k<t}(1-q_{\tau_k}).$$

If we integrate (5) from $s(t)$ to t, we obtain

$$\int_{s(t)}^{t} \frac{z'(r)}{z(r)}dr < -\int_{s(t)}^{t} q(r)dr,$$

or

$$\ln\frac{z(t)}{z(s(t))} + \sum_{s(t)<\tau_k<t} \ln\frac{z(\tau_k)}{z(\tau_k+0)} < -\int_{s(t)}^{t} q(r)dr. \tag{7}$$

Moreover, $z(\tau_k+0) - z(\tau_k) = -q_{\tau_k} z(s(\tau_k)) < -q_{\tau_k} z(\tau_k)$ and $z(\tau_k+0) < (1-q_{\tau_k})z(\tau_k)$, i.e. $\frac{1}{1-q_{\tau_k}} < \frac{z(\tau_k)}{z(\tau_k+0)}$. So, $\ln\frac{1}{1-q_{\tau_k}} < \ln\frac{z(\tau_k)}{z(\tau_k+0)}$ and from (6) and (7) we get

$$\ln[\frac{z(t)}{z(s(t))} \prod_{s(t)<\tau_k<t} \frac{1}{1-q_{\tau_k}}] < -\int_{s(t)}^{t} q(r)dr,$$

i.e.

$$\ln[\frac{z(s(t))}{z(t)} \prod_{s(t)<\tau_k<t} (1-q_{\tau_k})] > Lm.$$

Using the inequality $e^x > ex$, it follows from the last inequality that

$$\frac{z(s(t))}{z(t)} \prod_{s(t)<\tau_k<t} (1-q_{\tau_k}) > eLm,$$

which implies

$$\frac{z(s(t))}{z(t)} > eL.$$

Repeating the above procedure by induction on (5), we conclude that there exists a sequence $\{t_n\}$ where $t_n \to \infty$ as $n \to \infty$, such that

$$\frac{z(s(t))}{z(t)} > (e.L)^n,\ t \geq t_n. \tag{8}$$

Choose n such that

$$\left(\frac{2}{mL}\right)^2 < (eL)^n, \tag{9}$$

which is possible because $eL > 1$, by (6). Further, fix arbitrary chosen $\hat{t}$, where $\hat{t} \geq t_n$.

It follows from (6), that there exists a number $\xi \in [s(\hat{t}), \hat{t}]$, such that

$$\int_{s(\hat{t})}^{\xi} q(r)dr \geq \frac{mL}{2}, \qquad \int_{\xi}^{\hat{t}} q(r)dr > \frac{mL}{2}.$$

If we integrate (E_2) over the interval $[s(\hat{t}), \xi]$, we find

$$z(\xi) - z(s(\hat{t})) - \sum_{s(\hat{t})<\tau_i<\xi} \Delta z(\tau_i) + \int_{s(\hat{t})}^{\xi} q(r)z(s(r))dr = 0,$$

or

$$z(\xi) - z(s(\hat{t})) + \sum_{s(\hat{t})<\tau_i<\xi} q_{\tau_i} z(s(\tau_i)) + \int_{s(\hat{t})}^{\xi} q(s)z(s(r))dr = 0.$$

By omitting the first and the third terms and using the decreasing nature of $z(t)$ we find

$$z(s(\hat{t})) > z(s(\xi)) \int_{s(\hat{t})}^{\xi} q(r)dr,$$

i.e.

$$\frac{z(s(\hat{t}))}{z(s(\xi))} > \frac{mL}{2}. \tag{10}$$

Similarly, integrating (E_2) over the interval $(\xi, \hat{t}]$, we find

$$z(\hat{t}) - z(\xi) - \sum_{\xi<\tau_i<\hat{t}} \Delta z(\tau_i) + \int_{\xi}^{\hat{t}} q(r)z(s(r))dr = 0,$$

or

$$z(\hat{t}) - z(\xi) + \sum_{\xi<\tau_i<\hat{t}} q_{\tau_i} z(s(\tau_i)) + \int_{\xi}^{\hat{t}} q(r)z(s(r))dr = 0.$$

By omitting the first and the third terms and using the decreasing nature of $z(t)$ we find

$$z(\xi) > z(s(\hat{t})) \int_{\xi}^{\hat{t}} q(r)dr,$$

i.e.
$$\frac{z(\xi)}{z(s(\hat{t}))} > \frac{mL}{2}. \tag{11}$$

From (10) and (11) we conclude
$$\frac{z(s(\xi))}{z(\xi)} < \left(\frac{2}{mL}\right)^2,$$

which, together with (8), imply
$$(eL)^n < \frac{z(s(\xi))}{z(\xi)} < \left(\frac{2}{mL}\right)^2. \tag{12}$$

Note that (12) is in contradiction with (9). The proof of the lemma is complete.

In the next section we shall investigate the behavior of an eventually none-oscillatory solution $y(t)$ of equation (E_1). To this purpose, we introduce the following hypotheses, which describe the possible location of the points of impulse effect:

(**AH**) $\exists\, n \geq 1,\ n \in N\ :\ h(\tau_s) \in \{\tau_k\}_{k=1}^{\infty},\ \tau_s \in \{\tau_i\}_{i=n+1}^{\infty}$.

(**BH**) There exists a strictly increasing sequence $\{k_\nu\}_{\nu=1}^{\infty} \subseteq N$ (not obligatory consistent), for which $\{\tau_{k_\nu}\}_{\nu=1}^{\infty} \subseteq \{\tau_k\}_{k=1}^{\infty}$, but $h(\tau_{k_\nu}) \bar{\in} \{\tau_k\}_{k=1}^{\infty}$.

4. Asymptotic Behavior of Non-Oscillatory Solutions

The theorem in this section describes the asymptotic behavior of the eventually non-oscillatory solution y(t) of (E_1) under the hypothesis (H_5).

Theorem 1. *Let the hypotheses $(H_1) - (H_5)$ are satisfied. Then every non-oscillatory solution $y(t)$ of (E_1) tends to zero when t tends to $+\infty$, i.e.*
$$\lim_{t\to+\infty} y(t) = 0, \quad \lim_{\tau_k\to+\infty} |\Delta y(\tau_k)| = 0.$$

Proof. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering an eventually positive solution of (E_1), using Lemma 1. The case with eventually negative solutions can be considered respectively, by Lemma 2. So, let $y(t)$ be an eventually positive solution of the equation (E_1), i.e. $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$ such that $y(t) > 0$ for $t \geq \rho(T_0)$.

First of all, we claim that $y(t)$ is a bounded function. Assume for the sake of contradiction, that $y(t)$ is unbounded. Denote $y(t_n) = \max\limits_{t_0 \leq s \leq t_n} (y(s))$. Then, there exists a sequence $\{t_n\}$, such that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{t_n\to+\infty} y(t_n) = +\infty$. In view of Lemma 1, we have eventually $z(t) \le \delta$ for arbitrary positive constant δ. So, using (H_3), for sufficiently large n we get $y(t_n) \le \delta + c(t_n)y(h(t_n)) < \delta + c_0 y(t_n)$. This implies $(1-c_0)y(t_n) \le \delta$, which is a contradiction. Therefore, $y(t)$ is a bounded positive function.

From (1) and Lemma 1, we have $\lim_{\tau_k\to+\infty} |\Delta\left(y(\tau_k) - c_{\tau_k} y(h(\tau_k))\right)| = 0$, where $\Delta\left(y(\tau_k) - c_{\tau_k} y(h(\tau_k))\right) \le 0$, i.e. $\Delta y(\tau_k) \le c_{\tau_k} \Delta y(h(\tau_k))$, $k \in N$, $\tau_k \ge T_0$. (13)

Assume, hypothesis (**AH**) is valid. Then, there are two possibilities. The first one (let us call it Case (**AH**+)) is when there are no any down-jumps of $y(t)$ at the points $h(\tau_k)$, i.e. $\Delta y(h(\tau_k)) > 0$, for every $k \in N$, $k \ge n+1$. The second possibility (let us call it Case (**AH**−)) is when for some $i \ge k$, $i \in N$, we have $\Delta y(h(\tau_i)) < 0$, i.e. there is at least one down-jump of $y(t)$ at the point $h(\tau_i)$.

When Case (**AH**+) is valid, then $\Delta y(\tau_k) \le c_{\tau_k} \Delta y(h(\tau_k))$. Moreover, by (H_3), it follows $\Delta y(\tau_k) \le c_0 \Delta y(h(\tau_k))$. Note, that in view of (H_2), there exists the reverse function $h^{-n}(\tau_k)$. So, by iterations based on ${\tau_k}^* = h^{-n}(\tau_k)$, from the last inequality we can get $\Delta y({\tau_k}^*) \le (c_0)^{n+1} \Delta y(h(\tau_k))$, which in view of (H_3) implies that $\Delta y({\tau_k}^*)$ becomes less then every positive number, for all ${\tau_k}^*$ large enough. Hence, $\lim_{\tau_k\to+\infty} |\Delta y(\tau_k)| = 0$.

When Case (**AH**−) is valid, then for every $\varepsilon_\nu > 0$, it does exist $\tau_\nu \ge T_0 > 0$, such that $|\Delta\left(y(\tau_k) - c_{\tau_k} y(h(\tau_k))\right)| < \varepsilon_\nu$, $\tau_k \ge \tau_\nu, k \in N$. Thus, we have $-\Delta y(\tau_k) + c_{\tau_k} \Delta y(h(\tau_k)) < \varepsilon_\nu$, which implies $-\Delta y(\tau_k) < \varepsilon_\nu + c_0 \left(-\Delta y(h(\tau_k))\right)$. From the last inequality, by iterations based on ${\tau_k}^* = h^{-n}(\tau_k)$, $n \in N$, we can get

$$-\Delta y({\tau_k}^*) < \varepsilon_\nu (1 + c_0 + c_0^2 + ... + c_0^n) + c_0^{n+1} \left(-\Delta y(h(\tau_k))\right). \qquad (14)$$

In view of (H_3), (14) implies

$$-\Delta y({\tau_k}^*) < \frac{\varepsilon_\nu}{1-c_0}, \qquad t \ge t_\nu. \qquad (15)$$

From (15), it follows that $|\Delta y({\tau_k}^*)|$ becomes less then every positive number, for all ${\tau_k}^*$ large enough. Hence, $\lim_{\tau_k\to+\infty} |\Delta y(\tau_k)| = 0$.

Assume, hypotheses (**BH**) is valid. Then, $\{\Delta y(\tau_{k_\nu})\}_{\nu=1}^\infty$ is a sequence with negative numbers, for which, from (13), we have $\lim_{\nu\to+\infty} |\Delta\left(y(\tau_{k_\nu})\right)| = 0$. For the rest of the points of impulse effect, we have two possibilities. The first one (let us call it Case (**BH**+)) is when there are no any down-jumps of $y(t)$ at the points $h(\tau_k)$, i.e. $\Delta y(h(\tau_k)) > 0$, where $\tau_k \bar{\in} \{\tau_{k_\nu}\}_{\nu=1}^\infty$, $k \in N$. The second possibility (let us call it Case (**BH**$-$)) is when for some $i \geq k_1$, $i \in N$, we have $\Delta y(h(\tau_i)) < 0$, i.e. there is at least one down-jump of $y(t)$ at the point $h(\tau_i)$, where $\tau_i \bar{\in} \{\tau_{k_\nu}\}_{\nu=1}^{\infty}$, $i \in N$. Now, if Case (**BH**+) is valid, then by arguments, similar to those in Case (**AH**+), we get $\lim\limits_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0$. If Case (**BH**$-$) is valid, then by arguments, similar to those in Case (**AH**$-$), we conclude again $\lim\limits_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0$.

Therefore, $\lim\limits_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0$, at all.

Further, consider that by Lemma 1 we have $\lim\limits_{t \to +\infty} (y(t) - c(t)y(h(t))) = 0$, where in the meantime $y(t) - c(t)y(h(t)) > 0$ eventually. It means, that for every $\varepsilon_M > 0$ it does exist $M \geq T_0$, such that $|y(t) - c(t)y(h(t))| < \varepsilon_M$, or equivalently, in view of (H_3) and Lemma 1, $y(t) < \varepsilon_M + c_0 y(h(t))$ when $t > h^{-1}(M)$. From the last inequality, by iterations based on $t_* = h^{-n}(t)$, $n \in N$, we can get for every fixed $h(t) > M$

$$y(t_*) < \varepsilon_M(1 + c_0 + c_0^2 + ... + c_0^n) + c_0^{n+1} y(h(t)), \tag{16}$$

which implies that

$$y(t_*) < \frac{\varepsilon_M}{1 - c_0} \tag{17}$$

for all t_* large enough. So, we conclude that for every $\varepsilon_* = \frac{\varepsilon_M}{1-c_0} > 0$ it does exist $t_* > T_0 > 0$, such that $y(t) < \varepsilon_*$, when $t \geq t_*$, i.e. $y(t)$ becomes less then every positive number, for all t large enough. Therefore, $\lim\limits_{t \to +\infty} y(t) = 0$.

The proof of the theorem is complete.

5. Oscillation Criteria for the Solutions of (E_1)

In this section we study the oscillatory properties of the equation (E_1). The next theorems will establish sufficient conditions for oscillation of (E_1).

Theorem 2. *Let the hypotheses $(H_1) - (H_5)$ are satisfied. Suppose also that:*

(ii) $\quad p_{\tau_k} < \dfrac{1}{c_{\sigma(\tau_k)}}, \; k \in N;$

(iii) $$\frac{1}{e} \limsup_{t \to \infty} \Big[\prod_{h(\sigma(t)) < \tau_k < t} (1 - c_{\sigma(\tau_k)} p_{\tau_k}) \Big] < \liminf_{t \to \infty} \int_{h(\sigma(t))}^{t} c(\sigma(r)) p(r) dr \, .$$

Then, the equation (E_1) is oscillatory.

Proof. Assume, for the sake of contradiction, that equation (E_1) has a nonoscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering an eventually positive solution of (E_1).

So, let us suppose that $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$, such that $y(t) > 0$, for $t \geq \rho(T_0)$. From Lemma 1, it follows that $z(t)$ is a decreasing eventually positive function of t with not strict down-jumps, i.e. it satisfies the conditions

$$z(t) > 0, \qquad z'(t) \leq 0, \ \ t \geq T_0, \qquad \Delta z(\tau_k) \leq 0, \ \tau_k \geq T_0, \ k \in N. \tag{18}$$

Since we have eventually $y(t) - c(t)y(h(t)) = z(t) > 0$, it follows that $y(t) > c(t)y(h(t))$. Moreover, $y(t) > z(t)$. Obviously, then we have

$$y(\sigma(t)) > c(\sigma(t))y(h(\sigma(t))) > c(\sigma(t))z(h(\sigma(t))), \qquad \text{i.e.}$$

$$y(\sigma(t)) > c(\sigma(t))z(h(\sigma(t))). \tag{19}$$

Multiplying the both sides of (19) by $-p(t) < 0$, we obtain

$$z'(t) = -p(t)y(\sigma(t)) < -c(\sigma(t))p(t)z(h(\sigma(t))).$$

Hence,

$$z'(t) + c(\sigma(t))p(t)z(h(\sigma(t))) < 0. \tag{20}$$

Observe that from (19) we have also $c_{\sigma(\tau_k)}z(h(\sigma(\tau_k)) < y(\sigma(\tau_k)), \ k \in N$. Multiplying the both sides of the last inequality by $-p_{\tau_k} < 0, \ k \in N$, we obtain also

$$-c_{\sigma(\tau_k)}p_{\tau_k}z(h(\sigma(\tau_k))) > -p_{\tau_k}y(\sigma(\tau_k)) = \Delta z(\tau_k), \quad k \in N, \qquad \text{i.e.}$$

$$\Delta z(\tau_k) + c_{\sigma(\tau_k)}p_{\tau_k}z(h(\sigma(\tau_k))) < 0, \ k \in N. \tag{21}$$

From (20) and (21), when we denote $s(t) = h(\sigma(t)), \ q(t) = c(\sigma(t))p(t), \ q_{\tau_k} = c_{\sigma(\tau_k)}p_{\tau_k}$, we see that the positive function $z(t)$ satisfies the delay impulsive differential inequality

$$z(t)' + q(t)z(s(t)) < 0, \ t \neq \tau_k, \ k \in N \tag{22}$$

$$\Delta z(\tau_k) + q_{\tau_k}z(s(\tau_k)) < 0, \ k \in N,$$

which has the form of $(N_{2,\leq})$. But, the conclusion obtained under the conditions (ii) and (iii) of the theorem and in view of Lemma 3(b), contradicts (18). The proof is complete.

Corollary 1. *Let the conditions of Theorem 2 hold. Then:*

(i) the inequality $(N_{1,\leq})$ *has no eventually positive solutions;*

(ii) the inequality $(N_{1,\geq})$ *has no eventually negative solutions.*

The proof of the corollary is carried out analogously to the proofs of the respective theorems.

Next theorem will consider the oscillation of the equation (E_1) with oscillating coefficient $p(t)$. For convenience, in $E_1)$ we set $h(t) = t - h,\ h > 0$ and $\sigma(t) = t - \sigma,\ \sigma > 0$.

Theorem 3. *Let the hypotheses* $(H_1) - (H_3)$ *are satisfied. Suppose also that:*

1. There exists a number $l > 0$, *such that* $p(t) > 0$ *at least in the sequence of intervals* $\{[t_n - 2l, t_n + 2l]\}_{n=1}^{\infty}$, *where* $t_n < t_{n+1} - 2l$;

2. $\liminf\limits_{t\to+\infty} \int\limits_{t-\sigma}^{t} p(s)ds > \frac{1}{e}$, *where* $t \in U(t_\infty, 2l) = \bigcup\limits_{n=1}^{+\infty} [t_n - 2l, t_n + 2l],\ \rho < 2l$;

3. $p_k \geq 1 - \left[\dfrac{(1-c_0)c_0}{e}\right]^2,\ e = \exp(1),\ k \in N.$

Then all solutions of equation (E_1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that (E_1) has a nonoscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering an eventually positive solution of (E_1).

So, let us suppose that there exists a solution $y(t)$ of (E_1) and a number $\tilde{t} > 0$, such that $y(t)$ is defined for $t \geq \tilde{t}$, and $y(t) > 0,\ y(t-h) > 0,\ y(t-\sigma) > 0$ for $t \geq t_0 = \tilde{t} + 2l$.

Then, for the auxiliary function $z(t)$, which is defined by $(*)$, it is fulfilled

$$z'(t) \leq -p(t)y(t-\sigma) < 0,\ t \neq \tau_k,\ t \in U(t_\infty, 2l) \tag{23}$$

$$\Delta z(\tau_k) \leq -p_k y(\tau_k - \sigma) < 0,\ \tau_k \in U(t_\infty, 2l),\ k \in N.$$

From (23) and in accordance with Lemma 1, it follows that $z(t)$ is a decreasing eventually positive function with down-jumps in $U(t_\infty, 2l)$, i.e. there exists a number $t_1 \in U(t_\infty, 2l)$ such that $z(t) > 0$ for every $t \geq t_1 > t_0,\ t \in U(t_\infty, 2l)$.

Observe, that from $(*)$ we have $y(t-\sigma) = z(t-\sigma) + c(t-\sigma)y(t-\sigma-h)$, which yields $y(t-\sigma) > z(t-\sigma),\ t \in U(t_\infty, 2l)$. Then, from (23), it follows for every $t \in U(t_\infty, 2l)$

$$z'(t) + p(t)z(t-\sigma) < 0. \tag{24}$$

Moreover, $\Delta z(\tau_k) < -p_k y(\tau_k - \sigma)$ and $z(\tau_k) < z(\tau_k - \sigma) < y(\tau_k - \sigma)$, which leads us to $\Delta z(\tau_k) < -p_k z(\tau_k - \sigma)$ for every $\tau_k \in U(t_\infty, 2l)$. Hence, for every $\tau_k \in U(t_\infty, 2l)$ we have

$$\Delta z(\tau_k) + p_k z(\tau_k - \sigma) < 0. \tag{25}$$

From (24) and (25) we conclude that $z(t)$ satisfies the delay impulsive differential inequality

$$z'(t) + p(t)z(t-\sigma) < 0, \ t \neq \tau_k, \ t \in U(t_\infty, 2l), \tag{26}$$

$$\Delta z(\tau_k) + p_k z(\tau_k - \sigma) < 0, \ \tau_k \in U(t_\infty, 2l), \ k \in N.$$

We will prove that (26) has no eventually positive solutions. For that purpose let us define

$$l_n := \{l_n : \int_{t_n-\sigma}^{t_n-l_n} p(t)dt = \frac{1-c_0}{e}, t \in [t_n - 2l, t_n + 2l]\}. \tag{27}$$

Now, if we integrate (E_1) from $t_n - l_n$ to t_n, we obtain

$$\int_{t_n-l_n}^{t_n} z'(s)ds + \int_{t_n-l_n}^{t_n} p(s)z(s-\sigma)ds = 0.$$

Furthermore, if suppose there are some $\tau_k \in [t_n - l_n, t_n]$, then we have

$$z(t_n) - z(t_n - l_n) - \sum_{t_n - l_n \le \tau_k < t_n} \Delta z(\tau_k) + \int_{t_n-l_n}^{t_n} p(s)z(s-\sigma)ds = 0.$$

Here, by omitting the first and the third terms and using the decreasing nature of $z(t)$ we find $z(t_n - l_n) > z(t_n - \sigma) \int_{t_n-l_n}^{t_n} p(s)ds$. Then, in view of (13), it is easy to conclude

$$z(t_n - l_n) > z(t_n - \sigma)\frac{c_0}{e}. \tag{28}$$

Working in a similar way from $t_n + \sigma - 2l_n$ to $t_n + \sigma - l_n$, we can obtain

$$z(t_n + \sigma - 2l_n) > z(t_n - l_n)\frac{c_0}{e}. \tag{29}$$

From (28) and (29) we conclude

$$(\frac{e}{c_0})^2 > \frac{z(t_n - \sigma)}{z(t_n + \sigma - 2l_n)}. \tag{30}$$

Remark 1. It is obvious that $t_n - l_n < t_n + \sigma - 2l_n < t_n + \sigma - l_n$.

Using the decreasing nature of $z(t)$ we find $z'(t) < p(t)z(t-\sigma) < p(t)z(t)$. Thus, we have also $z'(t) + q(t)z(t) < 0$.

If we divide the last inequality by $-z(t) < 0$ and integrate from $t_n - \sigma$ to $t_n - l_n$, we obtain

$$-\int_{t_n-\sigma}^{t_n-l_n} \frac{dz(s)}{z(s)} > \int_{t_n-\sigma}^{t_n-l_n} q(s)ds = \frac{1-c_0}{e},$$

i.e.

$$-\left[ln\frac{z(t_n-l_n)}{z(t_n-\sigma)} - \sum_{\tau_k\in[t_n-\sigma,t_n-l_n]} ln\frac{z(\tau_k+0)}{z(\tau_k)}\right] > \frac{1-c_0}{e},$$

or

$$ln\left[\frac{z(t_n-\sigma)}{z(t_n-l_n)} \prod_{\tau_k\in[t_n-\sigma,t_n-l_n]} \frac{z(\tau_k+0)}{z(\tau_k)}\right] > \frac{1-c_0}{e}. \tag{31}$$

From (26), we have $\Delta z(\tau_k) + p_k z(\tau_k - \sigma) < 0$, which in view of the decreasing nature of the function $z(t)$ yields $z(\tau_k+0) < (1-p_k)z(\tau_k)$. Then, using the last fact in (31) we have

$$ln\left[\frac{z(t_n-\sigma)}{z(t_n-l_n)} \prod_{\tau_k\in[t_n-\sigma,t_n-l_n]} (1-p_k)\right] > \frac{1-c_0}{e},$$

i.e.

$$\frac{z(t_n-\sigma)}{z(t_n-l_n)} > \frac{1-c_0}{\prod\limits_{\tau_k\in[t_n-\sigma,t_n-l_n]} (1-p_k)} \tag{32}$$

Working in a similar way from $t_n - l_n$ to $t_n + \sigma - 2l_n$, we get

$$\frac{z(t_n-l_n)}{z(t_n+\sigma-2l_n)} > \frac{1-c_0}{\prod\limits_{\tau_k\in[t_n-l_n,t_n+\sigma-2l_n]} (1-p_k)} \tag{33}$$

From (32) and (33), it follows

$$\frac{z(t_n-\sigma)\prod\limits_{\tau_k\in[t_n-l_n,t_n+\sigma-2l_n]}(1-p_k)}{(1-c_0)z(t_n+\sigma-2l_n)} > \frac{(1-c_0)}{\prod\limits_{\tau_k\in[t_n-\sigma,t_n-l_n]}(1-p_k)}$$

i.e.

$$\frac{z(t_n-\sigma)}{z(t_n+\sigma-2l_n)} > \frac{(1-c_0)^2}{\prod\limits_{\tau_k\in[t_n-\sigma,t_n+\sigma-2l_n]}(1-p_k)} \tag{34}$$

From (30) and (34) we conclude

$$(\frac{e}{c_0})^2 > \frac{z(t_n-\sigma)}{z(t_n+\sigma-2l_n)} > \frac{(1-c_0)^2}{\prod\limits_{\tau_k\in[t_n-\sigma,t_n+\sigma-2l_n]}(1-p_k)}$$

i.e.

$$\prod_{\tau_k\in[t_n-\sigma,t_n+\sigma-2l_n]}(1-p_k) > \Big[\frac{(1-c_0)c_0}{e}\Big]^2$$

The last inequality contradicts to the condition 3 of the theorem. The proof is complete.

Corollary 2. *Let the conditions of Theorem 3 hold. Then:*

(i) the inequality $(N_{1,\leq})$ *has no eventually positive solutions;*

(ii) the inequality $(N_{1,\geq})$ *has no eventually negative solutions.*

The proof of the corollary is carried out analogously to the proof of the Theorem 3.

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