International Journal of Pure and Applied Mathematics
Volume 73 No. 3 2011, 289-298

GROUP DIVISIBLE DESIGNS WITH TWO ASSOCIATE CLASSES AND $\lambda_2 = 4$

Chariya Uiyyasathian[1], Nittiya Pabhapote[2 §]

[1]Department of Mathematics
Faculty of Science
Chulalongkorn University
Bangkok, 10330, THAILAND
[1]Center of Excellence in Mathematics, CHE
Sri Ayutthaya Rd., Bangkok, 10400, THAILAND
[2]School of Science and Technology
University of the Thai Chamber of Commerce
Dindaeng, Bangkok, 10400, THAILAND

Abstract: A *group divisible design* $\text{GDD}(v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2)$ is an ordered triple $(V, G, \mathcal{B})$, where V is a v-set of symbols, G is a partition of V into g sets of size $v_1, v_2, \ldots, v_g$, each set being called *group*, and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. Here, we focus on an existence problem of GDDs with two associate classes or when $g = 2$, and with blocks of size 3, when the required designs have two groups of unequal sizes and $\lambda_2 = 4$. We obtain the necessary conditions and prove that these conditions are sufficient.

AMS Subject Classification: 05B05, 05B07
Key Words: graph decomposition, group divisible design

1. Introduction

A *balanced incomplete block design* $\text{BIBD}(v, b, r, k, \lambda)$ is a set S of v elements together with a collection $\mathcal{B}$ of b k-subsets of S, called *blocks*, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks.

Received: August 19, 2011

§Correspondence author

The number $|S| = v$ is called the *order* of the BIBD. We often represent such BIBD by the order pair $(S, \mathcal{B})$. More details involve with BIBDs can be explored in [5], [6], and [8].

A *group divisible design* GDD$(v = v_1+v_2+\cdots+v_g, g, k, \lambda_1, \lambda_2)$ is an ordered triple $(V, G, \mathcal{B})$, where V is a v-set of symbols, G is a partition of V into g sets of size $v_1, v_2, \ldots, v_g$, each set being called *group*, and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [5], [6]). Elements occurring together in the same group are called *first associates*, and elements occurring in different groups we called *second associates*. We say that the GDD is defined on the set V. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs, see [1]. The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger [5], [6] when all groups are the same size.

In this paper, we continue to focus on blocks of size 3, solving the problem when the required designs having two groups of unequal size, namely, we consider the problem of determining necessary conditions for an existence of GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient. Since we are dealing on GDDs with two groups and block size 3, we will use GDD$(m, n; \lambda_1, \lambda_2)$ for GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as *triples*. We denote $(X, Y; \mathcal{B})$ for a GDD$(m, n; \lambda_1, \lambda_2)$ if X and Y are m-set and n-set, respectively. Chaiyasana, Hurd, Punnim and Sarvate [2] have written the first paper in this direction. In particular they have completely solved the problem of determining all pairs of integers (n, λ) in which a GDD$(1, n; 1, \lambda)$ exists. More work intends to solve the existence problem of a GDD$(m, n; \lambda_1, \lambda_2)$ for possible m, n, λ_1 and λ_2. Lapchinda and Pabhapote [7] solved the problem when the designs have unequal sizes and $\lambda_1 - \lambda_2 = 1$. Pabhapote and Punnim [9] investigate all triples of integers (m, n, λ) in which a GDD$(m, n; \lambda, 1)$ exists. Recently, the existence of the design GDD$(m, n; \lambda, 2)$ is completed solved when $\lambda \geq 2$, see [10]. Moreover, Chaiyasana and Pabhapote [3] investigate all triples of integers (m, n, λ) in which a GDD$(m, n; \lambda, 3), \lambda \geq 3$, exists. Analogously, in this paper, we continue to reveal all triples of integers (m, n, λ) in which a GDD$(m, n; \lambda, 4)$ exists for $\lambda \geq 4$. When $\lambda \leq 3$, a construction to prove the sufficiency seems to be much more complicated, which in fact remains an open problem in general case. Thus we here focus on $\lambda \geq 4$, and for convenience we present quick constructions based on known results from previous work to obtain the sufficiency of the following main theorem:

Theorem A. *Let m, n and $\lambda \geq 4$ be positive integers with $m \neq 2$ and*

$n \neq 2$. *There exists a* GDD$(m,\ n; \lambda, 4)$ *if and only if*

1. $3 \mid \lambda[m(m-1) + n(n-1)] + 2mn$, *and*

2. $2 \mid \lambda(m-1)$ *and* $2 \mid \lambda(n-1)$.

2. Preliminary Results

We will see that necessary conditions on the existence of a GDD$(m, n; \lambda_1, \lambda_2)$ can be easily obtained by describing it graphically as follows. Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD$(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m+n$ and size $\lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies

1. $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, and

2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

We will review some known results concerning triple designs that will be used in the sequel, most of which are taken from [8].

A BIBD$(v, 3, 1)$ is usually called *Steiner triple system* and is denoted by STS(v). Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v-1)/6$.

The next theorem concludes the conditions for the existence of a BIBD$(v, 3, 1)$.

Theorem 2.1. (see [8]) *Let v be a positive integer.*

1. *There exists an* STS(v) *if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

The following notations will be used for our constructions.

1. Let V be a v-set. Then there may be many different STS(v)s that can be constructed on the set V. Let STS(V) be defined as

$$\mathrm{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an } \mathrm{STS}(v)\}.$$

BIBD$(V, 3, \lambda)$ can be defined similarly, That is:

$$\text{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \text{BIBD}(v, 3, \lambda)\}.$$

Let X and Y be disjoint sets of cardinality m and n, respectively. We define GDD$(X, Y; \lambda_1, \lambda_2)$ as

$$\text{GDD}(X, Y; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y; \mathcal{B}) \text{ is a } \text{GDD}(m, n; \lambda_1, \lambda_2)\}.$$

2. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus " $\cup$ " in our construction will be used for the union of multi-sets.

When $k = 3$, a BIBD$(v, 3, \lambda)$ is also called a λ-*fold triple system* . The following results on existence of λ-fold triple systems are well-known (see e.g. [8]).

Theorem 2.2. *Let n be a positive integer. Then a* BIBD$(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases:*

(a) $\lambda \equiv 0 \pmod 6$ *and for all positive integers* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and for all n with* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and for all n with* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and for all odd integers n.*

The existence of a GDD$(m, n; \lambda_1, \lambda_2)$ for some specific 4-tuples $(m, n, \lambda_1, \lambda_2)$ satisfying the provided conditions is concluded in the next two theorems.

Theorem 2.3. (see [2]) *Let $v \geq 3$ be an integer. There exists a* GDD$(1, v;$ $1, \lambda)$ *if and only if* $2 \mid (v - 1 - \lambda)$ *and* $6 \mid v(v - 1 - \lambda)$.

Theorem 2.4. (see [9]) *Let m and n be positive integers with $m \neq 2$ and $n \neq 2$. There exists a* GDD$(m, n; \lambda, 1)$ *if and only if*

1. $3 \mid \lambda[m(m - 1) + n(n - 1)] + 2mn$, *and*

2. $2 \mid \lambda(m - 1) + n$ *and* $2 \mid \lambda(n - 1) + m$.

3. GDD$(m, n; \lambda, 4)$

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers $\{m, n\}$ in which a GDD$(m, n; \lambda, 4)$ exists. Recall that the existence of GDD$(m, n; \lambda, 4)$ implies $3 \mid \lambda[m(m-1)+n(n-1)]+2mn$, $2 \mid \lambda(m-1)$ and $2 \mid \lambda(n-1)$. Let

$$S_4(\lambda) := \{\{m, n\} : a\ \text{ GDD}(m, n; \lambda, 4)\ \textit{ exists}\}.$$

The next lemma shows the necessity of Theorem A, in other words, it provides all possible pairs of integers $\{m, n\}$ in which a GDD$(m, n; \lambda, 4)$ exists for a given λ.

Lemma 3.1. *Let t be a non-negative integer:*

(a) *If $\{m, n\} \in S_4(6t+4)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+2\}, \{6k+1, 6h+3\}, \{6k+1, 6h+5\}, \{6k+1, 6h+6\}, \{6k+2, 6h+2\}, \{6k+2, 6h+4\}, \{6k+2, 6h+5\}, \{6k+3, 6h+3\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+4, 6h+5\}, \{6k+4, 6h+6\}, \{6k+5, 6h+5\}, \{6k+6, 6h+6\}\}$.

(b) *If $\{m, n\} \in S_4(6t+5)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+3\}, \{6k+3, 6h+3\}\}$.

(c) *If $\{m, n\} \in S_4(6t+6)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+3\}, \{6k+1, 6h+6\}, \{6k+2, 6h+3\}, \{6k+2, 6h+6\}, \{6k+3, 6h+3\}, \{6k+3, 6h+4\}, \{6k+3, 6h+5\}, \{6k+3, 6h+6\}, \{6k+4, 6h+6\}, \{6k+5, 6h+6\}, \{6k+6, 6h+6\}\}$.

(d) *If $\{m, n\} \in S_4(6t+7)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+3\}, \{6k+1, 6h+5\}, \{6k+3, 6h+3\}, \{6k+5, 6h+5\}\}$.

(e) *If $\{m, n\} \in S_4(6t+8)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+3\}, \{6k+1, 6h+6\}, \{6k+3, 6h+3\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+4, 6h+6\}, \{6k+6, 6h+6\}\}$.

(f) *If $\{m, n\} \in S_4(6t+9)$, then there exist non-negative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+3\}, \{6k+3, 6h+3\}, \{6k+3, 6h+5\}\}$.

Proof. The proof follows from solving the corresponding systems of congruences. $\square$

The rest of the paper is devoted to show the existence of a GDD$(m, n; \lambda, 4)$ for each pair of integers (m, n) satisfying $\lambda \geq 4$ in Lemma 3.1. The following cases will be treated specifically while the rest cases have a similar construction.

Special case I : $\{6k+2, 6h+2\} \in S_4(6t+4)$.

Special case II : $\{6k+2, 6h+3\} \in S_4(6t+6)$.

Special case III : $\{6k+2, 6h+6\} \in S_4(6t+6)$.

Special case IV : $\{6k+3, 6h+5\} \in S_4(6t+6)$.

Special case V : $\{6k+5, 6h+6\} \in S_4(6t+6)$.

Special case VI : $\{6k+3, 6h+5\} \in S_4(6t+9)$.

Given two positive integers m and n such that $m+n \geq 4$, the construction of the remaining cases beyond three special cases above can be done by first building a BIBD$(m+n, 3, 4)$, and then completing the construction by building blocks for the first associate entries to occur together in more $\lambda - 4$ blocks. We thus can conclude the followings:

Lemma 3.2. *Let h, k and t be non-negative integers. Then*

(a) $\{6k+1, 6h+2\}$, $\{6k+1, 6h+3\}$, $\{6k+1, 6h+5\}$, $\{6k+1, 6h+6\}$, $\{6k+2, 6h+4\}$, $\{6k+2, 6h+5\}$, $\{6k+3, 6h+3\}$, $\{6k+3, 6h+4\}$, $\{6k+3, 6h+6\}$, $\{6k+4, 6h+5\}$, $\{6k+4, 6h+6\}$, $\{6k+5, 6h+5\}$, $\{6k+6, 6h+6\} \in S_4(6t+4)$,

(b) $\{6k+1, 6h+3\}$, $\{6k+3, 6h+3\} \in S_4(6t+5)$,

(c) $\{6k+1, 6h+3\}$, $\{6k+1, 6h+6\}$, $\{6k+3, 6h+3\}$, $\{6k+3, 6h+4\}$, $\{6k+3, 6h+6\}$, $\{6k+4, 6h+6\}$, $\{6k+6, 6h+6\} \in S_4(6t+6)$,

(d) $\{6k+1, 6h+3\}$, $\{6k+1, 6h+5\}\}$, $\{6k+3, 6h+3\}$, $\{6k+5, 6h+5\} \in S_4(6t+7)$,

(e) $\{6k+1, 6h+3\}$, $\{6k+1, 6h+6\}$, $\{6k+3, 6h+3\}$, $\{6k+3, 6h+4\}$, $\{6k+3, 6h+6\}$, $\{6k+4, 6h+6\}$, $\{6k+6, 6h+6\} \in S_4(6t+8)$,

(f) $\{6k+1, 6h+3\}, \{6k+3, 6h+3\} \in S_4(6t+9)$.

Proof. Let λ be given. Let $\{m, n\}$ be such a pair from the list above corresponding to the given λ. Let X be an m-set and Y be an n-set. By Theorem 2.2 for a 4-fold triple system, BIBD$(X \cup Y, 3, 4) \neq \emptyset$ since $m+n \equiv$

0 *or* 1 (mod 3). Let $\mathcal{B}_1 \in \text{BIBD}(X \cup Y, 3, 4)$. Furthermore, since $\lambda \geq 4$, Theorem 2.2 for a (λ-4)-fold triple system also guarantees that both $\text{BIBD}(X, 3, \lambda - 4)$ and $\text{BIBD}(Y, 3, \lambda - 4)$ are not empty. Let $\mathcal{B}_2 \in \text{BIBD}(X, 3, \lambda - 4)$ and $\mathcal{B}_3 \in \text{BIBD}(Y, 3, \lambda - 4)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Hence, $(X, Y; \mathcal{B})$ forms a $\text{GDD}(m, n; \lambda, 4)$ as desired. □

Next we consider the six special cases as follows.

Lemma 3.3. *Let h, k and t be non-negative integers. Then*

(a) $\{6k + 2, 6h + 2\} \in S_4(6t + 4)$,

(b) $\{6k + 2, 6h + 3\} \in S_4(6t + 6)$,

(c) $\{6k + 2, 6h + 6\} \in S_4(6t + 6)$,

(d) $\{6k + 3, 6h + 5\} \in S_4(6t + 6)$,

(e) $\{6k + 5, 6h + 6\} \in S_4(6t + 6)$, *and*

(f) $\{6k + 3, 6h + 5\} \in S_4(6t + 9)$.

Proof. Note that the proofs for (a)-(f) use the same technique though we spell all details out.

(a) Let X_k be a $(6k + 2)$-set and Y_h be a $(6h + 2)$-set. Since $X_k \cup Y_h$ is of size $6k + 6h + 4$, by Theorem 2.2 for a 4-fold triple system, $\text{BIBD}(X_k \cup Y_h, 3, 4)$ is not empty; so, let $\mathcal{B}_1$ be in $\text{BIBD}(X_k \cup Y_h, 3, 4)$. Besides, by Theorem 2.2 for a $6t$-fold triple system, we have $\mathcal{B}_2 \in \text{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_3 \in \text{BIBD}(Y_h, 3, 6t)$. By setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, we have that $(X_k, Y_h; \mathcal{B})$ forms a $\text{GDD}(6k+2, 6h+2; 6t + 4, 4)$.

(b) Let X_k be a $(6k+2)$-set containing a and Y_h be a $(6h+3)$-set. Since $(X_k \setminus \{a\}) \cup Y_h$ is of size $6k+6h+4$, it follows by Theorem 2.2 for a 4-fold triple system that $\text{BIBD}((X_k \setminus \{a\}) \cup Y_h, 3, 4)$ is not empty. Let $\mathcal{B}_1$ be in $\text{BIBD}((X_k \setminus \{a\}) \cup Y_h, 3, 4)$. Since $X_k \setminus \{a\}$ and Y_h are of size $6k+1$ and $6h+3$, respectively, Theorem 2.2 for a 1-fold triple system confirms that there exists $\mathcal{B}_2 \in \text{BIBD}(X_k \setminus \{a\}, 3, 1)$ and $\mathcal{B}_3 \in \text{BIBD}(Y_h, 3, 1)$. Moreover, Theorem 2.3 ensures that there exist $\mathcal{B}_4 \in \text{GDD}(\{a\}, X_k \setminus \{a\}; 1, 6)$ and $\mathcal{B}_5 \in \text{GDD}(\{a\}, Y_h; 1, 4)$. Lastly, use Theorem 2.2 for a $6t$-fold triple system to get $\mathcal{B}_6 \in \text{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_7 \in \text{BIBD}(Y_h, 3, 6t)$. Hence, when $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_7$, we have that $(X_k, Y_h; \mathcal{B})$ forms a $\text{GDD}(6k+2, 6h + 3; 6t + 6, 4)$ as desired.

(c) Let X_k be a $(6k + 2)$-set containing a and Y_h be a $(6h + 6)$-set. Since $(X_k \setminus \{a\}) \cup Y_h$ is of size $6k + 6h + 7$, by Theorem 2.2 for a 4-fold triple system, $\text{BIBD}((X_k \setminus \{a\} \cup Y_h), 3, 4)$ is not empty. Let $\mathcal{B}_1 \in \text{BIBD}((X_k \setminus \{a\}) \cup Y_h, 3, 4)$.

Since $(X_k \setminus \{a\})$ is of size $6k+1$, Theorem 2.2 for a 1-fold triple system confirms that there exists $\mathcal{B}_2 \in \mathrm{BIBD}(X_k \setminus \{a\}, 3, 1)$. Now consider $\{a\}$ with other elements, by Theorem 2.3, there exist $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, X_k \setminus \{a\}; 1, 6)$, $\mathcal{B}_4 \in \mathrm{GDD}(\{a\}, Y_h; 1, 1)$ and $\mathcal{B}_5 \in \mathrm{GDD}(\{a\}, Y_h; 1, 3)$. Besides, Theorem 2.2 for a $6t$-fold triple system gives us $\mathcal{B}_6 \in \mathrm{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_7 \in \mathrm{BIBD}(Y_h, 3, 6t)$. Therefore, setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_7$ yields that $(X_k, Y_h; \mathcal{B})$ forms our desired $\mathrm{GDD}(6k+2, 6h+6; 6t+6, 4)$.

(d) Let X_k be a $(6k+3)$-set and Y_h be a $(6h+5)$-set containing a. Since the sizes of K_k and $X_k \cup (Y_h \setminus \{a\})$, using Theorem 2.2 for 1-fold and 4-fold triple systems, we have $\mathcal{B}_1 \in \mathrm{BIBD}(X_k, 3, 1)$ and $\mathcal{B}_2 \in \mathrm{BIBD} X_k \cup (Y_h \setminus \{a\}), 3, 4)$. Next consider $\{a\}$ with other elements, by Theorem 2.3, there exist $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, X_k; 1, 4)$ and $\mathcal{B}_4 \in \mathrm{GDD}(\{a\}, Y_h \setminus \{a\}; 1, 3)$. Lastly, Since both X_k and Y_h have odd order, by Theorem 2.2 for a $6t$-fold triple system, there exist $\mathcal{B}_5 \in \mathrm{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_6 \in \mathrm{BIBD}(Y_h, 3, 6t)$. Therefore, setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$ yields that $(X_k, Y_h; \mathcal{B})$ forms our desired $\mathrm{GDD}(6k+3, 6h+5; 6t+6, 4)$.

(e) Let X_k be a $(6k+5)$-set containing a and Y_h be a $(6h+6)$-set. Since $(X_k \setminus \{a\}) \cup Y_h$ is of size $6k+6h+10$, by Theorem 2.2 for a 4-fold triple system, we have $\mathcal{B}_1 \in \mathrm{BIBD}((X_k \setminus \{a\}) \cup Y_h, 3, 4)$. Next consider $\{a\}$ with other elements, by Theorem 2.3, there exist $\mathcal{B}_2 \in \mathrm{GDD}(\{a\}, X_k \setminus \{a\}; 1, 3)$ and $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, Y_h; 1, 1)$ and $\mathcal{B}_4 \in \mathrm{GDD}(\{a\}, Y_h; 1, 3)$. Lastly, Since both X_k and Y_h have odd order, by Theorem 2.2 for a $6t$-fold triple system, there exist $\mathcal{B}_5 \in \mathrm{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_6 \in \mathrm{BIBD}(Y_h, 3, 6t)$. Therefore, setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$ yields that $(X_k, Y_h; \mathcal{B})$ forms our desired $\mathrm{GDD}(6k+5, 6h+6; 6t+6, 4)$.

(f) Let X_k be a $(6k+3)$-set and Y_h be a $(6h+5)$-set containing a. First, apply Theorem 2.2 for 4-fold triple systems $X_k \cup (Y_h \setminus \{a\})$, X_k and $Y_h \setminus \{a\}$, we have $\mathcal{B}_1 \in \mathrm{BIBD}(X_k \cup (Y_h \setminus \{a\}), 3, 4)$, $\mathcal{B}_2 \in \mathrm{BIBD}(X_k, 3, 4)$ and $\mathcal{B}_3 \in \mathrm{BIBD}(Y_h \setminus \{a\}, 3, 4)$, respectively. Next consider $\{a\}$ with other elements, by Theorem 2.3, there exist $\mathcal{B}_4 \in \mathrm{GDD}(\{a\}, X_k; 1, 4)$ and $\mathcal{B}_5 \in \mathrm{GDD}(\{a\}, Y_h \setminus \{a\}; 1, 9)$. Lastly, since both X_k and Y_h have odd order, by Theorem 2.2 for a $6t$-fold triple system, there exist $\mathcal{B}_6 \in \mathrm{BIBD}(X_k, 3, 6t)$ and $\mathcal{B}_7 \in \mathrm{BIBD}(Y_h, 3, 6t)$. Therefore, setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_7$ yields that $(X_k, Y_h; \mathcal{B})$ forms our desired $\mathrm{GDD}(6k+3, 6h+5; 6t+9, 4)$. □

Lemmas 3.2and 3.3 illustrate that the necessary conditions in Theorem A are sufficient. Therefore, our main theorem is completely proved.

Acknowledgments

The first author is supported by Center of Excellence in Mathematics, CHE, Sri Ayutthaya Rd., Bangkok 10400.

The second author is supported by University of the Thai Chamber of Commerce.

References

[1] R.C. Bose, T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Amer. Statist. Assoc.*, **47** (1952), 151-184.

[2] A. Chaiyasena, S.P. Hurd, N. Punnim, D.G. Sarvate, Group divisible designs with two associattion classes, *Journal of Combinatorial Mathematics and Combinatorial Computing*, To Appear.

[3] A. Chaiyasena, N. Pabhapote, Group divisible designs with two associattion classes and $\lambda_2 = 3$, *International Journal of Pure and Applied Mathematics*, **71**, No. 3 (2011), 455-463.

[4] S.I. El-Zanati, N. Punnim, C.A. Rodger, Gregarious GDDs with two associate classes, *Graphs and Combinatorics*, **26** (2010), 775-780.

[5] H.L. Fu, C.A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *Journal of Combinatorial Theory Series A*, **83** (1998), 94-117.

[6] H.L. Fu, C.A. Rodger, D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combinatoria*, **54** (2000), 33-50.

[7] W. Lapchinda, N. Pabhapote, Group divisible designs with two associate classes and $\lambda_1 - \lambda_2 = 1$, *International Journal of Pure and Applied Mathematics*, **54**, No. 4 (2009), 601-608.

[8] C.C. Lindner, C.A. Rodger, *Design Theory*, CRC Press, Boca Raton (1997).

[9] N. Pabhapote, N. Punnim, Group divisible designs with two associate classes and $\lambda_2 = 1$, *International Journal of Mathematics and Mathematical Sciences*, **2011**, Article ID 148580 (2011), 10 pages.

[10] C. Uiyyasathian, W. Lapchinda, Group divisible designs with two associate classes and $\lambda_2 = 2$, *International Journal of Pure and Applied Mathematics*, **55**, No. 4 (2009), 561-568.