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GENERALIZED DICHOTOMY FOR ORDINARY DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract: A dichotomy similar property for a class of homogeneous differential equations in an arbitrary Banach space is introduced. By help of them, existence of quasi bounded solutions of the appropriate nonhomogeneous equation is proved. A special roughness for such equations is considered.

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1. Introduction

The notion of exponential and ordinary dichotomy is fundamental in the qualitative theory of ordinary differential equations. It is considered in detail for example in the monographs [3]-[7].

In the given paper we introduce a (M, N, R) dichotomy, which is a generalisation of all dichotomies known by the authors. The aim of these paper is the research of this dichotomy and the connection between it and the existence of a special kind of solutions of the related nonhomogeneous differential equation.

A special kind of roughness of the (M, N, R) dichotomy is introduced and considered. Example for an equation, who is (M, N, R) dichotomous but not classical dichotomous is given.

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2. Problem Statement

Let X be an arbitrary Banach space with norm $|.|$ and identity I and let be $J = [c, \infty)$ where $c \in \mathbb{R}$. Let $L(X)$ be the space of all linear bounded operators acting in X with the norm $||.||$.

We consider the linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \tag{1}$$

where $A(t) \in L(X), t \in J$.

By $V(t)$ we will denote the Cauchy operator of (1).

We consider the nonhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t). \tag{2}$$

We say that the operators $D_1(t), D_2(t) : X \to X (t \in J)$ and the function $f(t) : J \to X (t \in J)$ satisfy the condition (H1) if the follow relation

$$\text{(H1).}\ f(t) \in FixV(t)(D_1(t) + D_2(t))(t \in J)$$

is fulfilled. By $FixT$ we denote the set of all fixed points of the map T, $T : X \to X$.

Remark 1. The condition (H1) is automatically fulfilled if for example $D_1(t) = R_1(t)V^{-1}(t)$, $D_2(t) = R_2(t)V^{-1}(t)$, where $R_1(t) + R_2(t) = I$.

Proof. We have

$$V(t)(D_1(t) + D_2(t))f(t) = f(t).$$

□

Lemma 1. *Let the condition (H1) holds. Then the function*

$$x(t) = \int_c^t V(t)D_1(s)f(s)\mathrm{d}s - \int_t^\infty V(t)D_2(s)f(s)\mathrm{d}s \tag{3}$$

is a solution of the equation (2) if the integrals in (3) exist.

Proof.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)\int_c^t V(t)D_1(s)f(s)\mathrm{d}s + V(t)D_1(t)f(t) + V(t)D_2(t)f(t)$$

$$- A(t) \int_t^{\infty} V(t) D_2(s) f(s) \mathrm{d}s,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + V(t)D_1(t)f(t) + V(t)D_2(t)f(t),$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + (V(t)D_1(t) + V(t)D_2(t))f(t),$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + V(t)(D_1(t) + D_2(t))f(t),$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + f(t).$$

□

We introduce the following conditions

$$\text{(H2)} \quad | V(t)D_1(s)z | \leq M(t,s,z), t \geq s, z \in X,$$

$$\text{(H3)} \quad | V(t)D_2(s)z | \leq N(t,s,z), t < s, z \in X.$$

For many important cases the right hand part of (H2) and (H3) has the form

$$\begin{cases} M(t,s,z) = \varphi_1(t)\varphi_2(s) \mid z \mid, \ (t \geq s), \ z \in X \\ N(t,s,z) = \psi_1(t)\psi_2(s) \mid z \mid, \ (t < s), \ z \in X \end{cases} \tag{4}$$

where $\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t)$ are positive scalar functions. We set

$$\alpha(t) = \max_{t \in J} \{\varphi_1(t), \psi_1(t)\},$$

$$\mu(t) = \min_{t \in J} \{\varphi_1(t), \psi_1(t)\},$$

$$\beta(t) = \max_{t \in J} \{\varphi_2(t), \psi_2(t)\}.$$

Definition 1. We call the equation (1) be (D_1, D_2, M, N) dichotomous if the conditions (H2), (H3) are fulfilled.

Such kind of dichotomy can be used for consideration of some equations in [1].

Definition 2. We call the equation (1) be (M, N, R) dichotomous if it is (D_1, D_2, M, N) dichotomous with

$$D_1(t) = R(t)V^{-1}(t),$$

$$D_2(t) = (I - R(t))V^{-1}(t),$$

where $R(t) : X \to X$ $(t \in J)$ is an arbitrary bounded operator.

Let $a(t)$ is an arbitrary positive scalar function. We consider the following Banach spaces :

$$K_a = \{g : J \to X : \sup_{t\in J} a(t) \int_c^t M(t,s,g(s))\mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{K_a} = \sup_{t\in J}\ a(t) \int_c^t M(t,s,g(s))\mathrm{d}s,$$

$$L_a = \{g : J \to X : \sup_{t\in J} a(t) \int_t^\infty N(t,s,g(s))\mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{L_a} = \sup_{t\in J}\ a(t) \int_t^\infty N(t,s,g(s))\mathrm{d}s,$$

$$C_a = \{g : J \to X : \sup_{t\in J} a(t) \mid g(t) \mid < \infty\}$$

with the norm

$$\mid g \mid_{C_a} = \sup_{t\in J}\ a(t) \mid g(t) \mid$$

and

$$T_a = \{g : J \to X : \int_c^\infty a(s) \mid g(s) \mid \mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{T_a} = \int_c^\infty a(s) \mid g(s) \mid \mathrm{d}s.$$

3. Main Results

3.1. Estimates of the Solutions of (1) and (2)

Theorem 1. *Let the equation (1) is (D_1, D_2, M, N) - dichotomous and let $D_1(t), D_2(t)$ and $f(t)$ fulfilled the condition (H1). Then for every function $f \in K_a \bigcap L_a$ the equation (2) has a solution in the space C_a.*

Proof. From Lemma 1 it follows

$$x(t) = \int_c^t V(t)D_1(s)f(s)\mathrm{d}s - \int_t^\infty V(t)D_2(s)f(s)\mathrm{d}s$$

$$|x(t)| \le \int_c^t |V(t)D_1(s)f(s)|\mathrm{d}s + \int_t^\infty |V(t)D_2(s)f(s)|\mathrm{d}s$$

$$a(t)|x(t)| \le a(t)\int_c^t M(t,s,f(s))\mathrm{d}s + a(t)\int_t^\infty N(t,s,f(s))\mathrm{d}s < \infty$$

□

Corollary 1. *Let the following conditions are fulfilled:*

1. The equation (1) be (D_1, D_2, M, N) *- dichotomous of the form (4).*

2. The operators $D_1(t), D_2(t)$ *and the function* $f(t)$ *satisfy the condition (H1).*

Then for every function $f \in K_{\mu^{-1}} \bigcap L_{\mu^{-1}}$ *the equation (2) has a solution in the space* $C_{\alpha^{-1}}$ *and the following estimate holds*

$$\sup_{t\in J} \alpha^{-1}(t) \mid x(t) \mid \le \int_c^t \beta(s) \mid f(s) \mid \mathrm{d}s + \int_t^\infty \beta(s) \mid f(s) \mid \mathrm{d}s < \infty$$

Proof. From Theorem 1 and the presentation (4) it follows

$$\mu^{-1}(t)|x(t)| \le \mu^{-1}(t)\varphi_1(t)\int_c^t \varphi_2(s)|f(s)|\mathrm{d}s + \mu^{-1}(t)\psi_1(t)\int_t^\infty \psi_2(s)|f(s)|\mathrm{d}s < \infty$$

By multiplying by $\mu(t)\alpha(t)^{-1} \le 1$ the left side and the one addent of the right side, whose multiplier in front of the integral is not equal to 1, we receive

$$\alpha^{-1}(t) \mid x(t) \mid \le \int_c^t \varphi_2(s) \mid f(s) \mid \mathrm{d}s + \int_t^\infty \psi_2(s) \mid f(s) \mid \mathrm{d}s \le$$

$$\le \int_c^t \beta(s) \mid f(s) \mid \mathrm{d}s + \int_t^\infty \beta(s) \mid f(s) \mid \mathrm{d}s < \infty$$

□

Theorem 2. *Let the equation (1) is* (M, N, R) *- dichotomous.*

Then following estimates hold

$$\mid x_1(t) \mid \le M(t, s, x_1(s)),\ t \ge s \ge c \tag{5}$$

for all solutions $x_1(t)$ *of (1) ,*$(t \ge c)$*, which started in the set*

$$\bigcap_{s\in J} Fix\ R(s)$$

and

$$\mid x_2(t) \mid \le N(t, s, x_2(s)),\ c \le t < s \tag{6}$$

for all solutions $x_2(t)$ of (1) ,$(t \geq c)$, which started in the set

$$\bigcap_{s \in J} Fix(I - R(s))$$

Proof. The equalities

$$x_1(t) = V(t)x_1(c) = V(t)R(s)x_1(c) = V(t)R(s)V^{-1}(s)x_1(s)$$

imply following estimates

$$\mid x_1(t) \mid = \mid V(t)R(s)V^{-1}(s)x_1(s) \mid \leq M(t, s, x_1(s)),\ (t \geq s)$$

The proof of (6) is analogously.

□

3.2. Roughness

We shall introduce and consider a special kind of roughness typical for (M, N, R) dichotomous equations.

Let Z be an arbitrary Banach space. In the set of all subspaces of Z we can define following metric (see [5]):

$$\rho(Z_1, Z_2) = \rho_H(\Sigma(Z_1), \Sigma(Z_2)) \tag{7}$$

where $\Sigma(Y)$ $(Y \subset Z)$ denotes the unit ball of the subspace Y and ρ_H is the Hausdorff metric.

We consider the following equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = B(t)x \tag{8}$$

Theorem 3. *Let the following conditions are fulfilled:*
1. The equation (1) is (M, N, R) - dichotomous of the form (4).
2. $R(t)$ is continuos operator for every $t \in J$ and

$$(R(t') - R(t''))x \underset{t'' \to t'}{\longrightarrow} 0$$

uniformly for every bounded subset of X and $t', t'' \in J$.

Then for every $\epsilon > 0$ there exists a $\bar{\delta} > 0$ such that for $\delta \in (0, \bar{\delta})$ all equations (8), for which

$$\sup_{t \in J} |(A(t) - B(t))x| \leq \delta |x| \quad (x \in X)$$

are $(\tilde{M}, \tilde{N}, \tilde{R})$ dichotomous and $\rho_1(C_{\alpha^{-1}}, C_{\tilde{\alpha}^{-1}}),\ \rho_2(T_\beta, T_{\tilde{\beta}}) \leq \epsilon$, where the functions $\tilde{M}, \tilde{N}$, the operator $\tilde{R}$ and the Banach spaces $C_{\alpha^{-1}}, C_{\tilde{\alpha}^{-1}}, T_\beta, T_{\tilde{\beta}}$ will be defined below and ρ_1, ρ_2 are metrics from (7).

Proof. Let $\tilde{V}(t)$ is the Cauchy operator of (8). Let $R_1 = R$ and $R_2 = I - R$. For sufficiently small $\bar{\delta} > 0$ the numbers

$$|V(t)R_i(s)V^{-1}(s)z| \ \text{ and } \ |\tilde{V}(t)\tilde{R}_i(s)\tilde{V}^{-1}(s)z| \ \ (i = 1, 2)$$

will have an arbitrary small difference for

$$| R_i(s)V^{-1}(s)z - \tilde{R}_i(s)\tilde{V}^{-1}(s)z | \leq \delta \mid z \mid, \ (z \in X), \ (i = 1, 2), \ \delta \in (0, \bar{\delta}), \ (s \geq c)$$

by suitable choice of $\tilde{R}_i(s)$ (for example if $\tilde{R}_i(s) = R_i(s)V^{-1}(s)\tilde{V}(s)$).

Because (1) is (M, N, R) - dichotomous of the form (4), we can find functions $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_1, \tilde{\psi}_2$ arbitrary close to $\varphi_1, \varphi_2, \psi_1, \psi_2$ respectively, for which

$$| \tilde{V}(t)\tilde{R}_1(s)\tilde{V}^{-1}(s)z | \leq \tilde{\varphi}_1(t)\tilde{\varphi}_2(s) \mid z \mid \ \ (t \geq s), \ z \in X$$

$$| \tilde{V}(t)\tilde{R}_2(s)\tilde{V}^{-1}(s)z | \leq \tilde{\psi}_1(t)\tilde{\psi}_2(s) \mid z \mid \ \ (s \geq t), \ z \in X$$

i.e. $\tilde{M}(t, s, z) = \tilde{\varphi}_1(t)\tilde{\varphi}_2(s)|z|$ and $\tilde{N}(t, s, z) = \tilde{\psi}_1(t)\tilde{\psi}_2(s)|z|$.

Let

$$\tilde{\alpha}(t) = \max_{t \in J}\{\tilde{\varphi}_1(t), \tilde{\psi}_1(t)\}$$

and

$$\tilde{\beta}(t) = \max_{t \in J}\{\tilde{\varphi}_2(t), \tilde{\psi}_2(t)\}.$$

We choose a function

$$a(t) \geq \max_{t \in J}\{\alpha(t), \tilde{\alpha}(t)\}$$

and consider the Banach space $C_{a^{-1}}$. Obviously $C_{\alpha^{-1}}$ and $C_{\tilde{\alpha}^{-1}}$ are subsets of $C_{a^{-1}}$. By straightforward verification we will prove that $C_{\alpha^{-1}}$ and $C_{\tilde{\alpha}^{-1}}$ are closed in $C_{a^{-1}}$, i.e. they are subspaces of $C_{a^{-1}}$.

Indeed, first we shall prove, that $C_{\alpha^{-1}}$ is closed in $C_{a^{-1}}$, when $a(t) > \alpha(t)$. Let

$$\lim_{n \to \infty} |g_n - g|_{C_{a^{-1}}} = 0$$

and let $g_n \in C_{\alpha^{-1}}$. We will prove, that $g \in C_{\alpha^{-1}}$. We have

$$\lim_{n \to \infty} (\sup_{t \in J} \frac{|g_n(t) - g(t)|}{a(t)}) = 0 \text{ , i.e. } \lim_{n \to \infty} (\sup_{t \in J} |g_n(t) - g(t)|) = 0$$

and

$$\sup_{t \in J} \frac{|g_n(t)|}{\alpha(t)} < \infty \ (n \in N).$$

Then from
$$\frac{|g(t)|}{\alpha(t)} \le \frac{|g_n(t)|}{\alpha(t)} + \frac{|g(t) - g_n(t)|}{\alpha(t)}$$
we obtain
$$\lim_{t\in J} \frac{|g(t)|}{\alpha(t)} \le \lim_{t\in J} \frac{|g_n(t)|}{\alpha(t)} + \lim_{t\in J} \frac{|g(t) - g_n(t)|}{\alpha(t)} < \infty$$
Hence $g \in C_{\alpha^{-1}}$, i.e. $C_{\alpha^{-1}}$ is closed in $C_{a^{-1}}$.

The proof for $C_{\tilde{\alpha}^{-1}}$ is analogously.

Let ρ_1 is the metric in the set of all subspaces of $C_{a^{-1}}$. To show that $\rho_1(C_{\alpha^{-1}}, C_{\tilde{\alpha}^{-1}}) = \rho_H(\Sigma(C_{\alpha^{-1}}), \Sigma(C_{\tilde{\alpha}^{-1}}) \le \epsilon$ it is necessary and sufficient that for every element $g \in \Sigma(C_{\alpha^{-1}})$ there exists an element $h \in \Sigma(C_{\tilde{\alpha}^{-1}})$, such that $\mid g - h \mid_{C_{a^{-1}}} \le \epsilon$ and vice versa.

We choose $\bar{\delta}$ so small, that
$$\sup_{t\in J} \mid \alpha(t) - \tilde{\alpha}(t) \mid = \eta \le \Delta\epsilon$$
where
$$\Delta = min\{\inf_{t\in J} \alpha(t), \inf_{t\in J} \tilde{\alpha}(t)\}$$
Without loss of generality we may assume that $\Delta > 0,\ \epsilon < 1$. Then we have
$$\alpha(t) - \eta \le \tilde{\alpha}(t) \le \alpha(t) + \eta \tag{9}$$

Let $g \in \Sigma(C_{\alpha^{-1}})$ is an arbitrary element, i.e.
$$\sup_{t\in J} \frac{\mid g(t) \mid}{a(t)} \le 1, \quad \sup_{t\in J} \frac{\mid g(t) \mid}{\alpha(t)} < \infty$$

We take
$$h(t) = \frac{\alpha(t) - \eta}{\alpha(t)} g(t)$$

Then $h \in C_{\tilde{\alpha}^{-1}}$ because
$$\sup_{t\in J} \frac{|h(t)|}{\tilde{\alpha}(t)} = \sup_{t\in J} \frac{(\alpha(t) - \eta)|g(t)|}{\tilde{\alpha}(t)\alpha(t)} \le \sup_{t\in J} \frac{(\tilde{\alpha}(t) + \eta - \eta)|g(t)|}{\tilde{\alpha}(t)\alpha(t)} = \sup_{t\in J} \frac{|g(t)|}{\alpha(t)} < \infty$$

We have $h \in \Sigma(C_{\tilde{\alpha}^{-1}})$ because
$$\mid h \mid_{C_{a^{-1}}} \le \mid g \mid_{C_{a^{-1}}} \sup_{t\in J} \frac{|\alpha(t) - \eta|}{\alpha(t)} \le \mid g \mid_{C_{a^{-1}}} \le 1$$

And $| \, g - h \, |_{C_{a^{-1}}} \le \epsilon$ because from

$$\frac{|g(t) - h(t)|}{a(t)} = \frac{1}{a(t)} |1 - \frac{\alpha(t) - \eta}{\alpha(t)}||g(t)|$$

we obtain

$$|g - h|_{C_{a^{-1}}} = \sup_{t \in J} \frac{|g(t) - h(t)|}{a(t)} = \sup_{t \in J} \frac{|g(t)|}{a(t)} \frac{\eta}{\alpha(t)} \le |g|_{C_{a^{-1}}} \eta \sup_{t \in J} \frac{1}{\alpha(t)} \le \frac{\Delta \epsilon}{\Delta} = \epsilon$$

Now vice versa - let $h \in \Sigma(C_{\tilde{\alpha}^{-1}})$ is an arbitrary element, i.e.

$$\sup_{t \in J} \frac{| \, h(t) \, |}{a(t)} \le 1, \quad \sup_{t \in J} \frac{| \, h(t) \, |}{\tilde{\alpha}(t)} < \infty$$

We take

$$g(t) = \frac{\tilde{\alpha}(t) - \eta}{\tilde{\alpha}(t)} h(t)$$

Then $g \in C_{\alpha^{-1}}$ because

$$\sup_{t \in J} \frac{|g(t)|}{\alpha(t)} = \sup_{t \in J} \frac{(\tilde{\alpha}(t) - \eta)|h(t)|}{\tilde{\alpha}(t)\alpha(t)} \le \sup_{t \in J} \frac{|h(t)|}{\tilde{\alpha}(t)} < \infty$$

We have $g \in \Sigma(C_{\alpha^{-1}})$ because

$$| \, g \, |_{C_{a^{-1}}} = \frac{| \, g(t) \, |}{a(t)} \le | \, h \, |_{C_{a^{-1}}} \sup_{t \in J} \frac{|\tilde{\alpha}(t) - \eta|}{\tilde{\alpha}(t)} \le | \, h \, |_{C_{a^{-1}}} \le 1$$

And $| \, g - h \, |_{C_{a^{-1}}} \le \epsilon$ because from

$$\frac{|g(t) - h(t)|}{a(t)} = \frac{1}{a(t)} | \frac{\tilde{\alpha}(t) - \eta}{\tilde{\alpha}(t)} - 1| \; |h(t)| = \frac{|h(t)|}{a(t)} \frac{\eta}{\tilde{\alpha}(t)}$$

we obtain

$$|g - h|_{C_{a^{-1}}} = \sup_{t \in J} \frac{|g(t) - h(t)|}{a(t)} \le |h|_{C_{a^{-1}}} \eta \sup_{t \in J} \frac{1}{\tilde{\alpha}(t)} \le \frac{\Delta \epsilon}{\Delta} = \epsilon$$

Hence $\rho_1(C_{\alpha^{-1}}, C_{\tilde{\alpha}^{-1}}) \le \epsilon$.

Now we choose a function

$$b(t) < \min_{t \in J} \{\beta(t), \tilde{\beta}(t)\}$$

and consider the Banach space T_b.

It is not diffucult to show that T_β and $T_{\tilde{\beta}}$ are closed in T_b and hence they are subspaces of T_b.

Let ρ_2 is the metric in the set of all subspaces of T_b. We choose $\bar{\delta}$ so small, that

$$\sup_{t\in J} |\beta(t) - \tilde{\beta}(t)| \leq \epsilon$$

Hence

$$\beta(t) - \epsilon \leq \tilde{\beta}(t) \leq \beta(t) + \epsilon \tag{10}$$

Let $g \in \Sigma(T_\beta)$ be an arbitrary element, i.e.

$$\int_J \beta(t) \mid g(t) \mid \mathrm{d}t < \infty, \int_J b(t) \mid g(t) \mid \mathrm{d}t \leq 1$$

We are searching for an element $h \in \Sigma(T_{\tilde{\beta}})$, such that $|g - h|_{T_b} \leq \epsilon$.

We set

$$h(t) = \frac{\beta(t)}{\beta(t) + \epsilon} g(t)$$

Then $h \in T_{\tilde{\beta}}$ because from (10) we have

$$\int_J \tilde{\beta}(t)|h(t)|\mathrm{d}t = \int_J \tilde{\beta}(t)\frac{\beta(t)}{\beta(t)+\epsilon}|g(t)|\mathrm{d}t \leq \int_J \frac{\beta(t)+\epsilon}{\beta(t)+\epsilon}\beta(t)|g(t)|\mathrm{d}t < \infty$$

Also $h \in \Sigma(T_{\tilde{\beta}})$ because

$$\int_J b(t)\frac{\beta(t)}{\beta(t)+\epsilon}|g(t)|\mathrm{d}t \leq \int_J b(t)|g(t)|\mathrm{d}t \leq 1$$

Without loss of generality we may assume that $\beta(t) \geq 1$ $(t \in J)$. Then $|g - h|_{T_b} \leq \epsilon$ because

$$\int_J b(t)|g(t) - \frac{\beta(t)}{\beta(t)+\epsilon}g(t)|\mathrm{d}t = \int_J b(t)|g(t)||\frac{\epsilon}{\beta(t)+\epsilon}|\mathrm{d}t \leq \epsilon \int_J b(t)|g(t)|\mathrm{d}t \leq \epsilon$$

Now vice versa - let $h \in \Sigma(T_{\tilde{\beta}})$ is an arbitrary element,i.e.

$$\int_J \tilde{\beta}(t) \mid h(t) \mid \mathrm{d}t < \infty, \int_J b(t) \mid h(t) \mid \mathrm{d}t \leq 1$$

We are searching for an element $g \in \Sigma(T_\beta)$, such that $\mid h - g \mid_{T_b} \leq \epsilon$.

We set

$$g(t) = \frac{\tilde{\beta}(t)}{\tilde{\beta}(t) + \epsilon} h(t)$$

From (10) follows
$$\tilde{\beta}(t) - \epsilon \leq \beta(t) \leq \tilde{\beta}(t) + \epsilon$$

Hence $g \in T_\beta$ because

$$\int_J \beta(t)|g(t)|\mathrm{d}t = \int_J \beta(t)\frac{\tilde{\beta}(t)}{\tilde{\beta}(t)+\epsilon}|h(t)|\mathrm{d}t \leq \int_J \tilde{\beta}(t)|h(t)|\mathrm{d}t < \infty$$

And $g \in \Sigma(T_\beta)$ because

$$\int_J b(t)\frac{\tilde{\beta}(t)}{\tilde{\beta}(t)+\epsilon}|h(t)|\mathrm{d}t \leq \int_J b(t)|g(t)|\mathrm{d}t \leq 1$$

Without loss of generality we may assume that $\beta(t) \geq 1$ $(t \in J)$. Then $\tilde{\beta}(t) + \epsilon \geq \beta(t) > 1$.

Then $|g-h|_{T_b} \leq \epsilon$ because

$$\int_J b(t)|h(t) - \frac{\tilde{\beta}(t)}{\tilde{\beta}(t)+\epsilon}h(t)|\mathrm{d}t = \int_J b(t)|h(t)||\frac{\epsilon}{\tilde{\beta}(t)+\epsilon}|\mathrm{d}t \leq \epsilon \int_J b(t)|h(t)|\mathrm{d}t \leq \epsilon$$

Hence $\rho_2(T_\beta, T_{\tilde{\beta}}) \leq \epsilon$. □

3.3. Considerations and an Example

Let $R(t) = P$, where $P : X \to X$ is a projector.

For
$$M(t,s,z) = K_1 e^{-\int_s^t \delta_1(\tau)\mathrm{d}\tau}|z| \ (t \geq s, z \in X)$$
$$N(t,s,z) = K_2 e^{-\int_t^s \delta_1(\tau)\mathrm{d}\tau}|z| \ (s > t, z \in X)$$

where K_1, K_2 are positive constants and δ_1, δ_2 are contnuous real-valued functions on J, we obtain the exponential dichotomy of [6] :

$$\| V(t)PV^{-1}(s) \| \leq K_1 e^{-\int_s^t \delta_1(\tau)\mathrm{d}\tau} \ (t \geq s)$$

$$\| V(t)(I-P)V^{-1}(s) \| \leq K_2 e^{-\int_t^s \delta_2(\tau)\mathrm{d}\tau} \ (s > t).$$

For $\delta_i(t) = 0$ $(c \leq t < \infty,\ i = 1,2)$ we obtain the exponential dichotomy of [3],[4],[5], for which case we have $K_a \bigcap L_a = C_a$ by $a(t) \equiv 1$.

For
$$M(t,s,z) = Kh(t)h^{-1}(s)|z| \ (t \geq s \geq c, , z \in X)$$
$$N(t,s,z) = Kk(t)k^{-1}(s)|z| \ (c \leq t \leq s, z \in X)$$

where K is a positive constant and $h, k : [0, \infty) \to (0, \infty)$ are two continuous functions, we obtain the dichotomy of [7]-[9]:

$$\| V(t)PV^{-1}(s) \| \leq Kh(t)h^{-1}(s) \ , (t \geq s \geq c)$$

$$\| V(t)(I-P)V^{-1}(s) \| \leq Kk(t)k^{-1}(s) \ , (c \leq t \leq s)$$

It may be also noted, that the dichotomies [2],[6]-[9] are a generalization of the dichotomy in [4].

Example. Let

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x$$

where $X = \mathbb{R}$ and $a(t)$ is a piecewise continuous function. In this case we have

$$V(t) = exp \int_c^t a(s)\mathrm{d}s.$$

It is well known, that this equation in several cases is not dichotomous or exponential dichotomous.

For the conditions (H2) and (H3) we take respectively the forms

$$| V(t)r_1(s)V^{-1}(s)\xi | \leq \alpha(t)\beta(s)|\xi| \ (t \geq s, \ \xi \in \mathbb{R}),$$

$$| V(t)r_2(s)V^{-1}(s)\xi | \leq \alpha(t)\beta(s)|\xi| \ (t < s, \ \xi \in \mathbb{R}).$$

We set $r_1(t) + r_2(t) = 1, (t \geq c)$. Now we have

$$|r_1(s)V^{-1}(s)\xi| \leq \beta(s)|\xi|, \quad |r_2(s)V^{-1}(s)\xi| \leq \beta(s)|\xi|,$$

i.e. $\beta(t) \geq \max\{|r_1(t)V^{-1}(t)|, |r_2(t)V^{-1}(t)|\}$.

We take $\alpha(t) \geq |V(t)|, (t \geq c)$. Because condition (H1) holds,

$$x(t) = \int_c^t V(t)r_1(s)V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty V(t)r_2(s)V^{-1}(s)f(s)\mathrm{d}s$$

is a solution of the nonhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x + f(t) \ .$$

For every positive function $b(t)$, the function x(t) belongs to the space C_b for every $f \in L_b \bigcap K_b$.

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