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POINT AVAILABILITY OF A WARM STAND-BY SYSTEM

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Abstract: We analyse the point availability of a duplex system characterized by warm stand-by and attended by 2 heterogeneous repairmen. In order to describe the random behaviour of the engineering system, we employ a stochastic process endowed with time dependent transition measures satisfying (coupled) partial differential equations. The solution procedure is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. As an application, we consider the particular but important case of Erlang repair.

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1. Introduction

A particular variant of Gaver's parallel system, e.g. [7], is Birolini's duplex system [2], henceforth called the **B**-system. The **B**-system consists of an active unit, the on-line unit (called the **o**-unit), sustained by an identical unit in warm stand-by (called the **s**-unit). The **B**-system is attended by a single repairman. The notion of warm stand-by signifies that the failure-free time of the **s**-unit is *stochastically* larger [5] than the failure free-time of the **o**-unit.

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Note that the warm stand-by mode is often indispensable to perform an instantaneous switch from stand-by into the operative state, allowing a continuous operation of an engineering system upon failure of the on-line unit. Furthermore, any switch from stand-by into the operative state changes the failure rate of the **s**-unit into the failure rate of the **o**-unit. The **B**-system is down if, and only if, both units are down. Otherwise, the **B**-system is up. The **B**-system acts as a closed queuing system evolving in time, i.e. any failed unit goes immediately into repair provided that the repairman is idle. Otherwise, the failed **o**-unit has to wait for repair. On the other hand, any repaired unit lines-up in warm stand-by if the remaining unit is still active. Otherwise, the repaired unit becomes immediately operative.

Industrial applications of warm stand-by systems have been presented by Shao et al. [6]. The analysis of duplex systems is far from complete! As a variant, we consider a **B**-system attended by 2 heterogeneous repairmen, henceforth called the **T**-system. Repairman R is skilled in repairing **o**-failures, whereas repairman R_s is an expert in repairing **s**-failures. Both repairmen are jointly busy if, and only if, the **o**-unit and the **s**-unit are down. Otherwise, at least one repairman is idle. Finally, note that any **o**-failure is always directed to repairman R, whereas a **s**-failure is always allocated to repairman R_s. Consequently, repairman R_s can be idle if the **T**-system is down.

The *long-run* availability of the **T**-system has been analysed by Vanderperre [9]. In order to derive the *point* (time-dependent) availability, we introduce a stochastic process endowed with time-dependent transition measures satisfying (coupled) partial differential equations. Our equations are generalizing the steady-state equations obtained in [9]. The solution procedure is based on a combination of the theory of sectionally holomorphic functions, e.g. [3] and the notion of dual transforms [8]. Finally, as an application, we consider the particular but important case of Erlang repair.

2. Formulation

Consider the **T**-system subjected to the following conditions and assumptions.

- The **o**-unit has a constant failure rate λ and a general repair time distribution $R(\cdot),\ R(0) = 0$ with finite mean. The failure-free time and the repair time are respectively denoted by f and r.

- The **s**-unit has a constant failure rate $0 < \lambda_s < \lambda$ and a general repair time distribution $R_s(\cdot),\ R_s(0) = 0$ with finite mean. The failure-free time and the repair time are respectively denoted by f_s and r_s. Note tat f_s is stochastically larger than f (as required). The random variables f, r, f_s, r_s are supposed to be *statistically* independent and any repair is perfect.

- The failure rate λ_s of the **s**-unit changes into the failure rate λ upon switch from stand-by into the operative state.

Characteristic functions (and their duals) are formulated in terms of a *complex* transform variable. For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x} dR(x), \quad \text{Im}\,\omega \geq 0.$$

Note that

$$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x} d\left\{1 - R((-x)-)\right\}, \quad \text{Im}\,\omega \leq 0.$$

The corresponding Fourier-Stieltjes-transforms are called *dual* transforms. Without loss of generality (see Remarks 5.1), we may assume that R and R_s have density functions (in the Radon-Nikodym sense) of bounded variation on $[0, \infty)$.

In order to describe the random behaviour of the **T**-system, we employ a stochastic process $\{N_t,\ t \geq 0\}$ with discrete state space $\{A, B, C, D, D_s\} \subset [0, \infty)$ characterized by the following mutually exclusive events:

$\{N_t = A\}$: "The **T**-system is up and both repairmen are idle at time t."
$\{N_t = B\}$: "The **T**-system is up and repairman R_s is busy at time t."
$\{N_t = C\}$: "The **T**-system is up and repairman R is busy at time t."
$\{N_t = D\}$: "The **T**-system is down and repairman R_s is idle at time t."
$\{N_t = D_s\}$: "The **T**-system is down and both repairmen are busy at time t."

State A is called the safe state. Note that the **T**-system is only available (functioning) in states A, B, C and fully available in state A. States D and D_s are called the system down states.

A (vector) Markov characterization of the non-Markovian process $\{N_t, t \geq 0\}$ is piecewise and conditionally defined by:

$\{N_t\}$, if $N_t = A$ (i.e. if the event $\{N_t = A\}$ occurs.)

$\{(N_t,\, Y_t)\}$, if $N_t = B$, where Y_t denotes the *remaining* repair time of the failed **s**-unit under progressive repair at time t.

$\{(N_t,\, X_t)\}$, if $N_t = C$ or D, where X_t denotes the *remaining* repair time of the failed **o**-unit under progressive repair at time t.

$\{(N_t,\, X_t,\, Y_t)\}$, if $N_t = D_s$.

The state space of the underlying Markov process is given by

$$\{A\}\bigcup\{(B,\, y)\}\bigcup\{(C,\, x)\}\bigcup\{(D,\, x)\}\bigcup\{(D_s,\, x,\, y)\}\,,$$

where $x \geq 0,\ y \geq 0$.

We assume that the **T**-system starts operating at some time origin $t = 0$ in state A, i.e. let $N_0 = A$, **P**-a.s.

For $K = A, B, C, D, D_s$ let $p_K(t) := \mathbf{P}\{N_t = K\}\,,\ t \geq 0$.

The function $p_A(t)$ is called the *point* availability of the safe state. Finally, we introduce the transition measures

$$\begin{aligned} p_B(t,\, y)dy &:= \mathbf{P}\{N_t = B,\ Y_t \in dy\}\,, \\ p_C(t,\, x)dx &:= \mathbf{P}\{N_t = C,\ X_t \in dx\}\,, \\ p_D(t,\, x)dx &:= \mathbf{P}\{N_t = D,\ X_t \in dx\}\,, \\ p_{D_s}(t,\, x,\, y)dx\, dy &:= \mathbf{P}\{N_t = D_s,\ X_t \in dx,\ Y_t \in dy\}\,. \end{aligned}$$

Note that, for instance,

$$p_{D_s}(t) = \int_0^\infty \int_0^\infty d_x d_y\, \mathbf{P}\{N_t = D_s,\, X_t \leq x,\, Y_t \leq y\} = \int_0^\infty \int_0^\infty p_{D_s}(t, x, y)dx\, dy.$$

3. Notations

Let $\{\Omega, \mathbf{B}, \mathbf{P}\}$ be a (complete) probability space.

- The indicator (function) of an event $\{N_t = K\} \in \mathbf{B}$ is denoted by $\mathbf{1}\{N_t = K\}$.

- The complex plane and the real line are respectively denoted by $\mathbf{C}$ and $\mathbf{R}$ with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance, $\mathbf{C}^+ := \{\omega \in \mathbf{C} : \text{Im } \omega > 0\}$.

- The Laplace-transform of any locally integrable and bounded function on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,
$$p_A^*(z) := \int_0^\infty e^{-zt} p_A(t)dt, \ \text{Re } z > 0.$$

  - Let $\varphi(\tau),\ \tau \in \mathbf{R}$ be a bounded and continuous function. φ is called Γ-integrable if
  $$\lim_{\substack{T\to\infty \\ \varepsilon\downarrow 0}} \int_{\Gamma_{T,\varepsilon}} \varphi(\tau)\frac{d\tau}{\tau - u}, \ u \in \mathbf{R}$$
  exists, where $\Gamma_{T,\varepsilon} := (-T, u-\varepsilon] \bigcup [u+\varepsilon, T)$. The corresponding integral, denoted by
  $$\frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau - u},$$
  is called a Cauchy principal value in double sense.

  - A function $\varphi(\tau), \tau \in \mathbf{R}$ is called Hölder-continuous on $\mathbf{R}$ if $\forall\, \tau_1, \tau_2 \in \mathbf{R},\ \exists(\beta, A);\ 0 < \beta \leq 1,\ A > 0:$
  $$|\varphi(\tau_2) - \varphi(\tau_1)| \leq A|\tau_2 - \tau_1|^\beta.$$
  The function $\varphi(\tau),\ \tau \in \mathbf{R}$ is called Hölder-continuous at infinity if $\exists\ \gamma > 0:$
  $$|\varphi(\tau)| = O\left(\frac{1}{|\tau|^\gamma}\right),\ |\tau| \to \infty.$$
  Hölder-continuous functions with exponent $\beta = \gamma = 1$ are called Lipschitz-continuous.

  - Note that the Hölder-continuity of $\varphi(\cdot)$ on $\mathbf{R}$ and at infinity is sufficient for the existence of the Cauchy-type integral
  $$\frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau - \omega},\ \omega \in \mathbf{C}.$$
  See [4] for further details.

4. Differential Equations

In order to derive a system of differential equations, we observe the random behaviour of the **T**-system in some time interval $(t, t+\Delta)$, $\Delta \downarrow 0$. Applying a general birth and death technique and grouping terms of $o(\Delta)$, yields the time-dependent balance equations

$$p_A(t+\Delta) = p_A(t)(1-(\lambda+\lambda_s)\Delta) + p_B(t,\,0)\Delta + p_C(t,\,0)\Delta + o(\Delta),$$

$$p_B(t+\Delta,\,y-\Delta) = p_B(t,\,y)(1-\lambda\Delta) + \lambda_s\, p_A(t)\frac{dR_s}{dy}(y)\Delta + p_{D_s}(t,0,y)\Delta + o(\Delta),$$

$$p_C(t+\Delta,\,x-\Delta) = p_C(t,\,x)(1-\lambda\Delta) + (\lambda\; p_A(t) + p_D\,(t,0))\frac{dR}{dx}(x)\Delta + p_{D_s}(t,x,0)\Delta + o\,(\Delta),$$

$$p_D(t+\Delta,\; x-\Delta) = p_D(t,x) + \lambda\; p_C\;(t,x)\Delta + o(\Delta),$$

$$p_{D_s}(t+\Delta,\; x-\Delta,\; y-\Delta) = p_{D_s}(t,x,y) + \lambda\; p_B(t,y)\frac{dR}{dx}(x)\Delta + o(\Delta).$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_{D_s}(t,x,y) = \lim_{\Delta\downarrow 0} \frac{p_{D_s}(t+\Delta,\,x-\Delta,\,y-\Delta) - p_{D_s}(t,x,y)}{\Delta},$$

yields for $t>0,\; x>0,\; y>0,$

$$\left(\lambda+\lambda_s+\frac{d}{dt}\right) p_A(t) = p_B(t,0) + p_C(t,0),$$

$$\left(\lambda+\frac{\partial}{\partial t}-\frac{\partial}{\partial y}\right) p_B(t,y) = \lambda_s\; p_A(t)\frac{dR_s}{dy}(y) + p_{D_s}(t,0,y),$$

$$\left(\lambda+\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) p_C(t,x) = (\lambda\; p_A(t) + p_D(t,0))\,\frac{dR}{dx}(x) + p_{D_s}(t,x,0),$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) p_D(t,x) = \lambda\, p_C(t,x),$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right) p_{D_s}(t,x,y) = \lambda\; p_B(t,y)\frac{dR}{dx}(x).$$

Observe that our equations are consistent with the general probability law $\sum_K p_K(t) = 1$ and that $p_A(0) = 1$.

5. Solution Procedure

First, we remark that our system of differential equations is well-adapted to a Laplace-Fourier transformation. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace-transform exists for $\operatorname{Re} z > 0$. Moreover, the obvious integrability of the repair time density functions and the transition functions with regard to x, y also implies the integrability of the corresponding partial derivatives. Applying a Laplace-Fourier-transform technique to the equations and taking the initial condition into account, reveals that for $\operatorname{Re} z > 0,\ \operatorname{Im} \omega \geq 0,\ \operatorname{Im} \eta \geq 0$,

$$(z + \lambda + \lambda_s) p_A^*(z) = p_B^*(z, 0) + p_C^*(z, 0) + 1, \tag{5.1}$$

$$(z + \lambda + i\eta) \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\eta Y_t} \mathbf{1} \{N_t = B\} \right) dt + p_B^*(z, 0)$$
$$= \lambda_s p_A^*(z) \mathbf{E} e^{i\eta r_s} + \int_0^\infty e^{i\eta y} p_{D_s}^*(z, 0, y) dy, \tag{5.2}$$

$$(z + \lambda + i\omega) \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\omega X_t} \mathbf{1} \{N_t = C\} \right) dt + p_C^*(z, 0)$$
$$= \left(\lambda p_A^*(z) + p_D^*(z, 0) \right) \mathbf{E} e^{i\omega r} + \int_0^\infty e^{i\omega x} p_{D_s}^*(z, x, 0) dx, \tag{5.3}$$

$$(z + i\omega) \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\omega X_t} \mathbf{1} \{N_t = D\} \right) dt + p_D^*(z, 0)$$
$$= \lambda \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\omega X_t} \mathbf{1} \{N_t = C\} \right) dt, \tag{5.4}$$

$$(z + i\omega + i\eta) \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\omega X_t} e^{i\eta Y_t} \mathbf{1} \{N_t = D_s\} \right) dt$$
$$+ \int_0^\infty e^{i\omega x} p_{D_s}^*(z, x, 0) dx + \int_0^\infty e^{i\eta y} p_{D_s}^*(z, 0, y) dy$$
$$= \lambda \mathbf{E} e^{i\omega r} \int_0^\infty e^{-zt} \mathbf{E} \left(e^{i\eta Y_t} \mathbf{1} \{N_t = B\} \right) dt. \tag{5.5}$$

Inserting $\omega = i\,z$, $\operatorname{Re} z > 0$ into Eq. (5.4) reveals that

$$p_D^*(z,0) = \lambda \psi_C^*(z), \tag{5.6}$$

where

$$\psi_C^*(z) := \int_0^\infty e^{-zt}\mathbf{E}\left(e^{-zX_t}\mathbf{1}\left\{N_t = C\right\}\right)dt.$$

Hence,

$$p_D^*(z) = \lambda \frac{p_C^*(z) - \psi_C^*(z)}{z}. \tag{5.7}$$

Adding the Eqs. (5.1), (5.2), (5.3), (5.5) and taking Eq. (5.6) into account, yields the functional equation

$$\begin{aligned}
&\left(z + \lambda\left(1 - \mathbf{E}e^{i\omega r}\right) + \lambda_s\left(1 - \mathbf{E}e^{i\eta r_s}\right)\right)p_A^*(z) - \lambda\ \psi_C^*(z)\mathbf{E}e^{i\omega r}\\
&\qquad + (z + \lambda + i\omega)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\omega X_t}\mathbf{1}\left\{N_t = C\right\}\right)dt\\
&\qquad + \left(z + \lambda\left(1 - \mathbf{E}e^{i\omega r}\right) + i\eta\right)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\eta Y_t}\mathbf{1}\left\{N_t = B\right\}\right)dt\\
&\qquad + (z + i\omega + i\eta)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\left\{N_t = D_s\right\}\right)dt = 1,
\end{aligned} \tag{5.8}$$

valid for $\operatorname{Re}\ z > 0,\ \operatorname{Im}\ \omega \geq 0,\ \operatorname{Im}\ \eta \geq 0$.

Substituting $\omega = iz$, $\eta = 0$ (respectively $\omega = 0$, $\eta = iz$) into Eq. (5.8) reveals that the functions $p_A^*(z)$, $p_B^*(z)$, $p_C^*(z)$, $\psi_C^*(z)$ are satisfying the relations

$$\left(z + \lambda(1 - \mathbf{E}e^{-zr})\right)\left(p_A^*(z) + p_B^*(z)\right) = 1 - \lambda(1 - \mathbf{E}e^{-zr})\psi_C^*(z), \tag{5.9}$$

$$\left(z + \lambda_s(1 - \mathbf{E}e^{-zr_s})\right)p_A^*(z) + (z + \lambda)p_C^*(z) = 1 + \lambda\ \psi_C^*(z). \tag{5.10}$$

In order to derive two additional (independent) relations, we transform the functional equation into a boundary value equation on the real line.

First, we eliminate the function

$$\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\left\{N_t = D_s\right\}\right)dt,\ \operatorname{Im}\ \omega \geq 0,\ \operatorname{Im}\ \eta \geq 0.$$

Substituting $\omega = \tau + iz$, $\eta = -\tau$, $\tau \in \mathbf{R}$, $\operatorname{Re}\ z > 0$ into Eq. (5.7) and noting that $z + i\omega + i\eta = 0$, yields

$$\left(z + \lambda(1 - \mathbf{E}e^{i(\tau+iz)r}) + \lambda_s(1 - \mathbf{E}e^{-i\tau r_s})\right)p_A^*(z) - (1 + \psi_C^*(z)\mathbf{E}e^{i(\tau+iz)r})$$

$$+ (\lambda + i\tau) \int_0^\infty e^{-zt}\mathbf{E}(e^{i(\tau+iz)X_t}\mathbf{1}\{N_t = C\})dt$$

$$- \varphi^+(\tau, z) \int_0^\infty e^{-zt}\mathbf{E}\left(e^{-i\tau Y_t}\mathbf{1}\{N_t = B\}\right) dt = 0, \quad (5.11)$$

where

$$\varphi^+(\tau, z) := i(\tau + iz) + \lambda(\mathbf{E}e^{i(\tau+iz)r} - 1).$$

A straightforward application of Rouché's theorem shows that the function $\varphi^+(\zeta, z)$, $\operatorname{Im}\zeta \geq 0$ has no zeros in $\mathbf{C}^+ \bigcup \mathbf{R}$.

Dividing Eq. (5.11) by $\varphi^+(\tau, z)$ and grouping appropriate functions according to their region of analycity entails that

$$\psi^+(\tau, z) - \psi^-(\tau, z) = \lambda_s p_A^*(z) K(\tau, z), \quad (5.12)$$

where

$$\psi^+(\zeta, z) := \frac{\phi^+(\zeta, z)}{\varphi^+(\zeta, z)}, \ \operatorname{Im}\zeta \geq 0,$$

$$\phi^+(\zeta, z) := \left(z + \lambda(1 - \mathbf{E}e^{i(\zeta+iz)r}) + \lambda_s\right) p_A^*(z) - \left(1 + \lambda\ \psi_C^*(z)\mathbf{E}e^{i(\zeta+iz)r}\right)$$
$$+ (\lambda + i\zeta) \int_0^\infty e^{-zt}\mathbf{E}(e^{i(\zeta+iz)X_t}\mathbf{1}\{N_t = C\})dt,$$

$$\psi^-(\zeta, z) := \int_0^\infty e^{-zt}\mathbf{E}(e^{-i\zeta Y_t}\mathbf{1}\{N_t = B\})dt, \ \operatorname{Im}\zeta \leq 0 \quad (5.13)$$

and where

$$K(\tau, z) := \frac{\mathbf{E}e^{-i\tau r_s}}{\varphi^+(\tau, z)}, \ \tau \in \mathbf{R}.$$

Eq. (5.12) constitutes a **z**-dependent Sokhotski-Plemelj problem on the real line, solvable by the theory of sectionally holomorphic functions [3]. First, we need the following

Property 5.1. *The function $K(\tau, z)$ is Lipschitz-continuous on $\boldsymbol{R}$ and at infinity.*

Proof. Clearly, $|\frac{\partial}{\partial\tau}\mathbf{E}e^{-i\tau r_s}| \leq \mathbf{E}r_s < \infty$, whereas $|\frac{\partial}{\partial\tau}\varphi^+(\tau, z)| \leq 1 + \lambda\mathbf{E}r < \infty$. Finally, the boundedness of $|\varphi^+(\tau, z)|^{-1}$ on $\mathbf{R}$ implies

$$\sup_{\tau\in\mathbf{R}} \left|\frac{\partial}{\partial\tau}K(\tau, z)\right| < \infty.$$

Consequently, by the mean value theorem, there exists a constant $|C(z)|$ such that for all τ_1, $\tau_2 \in \mathbf{R}$

$$|K(\tau_1, z) - K(\tau_2, z)| \leq |C(z)||\tau_1 - \tau_2|.$$

Therefore, $K(\tau, z)$ is Lipschitz-continuous on $\mathbf{R}$. On the other hand, the identity

$$\varphi^+(\tau, z) = \left(1 - \frac{i\lambda\ \mathbf{E}e^{i(\tau+iz)r}}{\tau + i(\lambda + z)}\right) i(\tau + i(\lambda + z)),$$

endowed with the inequality

$$\left|1 - \frac{i\lambda\ \mathbf{E}e^{i(\tau+iz)r}}{\tau + i(\lambda + z)}\right| \geq 1 - \left|\frac{\lambda}{\lambda + z}\right| > 0,$$

reveals that

$$|K(\tau, z)| \leq \left|\varphi^+(\tau, z)\right|^{-1} = O(|\tau|^{-1}), \quad |\tau| \to \infty.$$

Hence, $K(\tau, z)$ is also Lipschitz-continuous at infinity.

Corollary 5.1. *The function*

$$\frac{1}{2\pi i}\int_\Gamma K(\tau, z)\frac{d\tau}{\tau - \zeta}, \quad \zeta \in \mathbf{C},\ Re\, z > 0$$

is sectionally holomorphic and regular.

Moreover, by Eq. (5.12)

$$\psi^-(\zeta, z) = \lambda_s p_A^*(z)\frac{1}{2\pi i}\int_\Gamma K(\tau, z)\frac{d\tau}{\tau - \zeta}, \quad \zeta \in \mathbf{C}^-, \tag{5.14}$$

whereas

$$\psi^+(\zeta, z) = \lambda_s p_A^*(z)\frac{1}{2\pi i}\int_\Gamma K(\tau, z)\frac{d\tau}{\tau - \zeta}, \quad \zeta \in \mathbf{C}^+. \tag{5.15}$$

Note that Eq. (5.14) is only valid for $\operatorname{Im}\zeta < 0$.

However, by the Sokhotski-Plemelj formulas, e.g. [3, page 36],

$$\lim_{\substack{\zeta\to 0\\ \zeta\in\mathbf{C}^-}} \psi^-(\zeta, z) = \lambda_s\, p_A^*(z)\,\alpha(z),$$

where

$$\alpha(z) := \lim_{\substack{\zeta\to 0\\ \zeta\in\mathbf{C}^-}} \frac{1}{2\pi i}\int_\Gamma K(\tau, z)\frac{d\tau}{\tau - \zeta} = -\frac{1}{2}K(0, z) + \frac{1}{2\pi i}\int_\Gamma K(\tau, z)\frac{d\tau}{\tau}.$$

On the other hand, Eq. (5.13) entails that by continuity,

$$\lim_{\substack{\zeta\to 0\\ \zeta\in\mathbf{C}^-}} \psi^-(\zeta,z)=\psi^-(0,z)=p_B^*(z).$$

Hence,

$$p_B^*(z)=\lambda_s\,p_A^*(z)\alpha(z). \tag{5.16}$$

In order to obtain a relationship between $\psi_C^*(z)$ and $p_A^*(z)$, let

$$\alpha(\lambda,z):=\frac{1}{2\pi i}\int_\Gamma K(\tau,z)\frac{d\tau}{\tau-i\lambda}.$$

Substituting $\zeta=i\lambda$ into Eq. (5.15) and taking the definitions of $\phi^+(\zeta,z)$ and $\varphi^+(\zeta,z)$ into account, yields the (required) additional equation

$$(-\varphi^+(i\lambda,z)(1+\lambda_s\,\alpha(\lambda,z))+\lambda_s-\lambda)p_A^*(z)=1+\lambda\psi_C^*(z)\mathbf{E}e^{-(\lambda+z)r}. \tag{5.17}$$

The Laplace-transforms $p_A^*(z)$, $p_B^*(z)$, $p_C^*(z)$ and $p_D^*(z)$ are now completely determined by the Eqs. (5.7), (5.9), (5.10), (5.16) and (5.17).

Finally, note that $p_{D_s}^*(z)$ follows from the identity

$$z\sum_K p_K^*(z)=1.$$

Remarks 5.1. Clearly, Property 5.1 also holds for *arbitrary* R and R_s with finite mean. In fact, the existence of moments does not depend on the canonical structure (Lebesgue decomposition) of the underlying probability distribution. For instance, the inequality

$$\left|\frac{\partial}{\partial\tau}\mathbf{E}e^{i\tau r}\right|\le\mathbf{E}r$$

also holds for an *arbitrary* R with finite mean. The assumption of finite first moments for R and R_s is extremely mild. As a matter of fact, the current probability distribution of interest in Statistical Reliability Engineering, e.g.[2], [4] even have moments of *any* order.

Consequently, our initial assumption concerning the existence of repair time density functions is totally superfluous for the existence of $p_K^*(z)$.

We now focus on the inversion of $p_A^*(z)$.

6. Numerical Example

In order to present our numerical example, we first recall the main properties of a positive random variable ρ with Erlang-K distribution

$$\mathbf{P}\{\rho \leq u\} \equiv E_{K,\theta}(u) := 1 - e^{-\theta u} \sum_{k=0}^{K-1} \frac{(\theta u)^k}{k!}, \quad K \geq 1,\ \theta > 0.$$

This random variable has the following properties:

- ρ has an increasing hazard rate on $[0, \infty)$, e.g. [2, page 423].
- ρ is unimodal, i.e. the corresponding density function, denoted by $E_K^{'}(u)$, has a single maximum.
- $E_K^{'}(u)$ is strongly decreasing in a neighbourhood of infinity.
- ρ has moments of any order and $\mathbf{E}e^{-z\rho} = (\theta/\theta + z)^K$.
  Consequently, the Erlang-K distribution has fairly interesting engineering applications. For instance, ρ is suitable to model repair times.

As an example, let $R(\cdot) \equiv E_{1,\mu}(\cdot)$ and $R_s(\cdot) \equiv E_{K,\,\mu_s}(\cdot)$.

Observe that $\mathbf{E}r = 1/\mu$ and that $\mathbf{E}r_s = K/\mu_s$.

It should be noted that the particular case $R(\cdot) \equiv E_{1,\mu}(\cdot)$ allows to determine $p_A^*(z)$ by a simplified procedure. Indeed, the Markov property of the exponential distribution implies that $X_t \sim r$. Hence

$$\begin{aligned}
\psi_C^*(z) &= \int_0^\infty e^{-zt} \mathbf{E}\left(e^{-zr} \mathbf{1}\{N_t = C\}\right) dt \\
&= \mathbf{E}e^{-zr} \int_0^\infty e^{-zt} p_C(t) dt \\
&= \frac{\mu}{\mu + z} p_C^*(z),
\end{aligned}$$

whereas

$$p_C^*(z, 0) = \mu\ p_C^*(z).$$

Hence, from Eqs. (5.1) and (5.10) we obtain the simplified equations

$$(z + \lambda + \lambda_s) p_A^*(z) = p_B^*(z, 0) + \mu p_C^*(z) + 1, \tag{6.1}$$

$$\left(z + \lambda_s(1 - \mathbf{E}e^{-zr_s})\right) p_A^*(z) + (z + \lambda(1 - \frac{\mu}{\mu + z})) p_C^*(z) = 1. \tag{6.2}$$

Consequently, we only have to determine $p_B^*(z,0)$.

However, a Tauberian theorem for Fourier-transforms entails that

$$p_B^*(z,0) = \lim_{\substack{|\omega|\to\infty \\ \pi<\arg\omega<2\pi}} \left\{ i\omega \int_0^\infty e^{-zt}\mathbf{E}\left(e^{-i\omega Y_t}\mathbf{1}\left\{N_t = B\right\}\right) dt \right\}$$

We recall that for $\omega \in \mathbf{C}^-$,

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{-i\omega Y_t}\mathbf{1}\left\{N_t = B\right\})dt = \lambda_s\, p_A^*(z)\frac{1}{2\pi i}\int_\Gamma K(\tau,z)\frac{d\tau}{\tau-\omega}.$$

Evaluating the particular Cauchy integral by the methods of residues, see [8] for a general methodology, yields for Im $\omega \le 0,\ \omega \ne -iz$,

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{-i\omega Y_t}\mathbf{1}\left\{N_t = B\right\})dt$$
$$= \lambda_s\, p_A^*(z)\left\{\frac{i\omega-(\mu+z)}{(i\omega-z)(\lambda+\mu+z-i\omega)}\mathbf{E}e^{-i\omega r_s}\right.$$
$$\left. + \frac{\mu}{(\lambda+\mu)(i\omega-z)}\mathbf{E}e^{-zr_s} + \frac{\lambda}{(\lambda+\mu)(i\omega-(\lambda+\mu+z))}\mathbf{E}e^{-(\lambda+\mu+z)r_s}\right\}.$$

Note that the property

$$\lim_{\substack{|\omega|\to\infty \\ \pi<\arg\omega<2\pi}} \mathbf{E}e^{-i\omega r_s} = 0,$$

holds for an *arbitrary* r_s.

Hence,

$$p_B^*(z,0) = \lambda_s\, p_A^*(z)\frac{\mu\mathbf{E}e^{-zr_s} + \lambda\,\mathbf{E}e^{-(\lambda+\mu+z)r_s}}{\lambda+\mu}. \tag{6.3}$$

From Eqs. (6.1), (6.2) and (6.3), we finally obtain

$$p_A^*(z) = 1/T^*(z),\ \text{Re } z > 0, \tag{6.4}$$

where

$$T^*(z) := z+\lambda+\lambda_s - \frac{\lambda\mu(\lambda+\mu+\lambda_s(\mathbf{E}e^{-zr_s} - \mathbf{E}e^{-(\lambda+\mu+z)r_s}))}{(\lambda+\mu)(\lambda(1-\frac{\mu}{\mu+z})+\mu+z)}$$
$$- \frac{\lambda_s}{\lambda+\mu}(\mu\mathbf{E}e^{-zr_s} + \lambda\mathbf{E}e^{-(\lambda+\mu+z)r_s}).$$

As a *numerical* example, we take $\lambda = 0.5$, $\lambda_s = 0.1$, $\mu = 2$, $\mu_s = 1$, $K = 2$. Note that $\mathbf{E}e^{-zr_s} = 1/(1+z)^2$. Eq. (6.4) yields

$$p_A^*(z) = N(z)/z\ D(z),\ \mathrm{Re}\ z > 0,$$

where

$$N(z) = 49 + 181.12\ z + 263\ z^2 + 190.12\ z^3 + 71.75\ z^4 + 13.50\ z^5 + z^6$$

and

$$D(z) = 74.76 + 248.62\ z + 330.77\ z^2 + 222.07\ z^3 + 78.85\ z^4 + 14.10\ z^5 + z^6.$$

Clearly, $p_A(t)$ is continuous on $(0,\infty)$ and of bounded variation on $[0,\infty)$. Hence, by the inversion theorem,

$$p_A(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT+\delta}^{iT+\delta} e^{zt} \frac{N(z)}{D(z)} \frac{dz}{z},\ \ \delta > 0,\ t > 0.$$

Applying the residue theorem for Laplace-transforms, e.g. [1, page 438, Theorem 16.39], sustained by *Mathematica*, reveals that

$$\begin{aligned} p_A(t) =& 0.6561 + 0.0013\, e^{-3.29t} + 0.1502\, e^{-1.40t} \\ & + e^{-1.05t}(0.1174 \cos\ 0.2503\, t + 0.0590\ \sin\ 0.2503\, t) \\ & + e^{-3.61\, t}(0.0750\ \cos\ 0.1949\, t + 0.0384\ \sin\ 0.1942\, t). \end{aligned}$$

Clearly,

$$p_A(\infty) := \lim_{t\to\infty} p_A(t) = 0.6561.$$

$p_A(\infty)$ is called the long-run availability of the safe state. See [8] for further details.

Figure 1 shows the graph of $p_A(t)$, $t \in [0, 20]$.

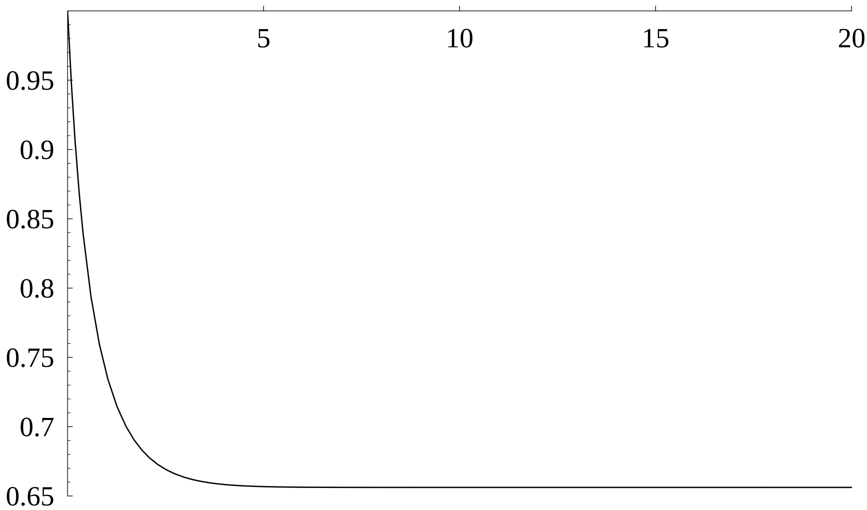

Figure 1: Point availability of the safe state

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