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THE APPROXIMATE AND EXACT SOLUTIONS OF THE FRACTIONAL-ORDER DELAY DIFFERENTIAL EQUATIONS USING LEGENDRE SEUDOSPECTRAL METHOD

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Abstract: Fractional differential equations have recently been applied in various area of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. In this paper, an efficient numerical method for solving the fractional delay differential equations (FDDEs) is considered. The fractional derivative is described in the Caputo sense. The method is based upon Legendre approximations. The properties of Legendre polynomials are utilized to reduce FDDEs to linear or nonlinear system of algebraic equations. Numerical simulation with the exact solutions of FDDEs is presented.

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1. Introduction

Ordinary and partial fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [1]. Consequently, considerable attention has been given to the solutions of fractional ordinary and partial differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques (see [3-7], [13], [21]), must be used. Recently, several approximated methods to solve fractional differential equations have been given such as variational iteration method (see [8], [10]), homotopy perturbation method (see [9], [19], [20]), Adomian's decomposition method [11], homotopy analysis method [6] and collocation method (see [12], [17]).

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. The Caputo fractional derivative operator D^α of order α is defined in the following form:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where: $m-1 < \alpha < m,\ m \in \mathbb{N},\ x > 0.$

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^\alpha (\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x),$$

where λ and μ are constants.

For the Caputo's derivative we have [14]:

$$D^\alpha C = 0, \quad \text{C is a constant,} \tag{1}$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \tag{2}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo fractional differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see (see [14], [18]).

The main goal in this article is concerned with the application of Legendre pseudospectral method to obtain the numerical solution of FDDEs of the form:

$$u^{(\alpha)}(x) = f\left(x, u(x), u(g(x))\right), \quad a \le x \le b, \quad 1 < \alpha \le 2, \tag{3}$$

with the following boundary conditions:

$$u(a) = \gamma_0, \quad u(b) = \gamma_1, \quad u(x) = \psi(x), \quad x \in [a^*, a], \tag{4}$$

where α refers to the fractional order of spatial derivatives. The fractional derivative is described in the Caputo sense. And the function g is called the delay function and it is assumed to be continuous on the interval $[a, b]$ and satisfies the inequality $a^* \le g(x) \le x$, $x \in [a, b]$ and $\psi \in C[a^*, a]$. In our examples in Section 4, the interval [0,1] is under consideration. Note that $\alpha = 2$, equation (3) is the classical second-order delay differential equation:

$$u''(x) = f\left(x, u(x), u(g(x))\right), \quad a \le x \le b. \tag{5}$$

In [15], the author studied the first order delay differential equation, using spline functions and studied the stability and the error analysis. Also, in [16], the authors studied the system of first order delay differential equations, using spline functions, and studied the stability and the error analysis. Our idea is to apply the Legendre collocation method to discretize (3) to get a linear or non-linear system of algebraic equations thus greatly simplifying the problem, and solve the resulting system.

Legendre polynomials are well known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications [17]. They are widely used because of their good properties in the approximation of functions. However, with our best knowledge, very little work was done to adapt this polynomials to the solution of fractional differential equations.

The organization of this paper is as follows. In the next section, the approximation of fractional derivative $D^{\alpha}u(x)$ is obtained. Section 3 summarizes the application of Legendre collocation method to solve (3). As a result, a system of linear or non-linear of algebraic equations is formed and the solution of the considered problem is introduced. In Section 4, two examples are given to clarify the advantages of the proposed method. Also, a conclusion is given in Section 5. The numerical results are computed using Matlab software.

2. Derivation an Approximate Formula for Fractional Derivatives. Using Legendre Series Expansion

The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula [2]:

$$L_{k+1}(z)=\frac{2k+1}{k+1}\,z\,L_k(z)-\frac{k}{k+1}L_{k-1}(z),\qquad k=1,2,...,$$

where $L_0(z)=1$ and $L_1(z)=z$. In order to use these polynomials on the interval $x\in[0,1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z=2x-1$. Let the shifted Legendre polynomials $L_k(2x-1)$ be denoted by $P_k(x)$. Then $P_k(x)$ can be obtained as follows:

$$P_{k+1}(x)=\frac{(2k+1)(2x-1)}{(k+1)}\,P_k(x)-\frac{k}{k+1}\,P_{k-1}(x),\qquad k=1,2,..., \tag{6}$$

where $P_0(x)=1$ and $P_1(x)=2x-1$. The analytic form of the shifted Legendre polynomials $P_k(x)$ of degree k given by:

$$P_k(x)=\sum_{i=0}^{k}(-1)^{k+i}\frac{(k+i)!\,x^i}{(k-i)(i!)^2}. \tag{7}$$

Note that $P_k(0)=(-1)^k$ and $P_k(1)=1$. The orthogonality condition is:

$$\int_0^1 P_i(x)P_j(x)\,dx=\begin{cases}\frac{1}{2i+1}, & \text{for}\quad i=j;\\ 0, & \text{for}\quad i\neq j.\end{cases} \tag{8}$$

The function $u(x)$ which is square integrable on $[0,1]$ may be expressed in terms of shifted Legendre polynomials as:

$$u(x)=\sum_{i=0}^{\infty}c_iP_i(x),$$

where the coefficients c_i are given by:

$$c_i=(2i+1)\int_0^1 u(x)P_i(x)\,dx,\quad i=1,2,\dots .$$

In practice, only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have:

$$u_m(x)=\sum_{i=0}^{m}c_iP_i(x). \tag{9}$$

In the following theorem we will approximate the fractional derivative of $u(x)$.

Theorem 1. *Let $u(x)$ be approximated by shifted Legendre polynomials as (9) and also suppose $\alpha > 0$ then:*

$$D^{\alpha}\Big(u_m(x)\Big) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{10}$$

where $w_{i,k}^{(\alpha)}$ is given by:

$$w_{i,k}^{(\alpha)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k+1-\alpha)}\,. \tag{11}$$

Proof. Since the Caputo's fractional differentiation is a linear operation we have:

$$D^{\alpha}\Big(u_m(x)\Big) = \sum_{i=0}^{m} c_i D^{\alpha}(P_i(x)). \tag{12}$$

Employing equations (1)-(2) in equation (7) we have:

$$D^{\alpha}P_i(x) = 0, \quad i = 0, 1, ..., \lceil\alpha\rceil - 1, \quad \alpha > 0. \tag{13}$$

Also, for $i = \lceil\alpha\rceil, ..., m$, by using equations (1)-(2) and (7) we get:

$$\begin{aligned} D^{\alpha}P_i(x) &= \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{\alpha}(x^k) \\ &= \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)} x^{k-\alpha}. \end{aligned} \tag{14}$$

A combination of equations (12), (13) and (14) leads to the desired result. □

2.1. Test Example

Consider the case when $u(x) = x^2$ and $m = 2$, the shifted series of x^2 is:

$$x^2 = \frac{1}{3}P_0(x) + \frac{1}{2}P_1(x) + \frac{1}{6}P_2(x).$$

Hence

$$D^{\frac{1}{2}}x^2 = \sum_{i=1}^{2}\sum_{k=1}^{i} c_i\, w_{i,k}^{(\frac{1}{2})} x^{k-\frac{1}{2}},$$

where

$$w_{1,1}^{(\frac{1}{2})} = \frac{2}{\Gamma(\frac{3}{2})}, \quad w_{2,1}^{(\frac{1}{2})} = \frac{-6}{\Gamma(\frac{3}{2})}, \quad w_{2,2}^{(\frac{1}{2})} = \frac{12}{\Gamma(\frac{5}{2})}.$$

Therefore

$$D^{\frac{1}{2}} x^2 = x^{-\frac{1}{2}} \left[c_1 \, w_{1,1}^{(\frac{1}{2})} x + c_2 \, w_{2,1}^{(\frac{1}{2})} x + c_2 \, w_{2,2}^{(\frac{1}{2})} x^2 \right] = \frac{2}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}}.$$

3. Procedure Solution of the Fractional Delay Differential Equation

Consider the fractional delay differential equation of type given in equation (3). In order to use Legendre collocation method, we first approximate $u(x)$ as:

$$u_m(x) = \sum_{i=0}^{m} c_i P_i(x). \tag{15}$$

From equations (3) and (15) and Theorem 1 we have:

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i \, w_{i,k}^{(\alpha)} x^{k-\alpha} = f\left(x, \sum_{i=0}^{m} c_i P_i(x), \sum_{i=0}^{m} c_i P_i(g(x)) \right). \tag{16}$$

We now collocate equation (16) at $(m+1-\lceil\alpha\rceil)$ points $x_p,\ p = 0, 1, ..., m-\lceil\alpha\rceil$ as:

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i \, w_{i,k}^{(\alpha)} x_p^{k-\alpha} = f\left(x_p, \sum_{i=0}^{m} c_i P_i(x_p), \sum_{i=0}^{m} c_i P_i(g(x_p)) \right). \tag{17}$$

For suitable collocation points we use roots of shifted Legendre polynomial $P_{m+1-\lceil\alpha\rceil}(x)$.

Also, by substituting equation (15) in the boundary conditions (4) we can obtain $\lceil\alpha\rceil$ equations as follows:

$$\sum_{i=0}^{m} (-1)^i c_i = \gamma_0, \quad \sum_{i=0}^{m} c_i = \gamma_1. \tag{18}$$

Equation (17), together with $\lceil\alpha\rceil$ equations of boundary conditions, give $(m+1)$ linear or non-linear algebraic equations which can be solved, for the unknown $c_n,\ n = 0, 1, 2, ..., m$. Consequently $u(x)$ given in equation (3) can be calculated.

4. Numerical Results

In this section, we implement the proposed method to solve linear and nonlinear examples of fractional delay differential equations. These examples can be used as a basis to illustrate the applicability of the proposed method.

Example 1. Consider the following linear fractional delay differential equation:

$$D^{1.5}u(x) = -u(x) + u\left(\frac{x}{2}\right) + \frac{7}{8}x^3 + \frac{6}{\Gamma(2.5)}x^{1.5}, \tag{19}$$

with the boundary conditions:

$$u(0) = 0, \quad u(1) = 1. \tag{20}$$

We implement the suggested method with $m = 3$, and we approximate solution as:

$$u_3(x) = \sum_{i=0}^{3} c_i P_i(x). \tag{21}$$

Using equation (17) we have:

$$\sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,k}^{(\alpha)}\, x_p^{k-\alpha} = -\sum_{i=0}^{3} c_i P_i(x_p) + \sum_{i=0}^{3} c_i P_i\left(\frac{x_p}{2}\right) + \frac{7}{8}x_p^3 + \frac{6}{\Gamma(2.5)}x_p^{1.5}, \tag{22}$$

with $p = 0, 1$, where x_p are roots of the shifted Chebyshev polynomial $P_2(x)$ and their values are:

$$x_0 = \tfrac{1}{6}(3 - \sqrt{3}), \quad x_1 = \tfrac{1}{6}(3 + \sqrt{3}).$$

By using equations (18) and (20) we get:

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{23}$$

$$c_0 + c_1 + c_2 + c_3 = 1. \tag{24}$$

Now solving equations (22) and (23)-(24) we get:

$$c_0 = 1/4, \quad c_1 = 9/20, \quad c_2 = 1/4, \quad c_3 = 1/20.$$

Thus using equation (21) we get:

$$u(x) = \tfrac{1}{4}P_0(x) + \tfrac{9}{20}P_1(x) + \tfrac{1}{4}P_2(x) + \tfrac{1}{20}P_3(x) = x^3,$$

which is the exact solution of equation (19).

Example 2. Consider the following nonlinear fractional delay differential equation:

$$D^{1.5}u(x) = u(x-0.5) + u^3(x) + f(x), \tag{25}$$

where $f(x) = \frac{2}{\Gamma(1.5)}x^{0.5} - (x-0.5)^2 - x^6$, with the boundary conditions:

$$u(0) = 0, \quad u(1) = 1. \tag{26}$$

We implement the suggested method with $m = 3$ and approximate the solution as:

$$u_3(x) = \sum_{i=0}^{3} c_i P_i(x). \tag{27}$$

Using equation (17) we have:

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)} x_p^{k-\alpha} = \sum_{i=0}^{3} c_i P_i(x_p - 0.5) + \left(\sum_{i=0}^{3} c_i P_i(x_p)\right)^3 + f(x_p), \tag{28}$$

with $p = 0, 1$, where x_p are roots of the shifted Legendre polynomial $P_2(x)$ and their values are:

$$x_0 = \tfrac{1}{6}(3-\sqrt{3}), \quad x_1 = \tfrac{1}{6}(3+\sqrt{3}).$$

By using equations (18), (26) and (28) we obtain the following non-linear system of algebraic equations:

$$c_2 w_{2,2}^{(1.5)} x_0^{0.5} + c_3\left(w_{3,2}^{(1.5)} x_0^{1.5} + w_{3,3}^{(1.5)}\, x_0^{1.5}\right) - \sum_{i=0}^{3} r_i c_i - \left(\sum_{i=0}^{3} s_i c_i\right)^3 - f(x_0) = 0, \tag{29}$$

$$c_2 w_{2,2}^{(1.5)} x_1^{0.5} + c_3\left(w_{3,2}^{(1.5)} x_1^{1.5} + w_{3,3}^{(1.5)}\, x_1^{1.5}\right) - \sum_{i=0}^{3} u_i c_i - \left(\sum_{i=0}^{3} v_i c_i\right)^3 - f(x_1) = 0, \tag{30}$$

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{31}$$

$$c_0 + c_1 + c_2 + c_3 = 1, \tag{32}$$

where:

$$r_i = P_i(x_0 - 0.5), \quad s_i = P_i(x_0), \quad u_i = P_i(x_1 - 0.5), \quad v_i = P_i(x_1).$$

Now solving the system of equations (29)-(32) we get:

$$c_0 = 1/3, \quad c_1 = 1/2, \quad c_2 = 1/6, \quad c_3 = 0.$$

Thus using equation (27) we get:

$$u(x) = \tfrac{1}{3}P_0(x) + \tfrac{1}{2}P_1(x) + \tfrac{1}{6}P_2(x) + 0P_3(x) = x^2,$$

which is the exact solution of equation (25).

5. Conclusion

The properties of the Legendre polynomials are used to reduce the fractional delay differential equations to the solution of system of linear or non-linear algebraic equations. The fractional derivative is considered in the Caputo sense. From the solutions obtained using the suggested method we can conclude that these solutions are in an excellent agreement with the already existing ones and show that this approach can solve the problem effectively. In our two examples we have the exact solution. All numerical results are obtained using Matlab 7.1.

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