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NEW MODULAR RELATIONS FOR RAMANUJAN'S PARAMETER $\mu(q)$

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Abstract: In his 'lost' notebook, S. Ramanujan introduced the parameter $\mu(q) := R(q)R(q^4)$ related to the Rogers - Ramanujan continued fraction $R(q)$. In this paper, we establish some new $P - Q$ modular equations of degree 5. We establish some general formulas for the explicit evaluations of the ratios of Ramanujan's theta function φ. We obtain several new modular relations connecting $\mu(q)$ with $\mu(q^n)$ for different positive integer $n > 1$, reciprocity theorems and also compute several new explicit evaluations.

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Key Words: continued fraction, modular equation, reciprocity theorem, theta function

1. Introduction

The Rogers-Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots, \quad |q| < 1, \tag{1}$$

was first studied by L.J. Rogers [19]. Later, this continued fraction was redis-

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covered by S. Ramanujan and recorded may interesting results involving $R(q)$. For more details on $R(q)$ one can see [2], [3], [6] and [20].

In his 'lost' notebook Ramanujan [18], introduced the parameters $\mu(q)$ and $\kappa(q) := R(q)R^2(q^2)$ which are related to Rogers-Ramanujan continued fraction. Ramanujan stated several interesting identities involving the parameters $\mu(q)$ and $\kappa(q)$. These results were studied in detail by S. -Y. Kang [9]. A similar work has been carried out by M. S. Mahadeva Naika [10], for Ramanujan cubic continued fraction. Recently, C. Gugg [8] established certain identities of Ramanujan using the parameter $\kappa(q)$. S. Cooper [7], also systematically studied several results involving the parameter $\kappa(q)$.

In Chapter 16, of his second notebook [17] Ramanujan defined his theta function as

$$\begin{aligned} f(a,b) :&= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \\ &= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty . \end{aligned} \tag{2}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1-aq^n), \qquad |q| < 1.$$

Now we define a modular equation in brief. The ordinary hypergeometric series
${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $\mid x \mid < 1$.
Let

$$z := z(x) := \ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{6}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{7}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{8}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (8). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In [21], J. Yi introduced two parameters $h_{k,n}$ and $h'_{k,n}$ as follows:

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \tag{9}$$

$$h'_{k,n} = \frac{\varphi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-\pi\sqrt{nk}})} \tag{10}$$

and established several properties as well as explicit evaluations of $h_{k,n}$ and $h'_{k,n}$ for different positive rational values of n and k by using the modular equations of Ramanujan. Mahadeva Naika and S. Chandankumar [11], Mahadeva Naika, Chandankumar and M. Manjunatha [13], Mahadeva Naika, K. Sushan Bairy and Chandankumar [15] have established several new modular equations of degree 2. They also established general formulas for the explicit evaluations of $h_{2,n}$ and several new explicit evaluations for Ramanujan -Göllnitz-Gordon continued fraction, Ramanujan-Selberg continued fraction and a continued fraction of Eisenstein. In [16], Mahadeva Naika, Bairy and Manjunatha have established several new modular equations of degree 4 and established general formulas for the explicit evaluations of $h_{4,n}$. In [14], Mahadeva Naika, Bairy and Chandankumar have established several new modular equations of degree 9 and established general formulas for the explicit evaluations of $h_{9,n}$ and they have established some explicit evaluations of Ramanujan's cubic continued fraction.

In Section 2, we collect the identities which are useful in proving our main results. In Section 3, we establish some new modular equations of degree 5. In Section 4, we establish some new modular relations connecting $h_{5,n}$ with h_{5,l^2n} for $l = 2, 4, 5, 7$ and 11. In Section 5, we establish several new modular relations connecting $\mu(q)$ with $\mu(q^n)$. In Section 6, we establish some new reciprocity theorems for $\mu(q)$ and obtain some new explicit evaluations of $\mu(q)$ using the values of $h_{5,n}$.

2. Preliminary Results

In this section, we collect some identities which are useful in establishing our main results.

Lemma 1. (see [18, p.56], [9]) *We have*

$$\frac{f^3(-q)}{f^3(-q^5)} = \frac{\psi(q)}{\psi(q^5)}\left(\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)}\right), \tag{11}$$

$$\frac{f^3(-q^2)}{qf^3(-q^{10})} = \frac{\varphi(q)}{\varphi(q^5)}\left(\frac{5\varphi^2(q^5)-\varphi^2(q)}{\varphi^2(q)-\varphi^2(q^5)}\right), \tag{12}$$

$$\frac{f^6(-q)}{qf^6(-q^5)} = \frac{\varphi^4(-q)}{\varphi^4(-q^5)}\left(\frac{5\varphi^2(-q^5)-\varphi^2(-q)}{\varphi^2(-q)-\varphi^2(-q^5)}\right). \tag{13}$$

Lemma 2. (see [18, p.26], [9]) *Let $\mu := \mu(q)$, then*

$$\frac{\varphi(q)}{\varphi(q^5)} = \frac{1+\mu}{1-\mu}. \tag{14}$$

Lemma 3. *Let $\omega := \omega(q) := -\mu(-q)$, then*

$$\frac{\varphi(-q)}{\varphi(-q^5)} = \frac{1+\omega}{1-\omega}. \tag{15}$$

Proof. Change q to $-q$ in the equation (14), we arrive at the equation (15). □

Lemma 4. (see [18, Entry 1.6.2(i), p.50]) *We have*

$$16qf^2(-q^2)f^2(-q^{10}) = (\varphi^2(q)-\varphi^2(q^5))(5\varphi^2(q^5)-\varphi^2(q)). \tag{16}$$

Lemma 5. (see [5, Ch. 16, Entry 24 (i), p.39]) *We have*

$$\frac{\psi^2(q)}{\psi^2(-q)} = \frac{\varphi(q)}{\varphi(-q)}. \tag{17}$$

Lemma 6. (see [5, Ch. 16, Corollary (ii), p.74]) *We have*

$$\psi(q^5)\psi(q^{11}) - q^5\psi(q)\psi(q^{55}) = \psi(-q^5)\psi(-q^{11}) + q^5\psi(q)\psi(q^{55}). \tag{18}$$

Lemma 7. (see [18, p.55]) *If* $x = \dfrac{f(-q)}{q^{\frac{1}{6}}f(-q^5)}$ *and* $y = \dfrac{f(-q^2)}{q^{\frac{1}{3}}f(-q^{10})}$, *then*

$$xy + \frac{5}{xy} = \left(\frac{x}{y}\right)^3 + \left(\frac{y}{x}\right)^3. \tag{19}$$

Lemma 8. (see [18, p.55]) *If* $x = \dfrac{f(-q)}{q^{\frac{1}{6}}f(-q^5)}$ *and* $y = \dfrac{f(-q^4)}{q^{\frac{2}{3}}f(-q^{20})}$, *then*

$$\begin{aligned}(xy)^3 + \left(\frac{5}{xy}\right)^3 &= \left(\frac{x}{y}\right)^5 + \left(\frac{y}{x}\right)^5 - 8\left\{\left(\frac{x}{y}\right)^3 + \left(\frac{y}{x}\right)^3\right\} \\ &+ 4\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{4}{\left(\frac{x}{y} + \frac{y}{x}\right)}.\end{aligned} \tag{20}$$

Lemma 9. (see [18, p.55]) *If* $x = \dfrac{f(-q)}{q^{\frac{1}{6}}f(-q^5)}$ *and* $y = \dfrac{f(-q^5)}{q^{\frac{5}{6}}f(-q^{25})}$, *then*

$$(xy)^2 + \left(\frac{5}{xy}\right)^2 + 5\left(xy + \frac{5}{xy}\right) + 15 = \left(\frac{y}{x}\right)^3. \tag{21}$$

Lemma 10. (see [5, Ch 20, Entry 18 (vi) and (vii), p.423]) *If* β, γ *and* δ *are of degrees 5, 7 and 35 respectively over* α, *then*

$$\begin{aligned}&\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} \\ &\qquad + 2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12} = \sqrt{\frac{m'}{m}},\end{aligned} \tag{22}$$

$$\begin{aligned}&\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} \\ &\qquad + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = -\sqrt{\frac{m}{m'}}.\end{aligned} \tag{23}$$

Lemma 11. (see [5, Ch. 17, Entry 10(i) and Entry 11(ii), pp. 122-123]) *We have*

$$\varphi(q) = \sqrt{z}, \tag{24}$$
$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}. \tag{25}$$

Lemma 12. (see [5, Ch. 16, Entry 27 (i), (iv), p.43]) *If* $\alpha\beta = \pi$, *then*

$$\sqrt{\alpha}\varphi(e^{-\alpha^2}) = \sqrt{\beta}\varphi(e^{-\beta^2}). \tag{26}$$

$$e^{-\alpha^2/12}\sqrt{\alpha}f(-e^{-2\alpha^2}) = e^{-\beta^2/12}\sqrt{\beta}f(-e^{-2\beta^2}). \tag{27}$$

Lemma 13. (see [1, Theorem 5.1]) *If* $P = \dfrac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$Q^2 + P^2Q^2 = 5 + P^2. \tag{28}$$

Lemma 14. (see [3, Ch. 25, Entry 66, p.233]) *If* $P = \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{15})}$, *then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \tag{29}$$

Lemma 15. (see [4, Theorem 2.16]) *If* $P = \dfrac{\varphi(-q)}{\varphi(-q^5)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$\left(\frac{P}{Q}\right) + \left(\frac{Q}{P}\right) = \left(PQ + \frac{5}{PQ}\right) - 4. \tag{30}$$

3. *P–Q* Modular Equations

In this section, we establish some new $P - Q$ modular equations of degree 5 for the ratios of Ramanujan's theta fuction.

Theorem 16. *If* $P := \dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}$ *and* $Q := \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = P^2 + \frac{5}{P^2}. \tag{31}$$

Proof. Cubing both sides of the equation (19) and using the equations (11) and (12), we find that

$$\begin{aligned}&(P^4-5P^2+4P^2Q^2-P^2Q^4+Q^4)(P^4Q^2-P^4-4P^2Q^2\\&+5Q^2-Q^4)(P^2Q^2-P^2+5-Q^2)(25-10P^2-10Q^2\\&+P^4-2P^4Q^2+P^4Q^4-4P^2Q^2-16P^3Q+16Q^3P+Q^4\\&-2P^2Q^4)(25-10P^2-10Q^2+P^4-2P^4Q^2+P^4Q^4+Q^4\\&-4P^2Q^2+16P^3Q-16Q^3P-2P^2Q^4)=0.\end{aligned} \tag{32}$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 17. *If* $P:=\dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q:=\dfrac{\varphi(q^2)}{\varphi(q^{10})}$, *then*

$$\begin{aligned}&\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2+(PQ)^2+\left(\frac{5}{PQ}\right)^2+16\left(\frac{P}{Q}-\frac{Q}{P}\right)\\&=2\left(P^2+\frac{5}{P^2}\right)+2\left(Q^2+\frac{5}{Q^2}\right)+4.\end{aligned} \tag{33}$$

Proof. Cubing both sides of the equation (20), we deduce that

$$\begin{aligned}&22500a^6b^{24}+1020a^{24}b^{18}+a^{30}b^{24}+7420a^{12}b^{24}+22500a^{24}b^6\\&+1020a^{18}b^{24}+7420a^{24}b^{12}+1953125a^{12}b^6+127500a^{12}b^{18}a^{36}\\&+1953125a^6b^{12}+127500a^{18}b^{12}+391b^6a^{30}+391a^6b^{30}+a^{24}b^{30}\\&+937500a^{12}b^{12}+180a^{30}b^{12}+375000a^{18}b^6+16380a^{18}b^{18}-b^{36}\\&+60a^{24}b^{24}+24a^{18}b^{30}+24a^{30}b^{18}+180a^{12}b^{30}+375000a^6b^{18}=0.\end{aligned} \tag{34}$$

where

$$a=\frac{f(-q)}{q^{\frac{1}{6}}f(-q^5)}\quad\text{and}\quad b=\frac{f(-q^4)}{q^{\frac{2}{3}}f(-q^{20})}.$$

Using the equations (12) and (13) in the above equation (34), we find that

$$\begin{aligned}
&(P^2Q^4 - Q^4 - 4P^2Q^2 + 5P^2 - P^4)(25 + 16Q^3P - 10Q^2 - 10P^2\\
&- 2P^2Q^4 + P^4Q^4 + Q^4 - 4P^2Q^2 - 16QP^3 + P^4 - 2P^4Q^2)(25\\
&- 16Q^3P - 10Q^2 - 10P^2 - 2P^2Q^4 + P^4Q^4 + Q^4 - 4P^2Q^2 + P^4\\
&+ 16QP^3 - 2P^4Q^2)(30P^4Q^4 - 25P^2Q^4 - P^2Q^6 + P^2Q^8 - 25P^4Q^2\\
&- 5P^4Q^6 - P^6Q^2 - 5P^6Q^4 + P^6Q^6 + 5P^6 + 5Q^6 - Q^8 + 25P^2Q^2\\
&- P^8 + P^8Q^2)(250000P^2Q^6 - 17976P^{12}Q^6 - 78125Q^2 + 368P^{10}Q^{14}\\
&- 112000P^2Q^8 - 325000P^4Q^2 + 600000P^4Q^4 + P^{16} + 270400P^4Q^8\\
&+ 230000P^6Q^2 - 512000P^6Q^4 + 527600P^6Q^6 - 339840P^6Q^8\\
&- 5120P^2Q^{12} + 33200P^2Q^{10} + 109375Q^4 - 65625Q^6 + 21875Q^8\\
&- 4375Q^{10} + 525Q^{12} + 250000P^2Q^2 + 14336P^4Q^{12} - 89880P^4Q^{10}\\
&- 35Q^{14} + 304P^2Q^{14} + 10816P^{12}Q^8 - 67404P^8Q^{10} - 856P^4Q^{14}\\
&- 469000P^4Q^6 - 17536P^6Q^{12} + 110672P^6Q^{10} + 277000P^8Q^4 + Q^{16}\\
&- 337020P^8Q^6 - 21P^{16}Q^{10} - 100500P^8Q^2 + 27600P^{10}Q^2 - 7P^{16}Q^2\\
&+ 110672P^{10}Q^6 - 4280P^{12}Q^2 + 14336P^{12}Q^4 - 804P^8Q^{14}\\
&- 4096P^{10}Q^{12} + 21104P^{10}Q^{10} - 104P^{12}Q^{14} + 960P^{12}Q^{12}\\
&- 1024P^{14}Q^4 - 896P^{14}Q^8 + 400P^{14}Q^{10} - 128P^{14}Q^{12} - 35P^{16}Q^6\\
&+ 7P^{16}Q^{12} + 16P^{14}Q^{14} + 21P^{16}Q^4 + 1328P^{14}Q^6 - P^{16}Q^{14}\\
&+ 11080P^8Q^{12} + 1104P^6Q^{14} + 304P^{14}Q^2 + 217648P^8Q^8\\
&+ 35P^{16}Q^8 - 67968P^{10}Q^8 - 3752P^{12}Q^{10} - 87680P^{10}Q^4\\
&- 400000P^2Q^4) = 0.
\end{aligned} \tag{35}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the third factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem third factor vanishes identically. This completes the proof. □

Theorem 18. *If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(-q^4)}{\varphi(-q^{20})}$, *then*

$$\begin{aligned}&\frac{P^4}{Q^4}+\frac{Q^4}{P^4}+24\left[\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right]+8\left[P^2Q^2+\frac{5^2}{P^2Q^2}\right]+120\\&+3\left[Q^4+\frac{5^2}{Q^4}\right]=20\left[P^2+\frac{5}{P^2}\right]+32\left[Q^2+\frac{5}{Q^2}\right]\\&+\left[P^2Q^4+\frac{125}{P^2Q^4}\right]+3\left[\frac{5P^2}{Q^4}+\frac{Q^4}{P^2}\right].\end{aligned}\tag{36}$$

Proof. Using the equations (13) and (20), we arrive at the equation (36). This completes the proof. □

Theorem 19. *If* $P := \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^5)\varphi(q^{20})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^5)\varphi(q^4)}$, *then*

$$\begin{aligned}&Q^4+\frac{1}{Q^4}-112\left[Q^3+\frac{1}{Q^3}\right]+1440\left[Q^2+\frac{1}{Q^2}\right]-3184\left[Q+\frac{1}{Q}\right]\\&+7316=8\left[P+\frac{1}{P}\right]\left[22\left[Q^2+\frac{1}{Q^2}\right]-31\left[Q+\frac{1}{Q}\right]+170\right]\\&-2\left[P^2+\frac{5^2}{P^2}\right]\left[3\left[Q^2+\frac{1}{Q^2}\right]+24\left[Q+\frac{1}{Q}\right]+64\right]\\&+4\left[P^3+\frac{5^3}{P^3}\right]\left[\left[Q+\frac{1}{Q}\right]+4\right].\end{aligned}\tag{37}$$

Proof. Using the equations (36) and (30), we arrive at the equation (37). □

Theorem 20. *If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^5)}{\varphi(q^{25})}$, *then*

$$\begin{aligned}&\frac{Q^3}{P^3}-\frac{5Q^2}{P^2}-\frac{15Q}{P}+5\left(PQ+\frac{5}{PQ}\right)+5\left(Q^2+\frac{5}{P^2}\right)\\&=P^2Q^2+\frac{5^2}{P^2Q^2}+15.\end{aligned}\tag{38}$$

Proof. Using the equations (13) and (21), we arrive at the equation (38). □

Theorem 21. *If* $P := \dfrac{\phi(q)\phi(q^7)}{\phi(q^5)\phi(q^{35})}$ *and* $Q := \dfrac{\phi(q)\phi(q^{35})}{\phi(q^5)\phi(q^7)}$, *then*

$$\begin{aligned} &Q^4 - \frac{1}{Q^4} - 14\left[\left(Q^3 + \frac{1}{Q^3}\right) - \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] \\ &+ P^3 + \frac{5^3}{P^3} = 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) - \left(P + \frac{5}{P}\right)\right. \\ &\left.\times\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\}. \end{aligned} \tag{39}$$

Proof. Using the equations (22), (23), (24) and (25), we deduce that

$$1 + r - 2Ar + sr - s + 2A^2r = 0, \tag{40}$$

where

$$r := \frac{q^3\psi(-q)\psi(-q^{35})}{\psi(-q^5)\psi(-q^7)}, \quad s := \frac{\varphi(q)\varphi(q^{35})}{\varphi(q^5)\varphi(q^7)} \text{ and } A := (s/r)^{1/3}.$$

On simplification of the equation (40), we find that

$$2A = 1 + m, \quad \text{where } m = \pm\sqrt{\frac{2s - r - 2sr - 2}{r}}. \tag{41}$$

Cubing both sides of the equation (41) and eliminating m, we deduce that

$$\begin{aligned} &15cf^2e - 6e^2cf^3 - e^4cf^3 + 8e^4cf + 6e^3df^2 + e^3df^4 - 8edf^4 \\ &- 14f^3de^2 - 6cf^4e + 3cf^4e^3 - 14cf^2e^3 + f^5de^2 + cf^5e^2 \\ &- 4f^5d - 2cf^5 - 4e^5c + 2e^5d + 5f^3d + 5cf^3 + 5e^3c - 5e^3d \\ &+ 15fde^2 + e^5cf^2 - e^5df^2 - 6fde^4 + 3f^3de^4 + 5edf^2 - 5e^2cf = 0, \end{aligned} \tag{42}$$

where

$$c := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)}, \quad d := \frac{\psi(-q^7)}{q^{7/2}\psi(-q^{35})}, \quad e := \frac{\varphi(q)}{\varphi(q^5)} \text{ and } f := \frac{\varphi(q^7)}{\varphi(q^{35})}.$$

Collecting the terms containing c on one side of the above equation (42) and squaring both sides and using the equation (28), we arrive at the equation (39). □

Theorem 22. *If* $P = \dfrac{\phi(q)\phi(q^{11})}{\phi(q^5)\phi(q^{55})}$ *and* $Q = \dfrac{\phi(q)\phi(q^{55})}{\phi(q^5)\phi(q^{11})}$, *then*

$$\begin{aligned} &Q^6 + \frac{1}{Q^6} + 33\left[Q^5 + \frac{1}{Q^5}\right] - 99\left[Q^4 + \frac{1}{Q^4}\right] + 1529\left[Q^3 + \frac{1}{Q^3}\right] \\ &- 1683\left[Q^2 + \frac{1}{Q^2}\right] + 8800\left[Q + \frac{1}{Q}\right] = 6534 + \left[P^5 + \frac{5^5}{P^5}\right] \\ &- 11\left\{\left[P^4 + \frac{5^4}{P^4}\right]\left[Q + \frac{1}{Q}\right] - \left[P^3 + \frac{5^3}{P^3}\right]\left[11 + 4\left[Q^2 + \frac{1}{Q^2}\right]\right]\right. \\ &- \left[P^2 + \frac{5^2}{P^2}\right]\left[18 - 56\left[Q + \frac{1}{Q}\right] + 3\left[Q^2 + \frac{1}{Q^2}\right] - 8\left[Q^3 + \frac{1}{Q^3}\right]\right] \\ &- \left[P + \frac{5}{P}\right]\left[-126\left[Q + \frac{1}{Q}\right] + 160\left[Q^2 + \frac{1}{Q^2}\right] - 18\left[Q^3 + \frac{1}{Q^3}\right]\right. \\ &\left.\left. +324 + 9\left[Q^4 + \frac{1}{Q^4}\right]\right] - \left[P^3 + \frac{5^3}{P^3}\right]\left[11 + 4\left[Q^2 + \frac{1}{Q^2}\right]\right]\right\}. \end{aligned} \tag{43}$$

Proof. Replacing q by $-q$ in the equation (16), we deduce that

$$\begin{aligned} &- 16qf^2(-q^2)f^2(-q^{10}) = \varphi^4(-q^5) \\ &\times \left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right]\left[5 - \frac{\varphi^2(-q)}{\varphi^2(-q^5)}\right]. \end{aligned} \tag{44}$$

Using the equations (44) and (16), we find that

$$\frac{\varphi^4(q^5)}{\varphi^4(-q^5)} = \frac{\left[\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right]\left[\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)} - 5\right]}{\left[\dfrac{\varphi^2(q)}{\varphi^2(q^5)} - 1\right]\left[5 - \dfrac{\varphi^2(q)}{\varphi^2(q^5)}\right]}. \tag{45}$$

Replacing q by q^{11} in the above equation (45), we deduce that

$$\frac{\varphi^4(q^{55})}{\varphi^4(-q^{55})} = \frac{\left[\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 1\right]\left[\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 5\right]}{\left[\dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})} - 1\right]\left[5 - \dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right]}. \tag{46}$$

Employing the equation (17) along with the equations (45) and (46), we deduce that

$$\left(\frac{\psi(q^5)\psi(q^{55})}{\psi(-q^5)\psi(-q^{55})}\right)^8=\frac{\left(\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)}-1\right)\left(\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)}-5\right)}{\left(\dfrac{\varphi^2(q)}{\varphi^2(q^5)}-1\right)\left(5-\dfrac{\varphi^2(q)}{\varphi^2(q^5)}\right)}\times\frac{\left(\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})}-1\right)\left(\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})}-5\right)}{\left(\dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})}-1\right)\left(5-\dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right)}. \tag{47}$$

The equation (18) can be re arranged as,

$$\left[\frac{\psi(q^5)\psi(q^{55})}{\psi(-q^5)\psi(-q^{55})}\right]^8=\frac{\left[\dfrac{\psi(-q^{11})}{q^{\frac{11}{2}}\psi(-q^{55})}+\dfrac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}\right]^8}{\left[\dfrac{\psi(q^{11})}{q^{\frac{11}{2}}\psi(q^{55})}-\dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}\right]^8}. \tag{48}$$

Using the equations (47) and (48), we deduce that

$$\frac{\left[\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)}-1\right]\left[\dfrac{\varphi^2(-q)}{\varphi^2(-q^5)}-5\right]}{\left[\dfrac{\varphi^2(q)}{\varphi^2(q^5)}-1\right]\left[5-\dfrac{\varphi^2(q)}{\varphi^2(q^5)}\right]}\frac{\left[\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})}-1\right]\left[\dfrac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})}-5\right]}{\left[\dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})}-1\right]\left[5-\dfrac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right]}=\frac{\left[\dfrac{\psi(-q^{11})}{q^{\frac{11}{2}}\psi(-q^{55})}+\dfrac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}\right]^8}{\left[\dfrac{\psi(q^{11})}{q^{\frac{11}{2}}\psi(q^{55})}-\dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}\right]^8}. \tag{49}$$

Using the equations (28), (30) and (31) in the above equation (49), we arrive at the equation (43). This completes the proof. □

4. Explicit Evaluations for $h_{5,n}$ and $h'_{5,n}$

In this section, we establish several general formulas for the explicit evaluations of $h_{5,n}$ by employing the $P-Q$ modular equations of degree 5 established in Section 3.

Theorem 23. *If* $X := h_{5,n}$ *and* $Y := h_{5,4n}$, *then*

$$\begin{aligned}&\left(\frac{X}{Y}\right)^2+\left(\frac{Y}{X}\right)^2+5\left((XY)^2+\frac{1}{(XY)^2}\right)+16\left(\frac{X}{Y}-\frac{Y}{X}\right)\\&=2\sqrt{5}\left\{\left(X^2+\frac{1}{X^2}\right)+\left(Y^2+\frac{1}{Y^2}\right)\right\}+4.\end{aligned}\tag{50}$$

Proof. Employing the equation (33) with $k=5$ in the equation (9), we arrive at the equation (50). □

Corollary 24. *We have*

$$h_{5,2}=(\sqrt{2}-1)\sqrt{\sqrt{5}+2},\tag{51}$$

$$h_{5,1/2}=(\sqrt{2}+1)\sqrt{\sqrt{5}-2},\tag{52}$$

$$\begin{aligned}h_{5,4}=&\sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}}\\&-\sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}},\end{aligned}\tag{53}$$

$$\begin{aligned}h_{5,1/4}=&\sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}}\\&+\sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}},\end{aligned}\tag{54}$$

$$\begin{aligned}h_{5,8}=&\sqrt{37\sqrt{5}+58\sqrt{2}+26\sqrt{10}+82}\\&-\sqrt{34\sqrt{5}+76+24\sqrt{10}+54\sqrt{2}},\end{aligned}\tag{55}$$

$$\begin{aligned}h_{5,1/8}=(3-2\sqrt{2})(\sqrt{5}-2)&\left[\sqrt{37\sqrt{5}+58\sqrt{2}+26\sqrt{10}+82}\right.\\&\left.+\sqrt{34\sqrt{5}+76+24\sqrt{10}+54\sqrt{2}}\right].\end{aligned}\tag{56}$$

Proofs of (51) *and* (52). Putting $n=1/2$ in the equation (50) and using the fact that $h_{5,2}h_{5,1/2}=1$, we deduce that

$$(h_{5,2}^4-12h_{5,2}^2-6\sqrt{5}h_{5,2}^2+9+4\sqrt{5})(h_{5,2}^2-2+\sqrt{5})^2=0.\tag{57}$$

Since the roots of the second factor are imaginary and $h_{5,2} > 0$, we deduce that

$$h_{5,2}^4 - (12 + 6\sqrt{5})h_{5,2}^2 + 9 + 4\sqrt{5} = 0. \tag{58}$$

On solving the above equation (58), we arrive at the equations (51) and (52). □

Proofs of (53) *and* (54). Putting $n = 1$ in the equation (50) and using the fact that $h_{5,1} = 1$, we deduce that

$$x^2 - (6 + 2\sqrt{5})x - 2\sqrt{5} - 2 = 0, \text{ where } x := h_{5,4} - \frac{1}{h_{5,4}}. \tag{59}$$

Since $0 < x < 1$, we deduce that

$$h_{5,4} - \frac{1}{h_{5,4}} = 3 + \sqrt{5} - 2\sqrt{4 + 2\sqrt{5}}. \tag{60}$$

On solving the above equation (60), we arrive at the equations (57) and (58). □

Proofs of (55) *and* (56). Using the equation (51) in the equation (50), we arrive at the equations (55) and (56). □

Theorem 25. *If $X = h_{5,n}$ and $Y = h'_{5,16n}$, then*

$$\begin{aligned}
&\frac{X^4}{Y^4} + \frac{Y^4}{X^4} + 24\left(\frac{X^2}{Y^2} + \frac{Y^2}{X^2}\right) + 40\left(X^2Y^2 + \frac{1}{X^2Y^2}\right) + 120 \\
&+ 15\left(Y^4 + \frac{1}{Y^4}\right) = \sqrt{5}\left[20\left(X^2 + \frac{1}{X^2}\right) + 32\left(Y^2 + \frac{1}{Y^2}\right)\right. \\
&\left. +5\left(X^2Y^4 + \frac{1}{X^2Y^4}\right) + 3\left(\frac{X^2}{Y^4} + \frac{Y^4}{X^2}\right)\right].
\end{aligned} \tag{61}$$

Proof. Using the equation (36) and then employing the equations (9) and (10) with $k = 5$, we arrive at the equation (61). □

Corollary 26.

$$h'_{5,2} = \sqrt{\sqrt{5} - 1}, \tag{62}$$

$$h'_{5,4} = \frac{\sqrt{\sqrt{5} + 1 - \sqrt{2}\sqrt{\sqrt{5} + 1}}}{\sqrt{2}}, \tag{63}$$

$$h'_{5,8} = \sqrt{\sqrt{2} - 1}, \tag{64}$$

$$h'_{5,16} = \left[\frac{\sqrt{11+5\sqrt{5}}}{2} - \frac{\sqrt{(\sqrt{11+5\sqrt{5}}-4)\sqrt{11+5\sqrt{5}}}}{2} - 1 \right]^{1/2}, \tag{65}$$

$$\begin{aligned} h'_{5,32} = \Bigg[& \sqrt{34+11\sqrt{10}+15\sqrt{5}+24\sqrt{2}} \\ & -\sqrt{32+10\sqrt{10}+14\sqrt{5}+22\sqrt{2}} \Bigg]^{1/2}. \end{aligned} \tag{66}$$

Proof. Putting $n = 1/8, 1/4, 1/2, 1, 2$ in the equation (61) and using the values of $h_{5,1/8}, h_{5,1/4}, h_{5,1/2}, h_{5,1}$ and $h_{5,2}$, we arrive at the equations (62), (63), (64), (65) and (66) respectively. This completes the proof. ☐

Theorem 27. *If $X = h_{5,n}$ and $Y = h_{5,25n}$, then*

$$\begin{aligned} & \frac{Y^3}{X^3} - \frac{5Y^2}{X^2} - \frac{15Y}{X} + 5\sqrt{5}\left(XY + \frac{1}{XY}\right) + 5\sqrt{5}\left(Y^2 + \frac{1}{X^2}\right) \\ & = 5\left(X^2Y^2 + \frac{1}{X^2Y^2}\right) + 15. \end{aligned} \tag{67}$$

Proof. Using the equation (9) with $k = 5$ in the equation (38), we obtain (67). ☐

Corollary 28. *We have*

$$h_{5,5} = \sqrt{5-2\sqrt{5}}, \tag{68}$$

$$h_{5,1/5} = \frac{\sqrt{5+2\sqrt{5}}}{\sqrt{5}}. \tag{69}$$

Proofs of (68) *and* (69). Putting $n = 1/5$ in the equation (67) and using the fact that $h_{5,5}h_{5,1/5} = 1$, we deduce that

$$(h_{5,5}^2 - 5 + 2\sqrt{5})(h_{5,5}^2 - \sqrt{5})^2 = 0. \tag{70}$$

Since $0 < h_{5,5} < 1$, Hence $h_{5,5}^2 - 5 + 2\sqrt{5} = 0$. ☐

Theorem 29. *If* $X = h_{5,n}h_{5,121n}$ *and* $Y = \dfrac{h_{5,n}}{h_{5,121n}}$, *then*

$$\begin{aligned}
&Y^6+\frac{1}{Y^6}+33\left[Y^5+\frac{1}{Y^5}\right]-99\left[Y^4+\frac{1}{Y^4}\right]+1529\left[Y^3+\frac{1}{Y^3}\right]\\
&-1683\left[Y^2+\frac{1}{Y^2}\right]+8800\left[Y+\frac{1}{Y}\right]=25\sqrt{5}\left[X^5+\frac{1}{X^5}\right]\\
&-11\sqrt{5}\left\{5\sqrt{5}\left[X^4+\frac{1}{X^4}\right]\left[Y+\frac{1}{Y}\right]-5\left[X^3+\frac{1}{X^3}\right][11\right.\\
&+4\left[Y^2+\frac{1}{Y^2}\right]]-\sqrt{5}\left[X^2+\frac{1}{X^2}\right]\left[18-56\left[Y+\frac{1}{Y}\right]\right.\\
&\left.+3\left[Y^2+\frac{1}{Y^2}\right]-8\left[Y^3+\frac{1}{Y^3}\right]\right]-\left[X+\frac{1}{X}\right]\left[324-126\left[Y+\frac{1}{Y}\right]\right.\\
&\left.\left.+160\left[Y^2+\frac{1}{Y^2}\right]-18\left[Y^3+\frac{1}{Y^3}\right]+9\left[Y^4+\frac{1}{Y^4}\right]\right]\right\}+6534.
\end{aligned} \tag{71}$$

Proof. Employing the equation (9) with $k = 5$ in the equation (43), we obtain (71). □

Corollary 30. *We have*

$$h_{5,11}=\frac{\sqrt{3+\sqrt{5}-\sqrt{6\sqrt{5}-2}}}{2}, \tag{72}$$

$$h_{5,1/11}=\frac{\sqrt{3+\sqrt{5}+\sqrt{6\sqrt{5}-2}}}{2}. \tag{73}$$

Proofs of (72) *and* (73). Putting $n = 1/5$ in the equation (71) and using the fact that $h_{5,11}h_{5,1/11} = 1$, we deduce that

$$\begin{aligned}
&(h_{5,11}^4+(24+10\sqrt{5})h_{5,11}^2+1)(2h_{5,11}^4-(3+\sqrt{5})h_{5,11}^2+2)\\
&(2h_{5,11}^8-(3+3\sqrt{5})h_{5,11}^6-(6\sqrt{5}-30)h_{5,11}^4-(3+3\sqrt{5})h_{5,11}^2+2)\\
&(h_{5,11}^4+(6-4\sqrt{5})h_{5,11}^2+1)^2=0.
\end{aligned} \tag{74}$$

By observing the behaviour of the factors of the above equation (74), the second factor vanishes for specific value of $q = e^{-\pi\sqrt{11/5}}$, where as the other factors does not vanish. Hence

$$2h_{5,11}^4-(3+\sqrt{5})h_{5,11}^2+2=0. \tag{75}$$

On solving the above equation (75) and $0 < h_{5,11} < 1$, we obtain the equation (72). □

5. Modular Relations between $\mu(q)$ and $\mu(q^n)$

In this section, we establish several new modular relations connecting $\mu(q)$ with $\mu(q^n)$ using the $P-Q$ modular equations obtained in Section 3.

Theorem 31. *If* $u := \mu(q)$ *and* $v := \omega(q)$, *then*

$$(v-1)u^2 + (v^2 - 4v + 1)u - v^2 + v = 0. \tag{76}$$

Proof. Using the equations (30), (14) we arrive at the equation (76). □

Theorem 32. *If* $u := \mu(q)$ *and* $v := \mu(q^2)$, *then*

$$\begin{aligned}&(u^4 - 3u^3 + 5u^2 - 3u)v^3 - (u^4 + 10u^2 - 5u - 5u^3 + 1)v^2\\&- (3u^3 - 5u^2 + 3u - 1)v - u^2 + u^3 + (u - u^2)v^4 = 0.\end{aligned} \tag{77}$$

Proof. Using the equations (33), (14) we arrive at the equation (77). □

Theorem 33. *If* $u := \mu(q)$ *and* $v := \mu(q^3)$, *then*

$$\begin{aligned}&(3u^2 - 3u^3 + u^4 + 3u)v^3 - 3(u + u^3)v^2\\&+ (1 - 3u + 3u^3 + 3u^2)v = uv^4 + u^3.\end{aligned} \tag{78}$$

Proof. Using the equations (29), (14) we arrive at the equation (78). □

Theorem 34. *If* $u := \mu(q)$ *and* $v := \mu(q^4)$, *then*

$$\begin{aligned}&u^7 - u^8 + (-u - u^5 + 1 + 6u^2 - 10u^3 + 5u^4)v^7 + (7u^5 - 31u^2 + 6u\\&+ 45u^3 - 27u^4)v^6 + (7u^6 - 42u^5 - 91u^3 - u^7 + 92u^4 - 10u + 45u^2)v^5\\&+ (u-1)v^8 + (92u^5 - 27u^6 - 27u^2 + 92u^3 - 142u^4 + 5u + 5u^7)v^4\\&+ (7u^2 - 42u^3 + 45u^6 + 92u^4 - u - 10u^7 - 91u^5)v^3 + (45u^5 - 31u^6\\&+ 7u^3 - 27u^4 + 6u^7)v^2 + (u^8 - u^3 + 6u^6 + 5u^4 - 10u^5 - u^7)v = 0.\end{aligned} \tag{79}$$

Proof. Using the equations (36), (14) we arrive at the equation (79). □

Theorem 35. *If* $u := \mu(q)$ *and* $v := \omega(q^4)$, *then*

$$\begin{aligned}&(u^8 - 4u^7 + 6u^2 + 6u^6 + 1 - 4u - 14u^4)(v^7 + v) + (6u^8 + 32u^3\\&+ 6 - 32u - 136u^4 + 48u^2 + 32u^5 - 32u^7 + 48u^6)(v^6 + v^2) + (15\\&+ 80u^3 + 15u^8 + 80u^5 - 434u^4 + 202u^6 - 108u - 108u^7 + 202u^2)\\&\times (v^5 + v^3) + (16u + 16u^7 - 2 - 32u^2 - 16u^3 + 75u^4 - 32u^6\\&- 2u^8 - 16u^5)10v^4 + u^4 + u^4v^8 = 0.\end{aligned} \tag{80}$$

Proof. Using the equations (37), (15) we arrive at the equation (80). □

Theorem 36. *If* $u := \mu(q)$ *and* $v := \mu(q^5)$, *then*

$$\begin{aligned}&(11v^2 - 6v^3 + v^4 - 6v + 1)u^5 + (-35v^2 - 5v^4 + 25v^3 + 10v)u^4\\&+ (5v + 20v^2 - 25v^3 + 10v^4)u^3 + (-5v^4 - 20v^3 - 10v + 25v^2)u^2\\&+ (35v^3 + 5v - 25v^2 - 10v^4)u - v + 6v^2 + 6v^4 - v^5 - 11v^3 = 0.\end{aligned} \tag{81}$$

Proof. Using the equations (38), (14) we arrive at the equation (81). □

Theorem 37. *If* $u := \mu(q)$ *and* $v := \mu(q^7)$, *then*

$$\begin{aligned}&v + u^8v^7 - v^8u - u^7 + 7\left\{(v - 3v^2 + 4v^3 - 4v^5 + 2v^6 - v^7)u^7\right.\\&+ (3v^7 - 2v + 14v^2 - 26v^3 + 29v^5 - 14v^6)u^6 + (-5v^4 - 55v^5\\&+ 4v - 29v^2 + 55v^3 + 26v^6 - 4v^7)u^5 + 5(v^5 + v^3)u^4 + (4v^7 - 4v\\&- 29v^6 + 55v^5 - 5v^4 + 26v^2 - 55v^3)u^3 + (14v^6 - 14v^2 + 29v^3\\&\left.+3v - 26v^5 - 2v^7)u^2 + (v^7 - v + 2v^2 - 4v^3 - 3v^6 + 4v^5)u\right\} = 0.\end{aligned} \tag{82}$$

Proof. Using the equations (39), (14) we arrive at the equation (82). □

Theorem 38. *If* $u := \mu(q)$ *and* $v := \mu(q^{11})$, *then*

$$\begin{aligned}&v^{12} - uv - u^{11}v^{11} + 11\{(14u^8 - 4u + 8u^2 + u^{10} - 21u^7 - 32u^4\\&+ 21u^5 + 8u^3 + 9u^6 - 5u^9)v^{11} + (8u - 214u^8 + 83u^9 - 15u^{10}\\&+ 247u^7 + 6u^2 - 211u^5 - 39u^6 + u^{11} - 128u^3 + 268u^4)v^{10}\\&+ (8u + 538u^3 - 489u^9 + 83u^{10} - 81u^6 + 1292u^8 - 1393u^7\\&+ 1321u^5 - 1166u^4 - 5u^{11} - 128u^2)v^9 + (268u^2 - 4284u^5\\&- 214u^{10} - 3638u^8 + 1292u^9 - 1166u^3 + 4176u^7 + 3251u^4\\&+ 14u^{11} - 32u + 378u^6)v^8 + (4176u^8 - 211u^2 - 5452u^7 + 21u\\&+ 5452u^5 - 1393u^9 - 21u^{11} - 4284u^4 + 247u^{10} + 1321u^3\\&+ 72u^6)v^7 + (9u - 81u^9 - 81u^3 - 39u^2 + 72u^7 + 72u^5 - 39u^{10}\\&+ 378u^8 + 378u^4 + 9u^{11} - 594u^6)v^6 + (21u^{11} + 247u^2 - 21u\end{aligned} \tag{83}$$

$$
\begin{aligned}
&- 5452u^5 + 5452u^7 + 4176u^4 + 1321u^9 - 1393u^3 - 211u^{10} \\
&- 4284u^8 + 72u^6)v^5 + (14u - 3638u^4 + 378u^6 - 32u^{11} - 214u^2 \\
&+ 3251u^8 + 268u^{10} + 1292u^3 + 4176u^5 - 4284u^7 - 1166u^9)v^4 \\
&+ (1292u^4 - 128u^{10} - 5u - 489u^3 - 1393u^5 + 1321u^7 + 8u^{11} \\
&- 1166u^8 + 83u^2 - 81u^6 + 538u^9)v^3 + (8u^{11} - 15u^2 - 211u^7 \\
&+ 6u^{10} + 83u^3 - 214u^4 + 268u^8 + u + 247u^5 - 39u^6 - 128u^9)v^2 \\
&+ (8u^{10} + 21u^7 - 32u^8 - 5u^3 - 21u^5 + 9u^6 + u^2 - 4u^{11} + 8u^9 \\
&+ 14u^4)v\} + u^{12} = 0.
\end{aligned}
$$

Proof. Using the equations (43), (14) we arrive at the equation (83). □

6. Reciprocity Theorems and Explicit Evaluations for $\mu(q)$

In this section, we establish new reciprocity theorems for $\mu(q)$ and establish several new explicit evaluations of $\mu(q)$ by using the values of $h_{5,n}$ established in Section 4.

Theorem 39. *If* $5\alpha\beta = 1$, *then*

$$\frac{\mu(e^{-\pi\alpha}) + \mu(e^{-\pi\beta})}{1 + \mu(e^{-\pi\alpha})\mu(e^{-\pi\beta})} = \frac{3 - \sqrt{5}}{2}. \tag{84}$$

Proof. Set $q = e^{-\pi\alpha}$ in the equation (14), we deduce that

$$\frac{\varphi(e^{-\pi\alpha})}{\varphi(e^{-5\pi\alpha})} = \frac{1 + \mu(e^{-\pi\alpha})}{1 - \mu(e^{-\pi\alpha})}. \tag{85}$$

Replace α by β in the above equation (85), we find that

$$\frac{\varphi(e^{-\pi\beta})}{\varphi(e^{-5\pi\beta})} = \frac{1 + \mu(e^{-\pi\beta})}{1 - \mu(e^{-\pi\beta})}. \tag{86}$$

Using the equations (26), (85) and (86), we arrive at the equation (84). □

Theorem 40. *If* $5\alpha\beta = 1$, *let* $x := \phi(e^{-\pi\alpha})$ *and* $y := \phi(e^{-\pi\beta})$ *where* $\phi(q) = \mu(-q)$, *then*

$$
\begin{aligned}
&1 + x + x^4y - 4x^2 - 4y^2 + 6x^3y - 14x^3y^2 - 14x^2y - 64x^2y^2 \\
&- 14xy^2 - 4x^4y^2 + 6x^3y^3 + x^3y^4 - 14x^2y^3 - 4x^2y^4 + 6xy^3 \\
&+ 6xy + xy^4 + x^4y^3 + x^4y^4 + x^3 + x^4 + y^3 + y^4 + y = 0.
\end{aligned} \tag{87}
$$

Proof. Putting $q = e^{-\pi\alpha}$ in (12) and change q to $-q$, we deduce that

$$\frac{f^3(-e^{-2\pi\alpha})}{e^{-\pi\alpha} f^3(-e^{-10\pi\alpha})} = \left(\frac{1-x}{1+x}\right)\left(\frac{x^2+3x+1}{x}\right). \tag{88}$$

Replacing α by β in (88), we find that

$$\frac{f^3(-e^{-2\pi\beta})}{e^{-\pi\beta} f^3(-e^{-10\pi\beta})} = \left(\frac{1-y}{1+y}\right)\left(\frac{y^2+3y+1}{y}\right). \tag{89}$$

Using (27) in (88) and (89), we find that

$$\begin{aligned}&(x^2y^2+3x^2y+x^2+3x+1+4xy+3y+3xy^2+y^2)\\&(1+x+y+x^4y-4x^2-4y^2+6x^3y-14x^3y^2-14x^2y\\&-64x^2y^2+6xy-14xy^2-4x^4y^2+6x^3y^3+x^3y^4-14x^2y^3\\&-4x^2y^4+6xy^3+xy^4+x^4y^3+x^4y^4+x^3+x^4+y^3+y^4)=0.\end{aligned} \tag{90}$$

As $x^2y^2+3x^2y+x^2+3x+1+4xy+3y+3xy^2+y^2 \neq 0$, hence the second factor is zero. This completes the proof □

Lemma 41. *We have*

$$\mu(e^{-\pi\sqrt{n/5}}) = \frac{5^{1/4}h_{5,n}-1}{5^{1/4}h_{5,n}+1}, \quad \textit{where } n \textit{ is any positive rational.} \tag{91}$$

Proof. Using the equations (9) with $n = 5$ and (14), we arrive the equation (91) □

Lemma 42.

$$\omega(e^{-\pi\sqrt{n/5}}) = \frac{5^{1/4}h'_{5,n}-1}{5^{1/4}h'_{5,n}+1}, \quad \textit{where } n \textit{ is any positive rational.} \tag{92}$$

Proof. Using the equations (10) with $n = 5$ and (15), we arrive the equation (92) □

We establish some explicit evaluations of $\mu(q)$ by using the values of $h_{5,n}$ in the following Theorem.

Theorem 43. *We have*

$$\mu(e^{-\pi/\sqrt{5}}) = \frac{5^{1/4}-1}{5^{1/4}+1},$$

$$\mu(e^{-\pi\sqrt{2/5}}) = \frac{5^{1/4}(\sqrt{2}-1)(\sqrt{2+\sqrt{5}})-1}{5^{1/4}(\sqrt{2}-1)(\sqrt{2+\sqrt{5}})+1},$$

$$\mu(e^{-\pi/\sqrt{10}}) = \frac{5^{1/4}(\sqrt{2}+1)(\sqrt{\sqrt{5}-2})-1}{5^{1/4}(\sqrt{2}+1)(\sqrt{\sqrt{5}-2})+1},$$

$$\mu(e^{-\pi}) = \frac{5^{1/4}(\sqrt{5-2\sqrt{5}})-1}{5^{1/4}(\sqrt{5-2\sqrt{5}})+1},$$

$$\mu(e^{-\pi/5}) = \frac{\sqrt{5+2\sqrt{5}}-5^{1/4}}{(\sqrt{5+2\sqrt{5}})+5^{1/4}},$$

$$\mu(e^{-\pi\sqrt{11/5}}) = \frac{5^{1/4}(3+\sqrt{5}-\sqrt{6\sqrt{5}-2})^{1/2}-2}{5^{1/4}(3+\sqrt{5}-\sqrt{6\sqrt{5}-2})^{1/2}+2},$$

$$\mu(e^{-\pi/\sqrt{55}}) = \frac{5^{1/4}(3+\sqrt{5}+\sqrt{6\sqrt{5}-2})^{1/2}-2}{5^{1/4}(3+\sqrt{5}+\sqrt{6\sqrt{5}-2})^{1/2}+2}.$$

Remark 1. Using the equation (92) and values of $h'_{5,n}$, one can evaluate the explicit evaluations of $\omega(q)$.

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