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ANTHOLOGY OF SPLINE BASED NUMERICAL TECHNIQUES FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract: Spline methods provide an important tool to solve singularly perturbed boundary value problems. The present paper gives a comprehensive review of computational methods based on splines used in the solution of various classes of singularly perturbed problems such as self adjoint, linear, non-linear, semi-linear, quasi-linear, singular, single parameter and multi parameter. The spline based numerical techniques used by various researchers included in this paper can be classified as polynomial spline methods, non polynomial spline methods, B-spline methods, exponential spline methods and spline in tension methods. The emphasis of present paper is to accumulate the precis of recent works published during the years 2006-mid 2011 on spline methods for singular perturbation problems.

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1. Introduction

A problem described by a differential equation involving a small parameter ε is called a singular perturbation problem, if the order of differential equation becomes lower for $\varepsilon = 0$ than for $\varepsilon \neq 0$. Obviously, the small parameter ε multiplies the highest derivative of the differential equation. On the contrary, in a regular perturbation problem the small parameter is not responsible for the reduction in order of the problem. An example of second order singularly perturbed boundary value problem can be written as

$$\varepsilon u''(x) = \varphi(x, u, u'),\ \alpha_0 u(a) + \alpha_1 u'(a) = \gamma_1,\ \ \beta_0 u(b) + \beta_1 u'(b) = \gamma_2. \qquad (1)$$

Many phenomena in physics, biology, chemistry, engineering, bionomics, and so on, can be described by singular perturbation problems associated with various types of differential equations. These singularly perturbed problems arise in the modeling of various modern complicated processes, such as fluid flow at high Reynolds numbers, chemical reactor theory, electromagnetic field problem in moving media, electro-analytical chemistry, water quality problems in rivers networks, convective heat transport problem with large Peclet numbers, drift diffusion equation of semiconductor device modeling, financial modeling of option pricing, turbulence model, simulation of oil extraction from underground reservoirs, theory of plates and shells, atmospheric pollution, groundwater transport etc. The mathematical models describing these phenomena contain a small parameter and the influence of this parameter reveals itself in a sudden change of the dependent variable, taking place within a small layer. The solution of the problem of the boundary layer flow and that of the elastic plate is characterized by the fact that the perturbation with small parameter has an observable effect only in the neighborhood of the boundary and, therefore, one uses the term “singular perturbations of boundary layer type”. However, it can also happen that the perturbation is observable in a thin layer not in the neighborhood of some boundary or edge and in this case we have a “singular perturbation of free layer type”. These thin layers are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, and transition points in quantum mechanics.

Present paper surveys the numerical methods using splines for singularly perturbed boundary value problems. Remaining part of the paper is organized as follows: Section 2 contains the description of nature of solution and available numerical methods for such problems. Section 3 depicts the development of solution of singular perturbation problems using splines, whereas section 4 comprises of recent developments in this area. Paper is concluded in section 5.

2. Numerical Methods for Solving Singular Perturbation Problems

It is a well known fact that the solution of singularly perturbed boundary value problem exhibits a multiscale character. That is, there is a thin layer, where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. Therefore usual numerical treatment of singular-perturbation problems gives major computational difficulties, and in recent years, a large number of special purpose methods have been developed to provide accurate numerical solution. The occurrence of sharp boundary layers as the coefficient of highest derivative approaches zero creates difficulty for most standard numerical methods. There are a wide variety of asymptotic expansion methods available for solving the problems of the above type. However, difficulties in applying these methods, such as finding the approximate asymptotic expansions in inner and outer regions are not easy but require insight and experimentations. There are two possibilities to obtain small truncation error inside the boundary layer. The first is to choose a fine mesh whereas the second one is to choose a different formula reflecting the behavior of solution inside the boundary layer.

Many approximate methods have been developed and refined, including the finite difference methods, the method of averaging, the boundary layer method, the method of matched asymptotic expansion, the method of multiple scales, the finite element methods, spline methods etc. Collection of various numerical methods for singularly perturbed boundary value problems could be found in [12] and [28].

3. Solution of Singular Perturbation Problems Using Splines

In last 25 years remarkable progress has been made in the theory, methods, and applications for the singular perturbation in the mathematical circles and a lot of new results have appeared. To be more accessible for practicing engineers and applied mathematicians there is a need for methods, which are easy and ready for computer implementation. The spline technique appears to be an ideal tool to attain these goals.

There has been a considerable amount of work using various spline methods for solution of singular perturbation problems. During the last two decades of 20^{th} century, major contribution was made by K. Surla, M. Stojanovic, D. Herceg, V. Jerkovic [10], [11], [43], [44], [46], [47], [48], [49], [50], [51], [52],

[53], [54], V. Strygin, I.A. Blatov, I.Y. Pokornaya [45], M. Marusic, M. Rogina [31], R. Sacco, M. Stynes [42], D. Funaro, and G. Pontrelli [9]. The literature published during the first five years of 21^{st} century includes the contributions of P.A. Farrell [8], M.K. Kadalbajoo, K.C. Patidar, V. K. Aggarwal [12], [13], [14], [15], [16] M. Marusic [32], T. Aziz, A. Khan [1], [2], K. Surla, Z. Uzelac [55] and R. K. Bawa [3], [4].

4. Recent Developments (2006 – mid 2011)

In this section we present the collection of precis of recent papers for solving singular perturbation problems using splines. These have been arranged in chronological order.

In year 2006, Bawa [5] proposed a spline based computational technique suitable for parallel computing for singularly perturbed reaction-diffusion problems of the form

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in D = (0,1) \text{ with } u(0) = A, \ u(1) = B, \quad (2)$$

where $\varepsilon>0$ is a small parameter, b and f are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\bar{D} = [0,1]$. Under these assumptions, the above problems possess a unique solution $u(x) \in C^2(\bar{D})$. In general, the solution $u(x)$may exhibit two boundary layers of exponential type at both end points $x = 0,\ 1$. To solve, authors first decompose the domain into three non overlapping sub domains (two boundary layer regions and one outer region). To determine the boundary conditions, the zeroth-order asymptotic approximate solution are required. Authors take the transition parameter as $k\sqrt{\varepsilon},\ where\ k = 2 * \ln(N)$, which is the transition parameter used in the Shishkin-type meshes. The sub problems corresponding to boundary layer regions are solved using adaptive spline scheme.

Khan et al. [27] in year 2006, take $p(x) = p$ =constant and $0 < \varepsilon << 1$ for

$$\begin{aligned} &Lu = -\varepsilon u'' + p(x)u = f(x), \quad p(x) \geq p > 0, \ 0 < x < 1 \text{ with} \\ &u(0) = \alpha_0, \ u(1) = \alpha_1 \end{aligned} \quad (3)$$

and describe a sixth-order method based on sextic splines. The advantage of this method is higher accuracy with the same computational effort. It is a computationally efficient method and the algorithm can easily be implemented on a computer. Comparison with other existing methods demonstrates the superiority of the proposed method. It is observed that the maximum error is reduced significantly by the proposed scheme. The implementation of the scheme is simple and it has inherent parallelism as, each sub problem can be solved on three different processors and depending upon availability of number of processors, the domain D^2, which is quite large as compared to others, can be further subdivided into appropriate number of sub domains and can be applied for parallel computing.

Surla et al. [56] in year 2006 constructed a quadratic spline collocation method for a singularly perturbed boundary value problem with two small parameters. The suitable choice of collocation points provides the inverse monotonicity enabling utilization of barrier function method in the error analysis.

Rao and Kumar [34] in year 2007 presented a higher order cubic B-spline collocation method for the numerical solution of self-adjoint singularly perturbed boundary value problem that is much easier and more efficient for computing. The essential idea in this method is to divide the domain of the differential equation into three non-overlapping sub domains and solve the regular problems obtained by transforming the differential equation with respective boundary conditions on these sub domains using the present higher order B-spline collocation method. The boundary conditions at the transition points are obtained using the zeroth-order asymptotic approximation to the solution of the problem. Authors consider the two step spline approximate method because a special type of tridiagonal system is obtained. Authors discuss the convergence analysis of the method through matrix analysis approach and prove the optimal order convergence of the cubic B-spline collocation method. Also this method is more efficient than classical finite difference scheme on piecewise uniform Shishkin meshes as given by Farrell et al. [8] and reasonably comparable with the quintic spline difference scheme of Bawa and Natesan [3], in which quintic splines are considered to obtain fourth order convergence of the scheme. However, the present method gives fourth order convergence (optimal order convergence) with cubic B-splines only. In the same year, the same authors [35], presented B-spline collocation method on fitted mesh of Shishkin type for non-linear singularly perturbed boundary value problem

$$\varepsilon y''(x) = f(x, y, y'),\ a_0 y(a) - a_1 y'(a) = \gamma_1,\ \ b_0 y(b) + b_1 y'(b) = \gamma_2 \tag{4}$$

They used quasi-linearization technique to linearize the original non-linear singular perturbation problem. They show that the method has second-order uniform convergence.

Rashidinia et al. [39] in year 2007 developed the class of methods for the numerical solution of singularly perturbed two-point boundary value problems using spline in compression. The proposed methods are second-order and fourth-order accurate and applicable to problems both in singular and non-singular cases. Convergence analyses of the methods are discussed and numerical results are given to illustrate the efficiency of methods.

Rashidinia et al. [40] again in the same year proposed a numerical technique for a class of singularly perturbed two-point singular boundary value problems of the form

$$\begin{aligned} &-\varepsilon u''(x) + \tfrac{k}{x}u'(x) + g(x)u(x) = f(x),\ 0 \le x \le 1, \\ &u(0) = \gamma_0,\ u(1) = \gamma_1,\ where\ \ 0 < \varepsilon << 1 \end{aligned} \tag{5}$$

on an uniform mesh using polynomial cubic spline. The scheme derived in this paper is second-order accurate. The resulting linear system of equations has been solved by using a tri-diagonal solver. Numerical results are provided to illustrate the proposed method and to compare with the method by Kadalbajoo and Agarwal [15].

In the paper of M.A. Ramadan et al. [33] in year 2007, quadratic non polynomial spline functions are used to develop a class of numerical methods for solving singularly perturbed boundary value problems. These non polynomial spline functions need less coefficients evaluations over other methods. Convergence analysis of the method is discussed. The presented approach leads to a generalized scheme that has third-order and fourth-order convergence depending on the choice of the parameters involved in the method.

Tirmizi et al. [59] in year 2008 used Quartic non-polynomial spline functions to develop a class of numerical methods for solving self adjoint singularly perturbed problems. The methods are computationally efficient and the algorithm can easily be implemented on a computer. Fourth- and sixth-order convergence is obtained. It has been shown that the relative errors in absolute value confirm the theoretical convergence.

In their article [36] Rao and Kumar in year 2008 presented the exponential B-spline collocation method for the numerical solution of self-adjoint singularly perturbed Dirichlet boundary value problem of the form (2). Authors claim the their method is much easier for computing than the other schemes available in the literature because it is relatively simple to collocate the boundary value problem at the nodal points of the uniform mesh, to set up the collocation system and to solve them. The essential idea in this method is to use the B-splines basis for the space of exponential spline to approximate the solution of given problems via collocation approach. The proposed method is implemented on two test examples to demonstrate its efficiency. Examples show that the present method is more efficient than the cubic B-spline collocation method on uniform mesh as well as the cubic B-spline collocation method on fitted mesh. In case of cubic B-spline collocation method on uniform mesh as we decrease the value of ε, to study the behavior of the solution at the boundary layer, we need large number of nodal points in that region. For the purpose, Kadalbajoo [16] considered the fitted mesh approach that consists of large number of nodal points in the boundary layer region. However, in the present scheme the exponential B-spline collocation method on uniform mesh is considered that consists of less number of nodal points in the boundary layer region.

In year 2008, Kadalbajoo and Sharma [17] presented a numerical scheme for a second order singularly perturbed delay boundary value problem, which works nicely in both the cases, i.e., when delay argument is the bigger one as well as the smaller one. To handle the delay argument, they constructed a special type of mesh so that the term containing delay lies on nodal points after discretization. The proposed method is analyzed for stability and convergence. In the same year Kadalbajoo with Yadaw [18] presented a B-spline collocation method for solving a class of two-parameter singularly perturbed boundary value problems and established second order uniform convergence. Kadalbajoo and D. Kumar [19] in 2008 constructed a fitted mesh B-spline collocation method for a singularly perturbed boundary value problem for a differential-difference equation containing negative shift in the differentiated term

$$\begin{aligned} &\varepsilon y''(x) + p(x)y'(x-\delta) + q(x)y(x) = r(x), \\ &\text{with } y(x) = \varphi(x) \text{ for } -\delta \le x \le 0,\ y(1) = \gamma. \end{aligned} \tag{6}$$

They show that the method has almost second-order parameter-uniform convergence. Again,Kadalbajoo et al. [20] presented a comparative study of fitted-mesh finite difference method, B-spline collocation method and finite element method for singularly perturbed two-point boundary value problem in the same year. Kadalbajoo again with Gupta and Awasthi [21] in the same year proposed a numerical method for solving singularly perturbed parabolic convection–diffusion problem. The method comprises a standard implicit finite difference scheme to discretize in temporal direction on uniform mesh by means of Rothe's method and B-spline collocation method in spatial direction on piecewise uniform mesh of Shishkin type. Authors show that the method is unconditionally stable.

K. Surla et al. [57] considered in year 2009, the two-parameter singularly perturbed convection-diffusion-reaction boundary value problem

$$\begin{aligned} &Ly := \varepsilon y''(x) + \mu\, a(x)y'(x) - b(x)y(x) = f(x),\ \ x \in (0,1), \\ &\text{with } y(0) = \gamma_0\,, y(1) = \gamma_1. \end{aligned} \tag{7}$$

with two small parameters $0 < \varepsilon, \mu \ll 1$. When $\mu = 1$, the problem (7) becomes a convection-diffusion problem with the boundary layer of width $O(\varepsilon)$ in the neighborhood of $x = 0$. If $\mu = 0$, we have the reaction–diffusion problem with boundary layers of width $O\left(\sqrt{\varepsilon}\right)$at $x = 0$ and $x = 1$. Numerical method constructed in this paper is based on a spline difference operator defined on layer-adapted Shishkin mesh. The collocation points are chosen to provide the monotonicity of the proposed method. The convergence of order $O(N^{-2}\ln^2 N)$ is proved for the given method in the case of $\varepsilon, \mu \leq CN^{-1}$, Where C denotes a generic positive constant independent of ε and μ and the mesh points N. The problem is numerically treated by a quadratic spline collocation method. The suitable choice of collocation points provides the discrete minimum principle. Error bounds for the numerical approximations are established. Numerical results give justification of the parameter-uniform convergence of the numerical approximations. Same authors in [58] in same year considered spline finite difference scheme for a singularly perturbed one-dimensional convection-diffusion two-point boundary value problem. They proved point wise convergence of order $O(N^{-2}\ln^2 N)$inside the boundary layer and second order convergence elsewhere.

Idris Dag and Ali Sahin [7] in year 2009, considered the solution of the singularly perturbed problems of the form

$$\begin{aligned} &-\varepsilon u'' + p(x)u' + q(x)u = f(x),\ 0 \leq x \leq 1, \\ &\text{with}\ \ u(0) = \lambda,\ u(1) = \beta \end{aligned} \tag{8}$$

In this article, they used the finite element method with the quadratic and the cubic B-splines. After giving the expressions of the mentioned B-splines over the geometrically graded mesh authors applied the collocation method. To be able to use the quadratic B-splines in the collocation method, the setting $-u' = v$ gives a first order system of equations for equation (8). This system can be solved by employing the quadratic B-spline collocation method. Numerical results are illustrated for some test problems. According to authors, in getting the numerical solution of the differential equations having boundary layers, B-spline collocation methods over the geometrically graded mesh are advisable.

In year 2009, Lin et al. [29] explained that any function of $L^2(R)$can be expressed by the dilation and translation of wavelet functions, so it has drawn a great deal of attention from scientists and engineers. The stiffness matrix is sparse when it is used as trial functions. Wavelets especially adapt to solve the equation with the singular solution and a local severe gradient. Wavelets have many excellent properties such as orthogonality, compact support, exact representation of polynomials to a certain degree, flexibility to represent functions at different levels of resolution. B-spline functions are useful wavelet basis functions and based on piece polynomials that possess attractive properties:

piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination, which leads to matrices that are easier to diagonalize. Authors consider a B-spline collocation method for singularly perturbed boundary value problems of form (8) arising in biology. The accuracy of the proposed method is demonstrated by test problems.

Kadalbajoo and Arora [22] in year 2009 developed a B-spline collocation method using artificial viscosity for solving singularly-perturbed equations of the form (8). They use the artificial viscosity to capture the exponential features of the exact solution on a uniform mesh and use B-spline collocation method which leads to a tridiagonal linear system. The convergence analysis is given and the method is shown to have uniform convergence of second order. The design of artificial viscosity parameter is confirmed to be a crucial ingredient for simulating the solution of the problem. Known test problems have been studied to demonstrate the accuracy of the method. Numerical results show the behavior of the method with emphasis on treatment of boundary conditions. Results shown by the method are found to be in good agreement with the exact solution. In the same year Kadalbajoo and Gupta [23] constructed a B-spline collocation method on piecewise-uniform Shishkin mesh to solve a singularly perturbed convection-diffusion problem of the similar type. Authors derived the bounds for the derivative of the analytical solution by decomposing the solution into regular and singular parts. They showed that the presented method is boundary layer resolving as well as second-order uniformly convergent in the maximum norm.

M.K. Kadalbajoo and Yadaw [24] in same calendar year presented a Ritz-Galerkin finite element method for the solution of two-parameter singularly perturbed boundary value problems similar to form (7). Due to these two small parameters boundary layers exist. To resolve the boundary layers a piecewise-uniform Shishkin mesh has been taken. It is relatively simple to collocate the solution at the mesh points. The results obtained using this method are more accurate than the stated existing method with same numbers of nodal points and gives the order of convergence to be almost two. Also, authors did not divide one problem into three equivalent boundary value problems, as previously done by other authors, which is not economical. Also, the maximum absolute error on the whole domain is presented. Authors discussed the solution behavior for fixed value of ε and different values of μ and also for fix value of μ and different values of ε. Presented numerical results agree with the theoretical results.

Rao and Kumar [37] in year 2009, considered a class of singularly perturbed semi-linear reaction-diffusion problem

$$-\varepsilon^2 u''(x) + f(x, u(x)) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0 \tag{9}$$

They used exponential spline difference scheme on the basis of splines in tension to discretize the problem on piecewise-uniform Shishkin mesh. They established almost second order uniform convergence in the discrete maximum norm. The same authors [38] in year 2010 developed a B-spline collocation method on a piecewise-uniform Shishkin mesh to solve a class of singularly perturbed semi linear reaction-diffusion problems. The convergence analysis is given and the method is shown to be almost second-order convergent, uniformly with respect to the perturbation parameter ε in the maximum norm.

Kadalbajoo and Kumar [25] in year 2010 considered a model problem for a class of singularly perturbed nonlinear differential equations with negative shift

$$\begin{aligned} &\varepsilon y''(x) = F(x, y(x), y'(x-\delta)), \quad x \in [0,1] \\ &\text{with } y(x) = \phi(x) \text{ for } -\delta \le x \le 0, \ y(1) = \gamma \end{aligned} \tag{10}$$

where ε is the small singular perturbation parameter and $\delta(\varepsilon)(0 < \delta \ll 1)$is the shift of $o(\varepsilon)$. The case when $\delta(\varepsilon)$ is of order $o(\varepsilon)$ is under consideration by the authors. The B-spline collocation method for singularly perturbed nonlinear differential-difference equations with negative shift in the convection term has been carried out. The piecewise-uniform mesh is chosen to grasp the better approximation to the exact solution in the boundary layer region. if we use the uniform mesh for solving the singular perturbation problem, not only it requires a larger number of mesh points in the given interval, but also the approximate solution oscillates about the exact solution in the boundary layer region when the value of ε decreases. So, authors considered piecewise-uniform mesh with more mesh points in the boundary layer region. Taylor's series is used to tackle the term containing shift. The proposed numerical method is accurate of order almost two.

In the year 2010, Loghmani and Ahmadinia [30] considered the more general form of m^{th} order linear and nonlinear singularly perturbed differential equations

$$\varepsilon y^{(m)}(t) = f(t, y(t), y'(t), ..., y^{(m-1)}(t)), \quad a \le t \le b, \quad 0 < \varepsilon \ll 1 \tag{11}$$

with the perturbed separated boundary conditions $\sum_{j=0}^{m-1} c_{ij} y^{(j)}(a_i, \varepsilon) = A_i \, 1 \le i \le m$. In this paper authors propose a scheme based on B-spline functions and least square method to find a sequence of functions $\{v_k\}$ such that the exact boundary conditions are satisfied. Also, up to an error δ_k, the function v_k

satisfies the differential equation, where $\delta_k \to 0$ as k $\to \infty$. Several numerical examples for linear and nonlinear boundary value problems are considered to illustrate the accuracy and performance of the presented method. The motivation of authors' work is to consider any order differential equation with vast types of boundary conditions utilizing numerical techniques with optimal control strategy.

Rashidinia et al. [41] in year 2010 considered the self-adjoint singularly perturbed boundary value problem of the form (2). In this paper, non-polynomial quintic spline relations have been derived using off-step points. The spline function authors proposed has the form $T_5 = Span\{1, x, x^2, x^3, \sin\tau x, \cos\tau x\}$ where τ is the frequency of the trigonometric part of the splines function, which can be real or pure imaginary. Using non-polynomial quintic spline in grid points we can obtain the fourth-order method only. But authors developed the quintic spline in off step points to raise the order of accuracy. Based on such spline, the proposed methods are fourth, sixth and eighth-order accurate. These methods are applicable to problems both in singular and non-singular cases. The convergence analysis of the fourth-order method is established.

In the present year 2011, Kadalbajoo and Yadaw [26] have presented a comparative study of fitted-mesh finite difference method, Ritz-Galerkin finite element method and B-spline collocation method for a two-parameter singularly perturbed boundary value problem. Authors have taken a piecewise-uniform fitted mesh to resolve the boundary layers and shown that fitted-mesh finite difference method has almost first order parameter-uniform convergence, Ritz-Galerkin finite element method has almost second order parameter-uniform convergence and B-spline collocation method has second order parameter-uniform convergence.

recently Bisht and Khan [6] have applied the difference scheme using adaptive cubic spline for solving a self adjoint singularly perturbed two point boundary value problem of the form (2). Their scheme leads to a tri diagonal linear system. The convergence analysis is given which shows the methods is second and fourth order convergent depending upon the choice of parameters.

5. Conclusion

Studying the singular perturbation problems is a very attractive object in the contemporary mathematical circles. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed boundary value problems. To be more accessible for practicing engineers and applied mathematicians there is a need for methods, which are easy and ready for computer implementation. The spline techniques appear to be an ideal tool to attain these goals. This survey paper presented the development and chronological advancement of various spline methods for different classes of singularly perturbed problems. This paper will be extensively helpful to researchers working in this area to construct their new improved numerical methods for the solution of singularly perturbed problems.

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