International Journal of Pure and Applied Mathematics
Volume 74 No. 4 2012, 455-463
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A NEW CHARACTERIZATION FOR INCLINED CURVES BY THE HELP OF SPHERICAL REPRESENTATIONS ACCORDING TO BISHOP FRAME

Raheleh Ghadami[1], Yusuf Yayli[2 §]

[1]Department of Mathematics
Islamic Azad University Urmia Branch
Urmia, IRAN

[2]Department of Mathematics
Faculty of Science
University of Ankara
Tandoğan, Ankara, TURKEY

Abstract: In this paper we investigate spherical images the $N_1$and N_2 indicatrix of a slant helix. We obtain that the spherical images are spherical helices. Moreover, arc lengths of spherical representations of tangent vector field T, vector field N_1, vector field N_2 and the vector field $\overrightarrow{C} = \frac{\overrightarrow{w}}{\|\overrightarrow{w}\|}$, where $\overrightarrow{w} = -k_2N_1 + k_1N_2$ is the Darboux vector field of a space curve α in $E^3$are calculated. Let us denote the spherical representation of $\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}$ and $\overrightarrow{C}$ by $\left(\overrightarrow{T}\right), \left(\overrightarrow{N_1}\right), \left(\overrightarrow{N_2}\right)$ and $\overrightarrow{C}$ respectively.The arc element ds_c of the spherical representation $\left(\overrightarrow{C}\right)$ expressed in terms of the harmonic curvature $H = \frac{k_2}{k_1} = const$ is slant helix of bishop frame. Thus the following characterization is given. The curve $\alpha \subset E^3$ is an inclined curve if and only if the arc length s_c of the Darboux spherical representation $\left(\overrightarrow{C}\right)$ of α is constant.

AMS Subject Classification: 53A04, 53A99
Key Words: inclined curve, harmonic curvature, ordinary helix, slant helix, spherical helix

Received: October 31, 2011

[§]Correspondence author

1. Introduction

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique,we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point.If the derivatives of $(N_1(s), N_2(s))$ depond only on $T(s)$ and not each other we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},$$

$$\theta(s) = \arctan\left(\frac{k_2}{k_1}\right), \quad k_1 \neq 0,$$

$$\tau(s) = -\frac{d\theta(s)}{ds}.$$

So that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)\, ds\, \theta$. The orientation of the parallel tran sport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation (see [6]).In recent years, many important and intensive studies are seen about inclined curves. Papers in (see [1, 2, 3, 5, 6, 7]) show that how important field of interest inclined curves have. For this reason, we have given a new characterization for the inclined curves which satisfy $k_1 \neq e^{te}$ and $k_2 \neq e^{te}$, but $H = \frac{k_2}{k_1} = e^{te}$. This comes into light by means of spherical representations of α (see [2, 3, 4, 6]).

2. Characterization of Slant Helices According to Bishop Frame

2.1. The Arc Length of Tangentian Representation of the Curve $\alpha \subset E^3$

Let $T = T(s)$ be the tangent vector field of the curve

$$\begin{aligned} \alpha &: I \subset R \to E^3 \\ s &\to \alpha(s) \end{aligned}$$

The spherical curve $\alpha_T = \overrightarrow{T}$ on S^2 is called I.st spherical representation of the tangents of α.Let s be the arc length parameter of α. If we denote the arc length of the curve α_T by s_T, then we may write

$$\alpha_T(s_T) = \overrightarrow{T}(s)$$

Letting $\frac{d\alpha_T}{ds_T} = T_T$ we have $T_T = \left(k_1\overrightarrow{N_1} + k_2\overrightarrow{N_2}\right)\frac{ds}{ds_T}$.Hence we obtain $\frac{ds_T}{ds} = \kappa$. thus we give the following result. If κ is the first curvature of the curve $\alpha : I \to E^3$,then the arc length s_T of the tangentian α_T of α is

$$s_T = \int \kappa ds + c$$

If the harmonic curvature of α is $H = \frac{k_2}{k_1}$, we get

$$s_T = \int k_1\sqrt{1 + H^2}ds + c$$

where c is an integral constant. thus we have the following theorem.

Theorem 1. *$\alpha \subset E^3$ is an slant helix of Bishop frame if and only if*

$$s_T = k_1\sqrt{1 + H^2}s + c$$

2.2. The Arc Length of the $\overrightarrow{N_1}$ Representation of the Curve $\alpha \subset E^3$

Let $\overrightarrow{N_1} = \overrightarrow{N_1}(s)$ be first normal vector field of the curve

$$\begin{aligned} \alpha &: I \subset R \to E^3 \\ s &\to \alpha(s) \end{aligned}$$

The spherical curve $\alpha_{N_1} = \overrightarrow{N_1}$ on S^2 is called II.nd spherical representation for α or is called the spherical representation of the $\overrightarrow{N_1}$ of α.Let $s \in I$ be the arc lenght of α.If we denote the arc length of α_{N_1} by s_{N_1} we may write

$$\alpha_{N_1}\left(s_{N_1}\right) = \overrightarrow{N_1}\left(s\right)$$

Moreover letting $\frac{d\alpha_{N_1}}{ds_{N_1}} = T_{N_1}$, we obtain

$$T_{N_1} = \left(-k_1\overrightarrow{T}\right)\frac{ds}{ds_{N_1}}$$

Hence we have

$$\frac{ds_{N_1}}{ds} = k_1$$

If the harmonic curvature of α is $H = \frac{k_2}{k_1}$, we get

$$s_{N_1} = \int \frac{k_2}{H} ds + c$$

Thus we have the following theorem:

Theorem 2. *$\alpha \subset E^3$ is a slant helix of Bishop frame if and only if*

$$s_{N_1} = \frac{k_2}{H}s + c.$$

2.3. The Arc Length of $\overrightarrow{N_2}$ Representation of the Curve $\alpha \subset E^3$

Let $\overrightarrow{N_2} = \overrightarrow{N_2}\left(s\right)$ be second vector field of the curve

$$\begin{aligned} \alpha \quad &: \quad I \subset R \to E^3 \\ s \quad &\to \quad \alpha\left(s\right) \end{aligned}$$

The sperical curve $\alpha_{N_2} = \overrightarrow{N_2}$ on S^2 is called III.rd spherical representation of the $\overrightarrow{N_2}$ of α.Let $s \in I$ be the arc length parameter of α. If we denote the arc length parameter of α_{N_2} by s_{N_2}, we may write

$$\alpha_{N_2}\left(s_{N_2}\right) = \overrightarrow{N_2}\left(s\right)$$

Moreover letting $\frac{d\alpha_{N_2}}{ds_{N_2}} = T_{N_2}$, we obtain $T_{N_2} = \left(-k_2\overrightarrow{T}\right)\frac{ds}{ds_{N_2}}$. Hence we have $\frac{ds_{N_2}}{ds} = k_2$ and $s_{N_2} = \int k_2 ds + c$ or in terms of the harmonic curvature of α we obtain

$$s_{N_2} = \int Hk_1 ds + c$$

Thus we give the following theorem:

Theorem 3. *$\alpha \subset E^3$ is a slant helix of Bishop frame if and only if*

$$s_{N_2} = Hk_1 s + c$$

2.4. The Arc Length of Darboux Spherical Representation of the Curve $\alpha \subset E^3$

Let $\overrightarrow{w} = -k_2\overrightarrow{N_1} + k_1\overrightarrow{N_2}$ be the Darboux vector field of the curve

$$\begin{array}{ccc} \alpha & : & I \subset R \to E^3 \\ s & \to & \alpha(s) \end{array}$$

Let us define the curve $\alpha_C = \overrightarrow{C}$ on S^2 by the help of the vector field $\overrightarrow{C} = \frac{\overrightarrow{w}}{\|\overrightarrow{w}\|}$. This curve is called IV.th sperical representation of α or is called Darboux representation of α. Let s_C be the length of α_C. Then we have $\alpha_C = \overrightarrow{C}(s_c) = \frac{\overrightarrow{W}}{\|\overrightarrow{w}\|}$. Let us denote the angle between $\overrightarrow{w}$ and $(-\overrightarrow{N_1})$ by φ.

Hence

$$k_1 = \|\overrightarrow{w}\| \sin\varphi \quad \text{and} \quad k_2 = \|\overrightarrow{w}\| \cos\varphi \tag{1}$$

Therefore we may write

$$\overrightarrow{C} = \left(-\overrightarrow{N_1}\right)\cos\varphi + \left(\overrightarrow{N_2}\right)\sin\varphi$$

From this last equality we get

$$\frac{d\overrightarrow{C}}{ds_C} = \frac{d\overrightarrow{C}}{ds} \cdot \frac{ds}{ds_C}$$

or

$$\|\frac{d\overrightarrow{C}}{ds}\| = \frac{ds_C}{ds}$$

or

$$\begin{aligned} \frac{d\overrightarrow{C}}{ds} &= \left(-\overrightarrow{N_1}\right)(\cos\varphi)' + \left(\overrightarrow{N_2}\right)(\sin\varphi)' \\ &= \left(\left(\overrightarrow{N_1}\right)\sin\varphi + \left(\overrightarrow{N_2}\right)\cos\varphi\right)\frac{d\varphi}{ds} \end{aligned}$$

Hence we have

$$\|\frac{d\overrightarrow{C}}{ds}\| = \frac{d\varphi}{ds} = \frac{ds_C}{ds} \tag{2}$$

From this equations, in (1) we obtain

$$\frac{k_2}{k_1} = \tan\varphi \tag{3}$$

Therefore, differentiating with respect to s we have

$$\left(\frac{k_2}{k_1}\right)' = \left(1+\tan^2\varphi\right)\frac{d\varphi}{ds}$$

or

$$\left(\frac{k_2}{k_1}\right)' = \left[1+\left(\frac{k_2}{k_1}\right)^2\right]\frac{d\varphi}{ds}$$

From (3), since we have

$$\frac{d\varphi}{ds} = \frac{\left(\frac{k_2}{k_1}\right)'}{1+\left(\frac{k_2}{k_1}\right)^2}$$

and since we have $H = \frac{k_2}{k_1}$, we get

$$\frac{d\varphi}{ds} = \frac{H'}{1+H^2}$$

Hence from (2), we obtain

$$\frac{ds_C}{ds} = \frac{H'}{1+H^2}$$

or hence

$$ds_C = \frac{H'}{1+H^2}ds$$

$ds_C = \frac{H'}{1+H^2}ds$ implies that

$$s_c = \int \frac{H'}{1+H^2}ds + c$$

since $H' = \frac{dH}{ds}$implies $H'ds = dH$, then we have

$$s_C = \arctan H + c$$

Thus we give the following theorem:

Theorem 4. *The curve $\alpha \subset E^3$ is an inclined curve if and only if s_C =const.*

3. The Spherical Indicatricies of Slant Helices

In this section, we investigate spherical images the N_1 indicatrix and N_2 indicatrix of a slant heix.using of the (Moreover the curvatures of (see [4, 7]) We obtain that spherical images are spherical helices.

Theorem 5. *Let $\alpha, \beta : I \to E^3$ be two curves.Moreover the curvatures of α are κ, τ and the Bishop curvatures of α are $k_1, k_2$and the curvatures of β are k_1, k_2.α is a helix if and only if $\beta$is a slant helix.*

Proof. We denote the curvature of α and the Bishop curvatures of α and the curvatures of β are τ, κ and k_1, k_2 and k_1, k_2 respectively.

$$\tau = \left(\arctan \frac{k_2}{k_1}\right)' \tag{4}$$

and

$$\kappa\left(s\right) = \sqrt{k_1^2 + k_2^2} \tag{5}$$

Therefore we can calculate that

$$\frac{\tau}{\kappa} = -\frac{k_1^2}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}} \cdot \left(\frac{k_2}{k_1}\right)'$$

Since $\frac{\tau_{N_1}}{\kappa_{N_1}}$ =constant.This completes the proof. □

Theorem 6. *Let $\alpha, \beta : I \to E^3$ be two curves.The Bishop curvature of α is k_1 and k_2. Let curvature of β is k_1 and torsion of β is k_2.$\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$ is a Bishop frame of α Moreover the spherical image of the N_1 is (N_1). (N_1) is a spherical helix if and only if β is a slant helix.*

Proof. We denote the curvature of (N_1) by κ_{N_1} and the torsion of (N_1) by τ_{N_1}.Then we have

$$\kappa_{N_1} = \frac{\sqrt{k_1^2 + k_2^2}}{k_2} \tag{6}$$

and

$$\tau_{N_1} = \frac{\left(k_1' k_2 - k_1 k_2'\right)}{k_2\left(k_1^2 + k_2^2\right)} \tag{7}$$

Therefore we can calculate that

$$\frac{\tau_{N_1}}{\kappa_{N_1}} = \frac{k_1}{\left(k_1^2 + k_2^2\right)} \cdot \left(\frac{k_2}{k_1}\right)'$$

Since $\frac{\tau_{N_1}}{\kappa_{N_1}}$ =constant.This completes the proof. □

Theorem 7. *Let $\alpha, \beta : I \to E^3$ be two curves. The Bishop curvature of α is k_1 and k_2.Let curvature of β is k_1 and torsion of β is k_2.$\{T, N_1, N_2\}$ is a Bishop frame of α. The spherical image of the N_2 is (N_2). (N_2) is a spherical helix if and only if β is a slant helix.*

Proof. We denote the curvature of (N_2) by κ_{N_2} and the torsion of (N_2) by τ_{N_2}.Then we have

$$\kappa_{N_2} = \frac{\sqrt{k_1^2 + k_2^2}}{k_2} \tag{8}$$

and

$$\tau_{N_2} = \frac{(k_1' k_2 - k_1 k_2')}{k_2 \left(k_1^2 + k_2^2\right)} \tag{9}$$

Therefore we can calculate that

$$\frac{\tau_{N_2}}{\kappa_{N_2}} = -\frac{k_1^2}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}} \cdot \left(\frac{k_2}{k_1}\right)'$$

Since $\frac{\tau_{N_1}}{\kappa_{N_1}}$ =constant. This completes the proof. □

Theorem 8. *Let $\alpha : I \to E^3$ be a curve. The Bishop curvature of α is k_1 and k_2. Let curvature of (α) ,is k_1 and torsion of (α) is k_2 . Then β is a slant helix if and only if the curve $\gamma : I \subset R \to R^2$, $\gamma(s) = (\gamma_2(s), \gamma_2(s))$ is a circle. where $\gamma_1(s) = \int k_1(s)\, ds$ and $\gamma_2(s) = \int k_2(s)\, ds$.*

Proof. We can calculate that the curvature of the curve γ

$$\kappa_\gamma = \frac{\alpha_1' \alpha_2'' - \alpha_1'' \alpha_2'}{\left((\alpha_1')^2 + (\alpha_2')^2\right)^{\frac{3}{2}}} = \frac{k_1^2}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}} \cdot \left(\frac{k_2}{k_1}\right)'$$

This complete the proof. □

References

[1] M. Barros, General helices and a theorem of Lancert, *Proc. AMS*, **125** (1997), 1503-1509.

[2] H.H. Hacısalihoğlu, A new characterization for inclined curves by the help of spherical representations, *International Electronic Journal of Geometry*, **2**, No. 2 (2009), 71-75.

[3] J. Hanson Andrew, Hui Ma, Parallel transport approach to curve framing, *Indiana University, Techreports*, **TR425** (1995).

[4] M.K. Karacan, B. Bukcu, The slant helices according to Bishop frame, *International Journal of Computational and Mathematical Sciences*, **3**, No. 2 (2009).

[5] K. Sakomato, Helical immersions into a unit sphere, *Math. Ann.*, **261** (1982), 63-80.

[6] L.R. Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly*, **82**, No. 3 (1975), 246-251.

[7] L. Kula, Y. Yayli, On slant helix and its spherical indicatrix, *Applied Mathematics and Computation*, **169** (2005), 600-607.