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THE FEJER SUM OF THE BESSEL SERIES ON THE UNIT CIRCLE

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Abstract: This paper discusses the Fejer Summation of the Bessel series, and gives a new integral representation for the Fejer sum of the Bessel series.

Key Words: Fejer sum, Bessel series

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Suppose that $f(z)$ is a Lebesque inegrable on the unit circle Γ. The Bessel series on the unit circle is:

$$\frac{A_0(f)}{2}J_0(z)+\sum_{n=1‘}^{\infty}A_n(f)J_n(z), \quad z\in\Gamma, \tag{1}$$

where

$$A_n(f)=\frac{1}{\pi i}\oint_{\Gamma}f(\zeta)O_n(\zeta)d\zeta \quad (n=0,1,2,\cdots).$$

Denote the partial sum of the Bessel series (0.1) by $S_n^{(B)}(f;z)$

$$S_n^{(B)}(f;z)=\frac{A_0(f)}{2}J_0(z)+\sum_{K=1‘}^{n}A_k(f)J_k(z), \quad z\in\Gamma.$$

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We recall

$$\sigma_n^{(B)}(f;z) = \frac{1}{n\ +\ 1}\sum_{k=0}^{n} S_k^{(B)}(f;z), \quad z \in \Gamma$$

the Fejer sum of the Bessel series (0.1).

Theorem 1. *Suppose that $f(z) \in B.V(\Gamma)$ and the Cauchy principle value integral $(P.V)\frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z} d\zeta$ on $z \in \Gamma$ exists, then the Fejer summation ofthe Bessel series of $f(z)$ on $z \in \Gamma$ converges to:*

$$\frac{1}{2} f(z) + \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

i.e.

$$\lim_{n\ \to \infty} \sigma_n^{(B)}(f;z) = \frac{1}{2} f(z) + \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} dz, \quad z \in \Gamma.$$

Proof. By definition

$$\sigma_n^{(B)}(f;z) = \frac{1}{n\ +1}\sum_{k=o}^{n} S_k^{(B)}(f;z).$$

Hence

$$\sigma_n^{(B)}(f;z) - (\frac{1}{2} f(z) + \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta)$$

$$= \frac{1}{n\ +1}\sum_{k=o}^{n} S_k^{(B)}(f;z) - \frac{1}{n\ +1}\sum_{k=o}^{n} (\frac{1}{2} f(z) + \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta)$$

$$= \frac{1}{n\ +1}\sum_{k=o}^{n} \{S_k^{(B)}(f\ ;z) - (\frac{1}{2} f(z) + \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta)\}. \tag{2}$$

By Corollary 1 (see [2]), suppose that $f(z) \in \Gamma$ and (P.V.)$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$ on $z \in \Gamma$ exists. Then

$$\lim_{n\ \to \infty} S_n^{(B)}(f\ ;z) = \tfrac{1}{2} f(z) + \tfrac{1}{2\pi i}\oint_{\Gamma} \tfrac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Gamma.$$

By the definition of the limit, we know for every $\epsilon > 0, \exists N_1 > 0$, when $n > N_1$, we have

$$|S_n^{(B)}(f\ ;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)| < \frac{\epsilon}{2}. \tag{3}$$

Divide equation (2) into two parts: denoted I_1 and I_2, respectively:

$$\sigma_n^{(B)}(f;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)$$
$$= \frac{1}{n\ +1}(\sum_{k=0}^{N_1} + \sum_{k=N_1+1}^{n})\ \{S_k^{(B)}(f;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)\}$$
$$= I_1 + I_2. \tag{4}$$

Using (3) we have

$$|I_2| \le \frac{1}{n\ +1}\sum_{k=N_1+1}^{n}|S_k^{(B)}(f\ ;z) - (\tfrac{1}{2}f(z) + \tfrac{1}{2\pi i}\oint\limits_{\Gamma}\tfrac{f(\zeta)}{\zeta - z}d\zeta)|$$
$$\le \frac{1}{n\ +1}\sum_{k=N_1+1}^{n}\frac{\epsilon}{2} = \frac{\frac{\epsilon}{2}(n - N_1)}{n+1} < \frac{\epsilon}{2}. \tag{5}$$

Because I_1 is the expression of the sum of finite terms. Hence

$$\lim_{n\ \to\infty}\frac{1}{n\ +1}\sum_{k=0}^{N_1}\{S_k^{(B)}(f;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)\} = 0.$$

Therefore, for the above $\epsilon > 0$, there exists $N > N_1$, when $n > N$. Therefore

$$|I_1| = |\frac{1}{n\ +1}\sum_{k=0}^{N}S_k^{(B)}(f;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)| < \frac{\epsilon}{2} \tag{6}$$

By equation (4), (5) and (6), when $n > N$, we have

$$|\sigma_k^{(B)}(f;z) - (\frac{1}{2}f(z) + \frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta)| < \epsilon,$$

i.e.

$$\lim_{n\to\infty}\sigma_n^{(B)}(f;z)=\frac{1}{2}f(z)+\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta,\quad z\in\Gamma.$$

The proof of Theorem 1 is completed. □

Next we give a representation of the Fejer sum of the Bessel series in the following theorem.

Theorem 2. *Suppose that $f(z)\in B.V.(\Gamma)$, then the Fejer sum $\sigma_n^{(B)}(f\ ;z)$ of the Bessel series (1) can be represented by an integral as follows:*

$$\sigma_n^{(B)}(f\ ;z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{is})\frac{e^{is}}{e^{is}-e^{i\theta}}ds+\frac{i}{8\pi(n+1)}\int_0^{2\pi}f(e^{is})\frac{\sin(n+1)(s-\theta)}{(\sin\frac{s-\theta}{2})^2}ds$$
$$+\frac{1}{4\pi(n+1)}\int_0^{2\pi}f(e^{is})(\frac{\sin\frac{n+1}{2}(s-\theta)}{\sin\frac{s-\theta}{2}})^2ds+O(\frac{1}{n}),\quad z\in\Gamma.$$

Proof. By Theorem 1 of [2] and

$$S_{n:}^{(B)}(f;z)=\frac{1}{2\pi i}\oint_{\Gamma}f(\zeta)K_n^{(B)}(z,\zeta)d\zeta,\quad z\in\Gamma,$$

when $z=e^{i\theta}$. We have (see [1]):

$$S_n^{(B)}(f;z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{is})K_n^{(B)}(e^{i\theta},e^{is})e^{is}ds$$
$$=\frac{1}{2\pi}\int_0^{2\pi}f(e^{is})(\frac{e^{is}}{e^{is}-e^{i\theta}}+\frac{i}{2}\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}e^{-(n+1)(s-\theta)})ds$$
$$-\frac{1}{8\pi n}\int_0^{2\pi}f(e^{is})(e^{is}+e^{i\theta})e^{is}e^{-i(n+1)(s-\theta)}ds+O(\frac{1}{n^2}).$$

Again by the definition of the Fejer sum:

$$\sigma_n^{(B)}(f;z)=\frac{1}{n+1}\sum_{k=0}^{n}S_k^{(B)}(f;z)$$

we obtain

$$\sigma_n^{(B)}(f;z) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_0^{2\pi} f(e^{is})\{(\frac{e^{is}}{e^{is} - e^{i\theta}} + \frac{i}{2}\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}e^{-i(k+1)(s-\theta)}ds$$

$$-\frac{1}{8\pi n}\int_0^{2\pi} f(e^{is})(e^{is}+e^{i\theta})e^{is}e^{-i(k+1)(s-\theta)}ds + O(\frac{1}{k^2})\} = S_1 + S_2. \quad (7)$$

For the integral S_1, we have

$$S_1 = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_0^{2\pi} f(e^{is})\{(\frac{e^{is}}{e^{is} - e^{i\theta}} + \frac{i}{2}\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}e^{-i(k+1)(s-\theta)}ds$$

$$= \frac{1}{2\pi(n+1)}\int_0^{2\pi}\sum_{k=0}^{n} f(e^{is})\{(\frac{e^{is}}{e^{is} - e^{i\theta}} + \frac{i}{2}\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}e^{-i(k+1)(s-\theta)}ds$$

$$= \frac{1}{2\pi(n+1)}\int_0^{2\pi}\sum_{k=0}^{n} f(e^{is})\frac{e^{is}}{e^{is} - e^{i\theta}}ds$$

$$+ \frac{i}{4\pi(n+1)}\int_0^{2\pi}\sum_{k=0}^{n} f(e^{is})\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}e^{-i(k+1)(s-\theta)}ds = I_1 + I_2.$$

Firstly, estimate the integral I_1 :

$$I_1 = \frac{1}{2\pi(n+1)}\int_0^{2\pi}\sum_{k=0}^{n} f(e^{is})\frac{e^{is}}{e^{is} - e^{i\theta}}ds$$

$$= \frac{(n+1)}{2\pi(n+1)}\int_0^{2\pi} f(e^{is})\frac{e^{is}}{e^{is} - e^{i\theta}}ds = \frac{1}{2\pi}\int_0^{2\pi} f(e^{is})\frac{e^{is}}{e^{is} - e^{i\theta}}ds.$$

Next, for I_2, we obtain:

$$I_2 = \frac{i}{4\pi(n+1)}\int_{\substack{0\\0}}^{2\pi} f(e^{is})\frac{e^{i\frac{s-\theta}{2}}}{\sin\frac{s-\theta}{2}}\sum_{k=0}^{n} e^{-i(k+1)(s-\theta)}ds.$$

For the summation of the integral of above equation:

$$\sum_{k=0}^{n} e^{-i(k+1)(s-\theta)},$$

using the Euler formula. Through calculation we obtain:

$$\sum_{k=0}^{n} e^{-i(k+1)(s-\theta)} = e^{-i(s-\theta)} + e^{-2i(s-\theta)} + \cdots + e^{-in(s-\theta)} + e^{-(n+1)i(s-\theta)}$$

$$= \frac{e^{-i(s-\theta)} - e^{-i(n+1)(s-\theta)} e^{-i(s-\theta)}}{1 - e^{-i(s-\theta)}} = \frac{e^{-i(s-\theta)}(1 - e^{-i(n+1)} e^{-i(s-\theta)})}{1 - e^{-i(s-\theta)}}$$

$$= \left(\frac{e^{-i(s-\theta)} e^{-i\frac{n+1}{2}(s-\theta)}}{e^{-\frac{i}{2}(s-\theta)}}\right)\left(\frac{e^{i\frac{n+1}{2}(s-\theta)} - e^{i\frac{n+1}{2}(s-\theta)}}{e^{\frac{i}{2}(s-\theta)} - e^{-\frac{i}{2}(s-\theta)}}\right)$$

$$= \frac{-2ie^{-\frac{i}{2}(s-\theta)} e^{-\frac{i(n+1)}{2}(s-\theta)} \sin\frac{n+1}{2}(s-\theta)}{-2i\sin(\frac{s-\theta}{2})} = \frac{e^{-i(\frac{n}{2}+1)(s-\theta)} \sin\frac{n+1}{2}(s-\theta)}{\sin(\frac{s-\theta}{2})}.$$

Hence, integral I_2 can be rewritten as:

$$I_2 = \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{e^{\frac{i}{2}(s-\theta)} e^{-i(\frac{n}{2}+1)(s-\theta)} \sin\frac{n+1}{2}(s-\theta)}{\sin\frac{s-\theta}{2}\sin\frac{s-\theta}{2}} ds$$

$$= \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{e^{-i(\frac{n+1}{2})(s-\theta)} \sin\frac{n+1}{2}(s-\theta)}{(\sin\frac{s-\theta}{2})^2} ds$$

$$= \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{\cos\frac{n+1}{2}(s-\theta) - i\sin\frac{n+1}{2}(s-\theta)}{(\sin\frac{s-\theta}{2})^2} \times \sin\frac{n+1}{2}(s-\theta) ds$$

$$= \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{\cos\frac{n+1}{2}(s-\theta).\sin\frac{n+1}{2}(s-\theta)}{(\sin\frac{s-\theta}{2})^2} ds$$

$$+ \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{\sin^2\frac{n+1}{2}(s-\theta)}{(\sin\frac{s-\theta}{2})^2} ds$$

$$= \frac{i}{8\pi(n+1)} \int_0^{2\pi} f(e^{is}) \frac{\sin(n+1)(s-\theta)}{(\sin\frac{s-\theta}{2})^2} ds + \frac{i}{4\pi(n+1)} \int_0^{2\pi} f(e^{is}) \left(\frac{\sin\frac{n+1}{2}(s-\theta)}{\sin\frac{s-\theta}{2}}\right)^2 ds.$$

It is well known that $f \in B.V.(\Gamma)$. Then $f(e^{is})e^{2is} \in B.V.(0, 2\pi)$ and $f(e^{is})e^{i(s+\theta)} \in B.V.(0, 2\pi)$. Thus $f(e^{is})e^{i(\theta)}(e^{is} + e^{i\theta}) \in B.V.(0, 2\pi)$. Using one known result of the Fourier coefficient (see [4]), we have

$$\int_0^{2\pi} f(e^{is})e^{is}(e^{is} + e^{i\theta})e^{-i(k+1)(s-\theta)}ds = O(\frac{1}{(k+1)^2}).$$

Therefore, the integral S_2 can be rewritten as:

$$S_2 = -\frac{1}{8\pi n}\sum_{k=0}^{n}\{\int_0^{2\pi} f(e^{is})e^{is}(e^{is} + e^{i\theta})e^{-i(k+1)(s-\theta)}ds + O(\frac{1}{k^2})\}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} O(\frac{1}{(k+1)^2})$$

and

$$\frac{1}{n+1}\sum_{k=0}^{n} O(\frac{1}{(k+1)^2}) = O(1)\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{(k+1)^2}$$

$$= O(1)\frac{1}{n+1}\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2}\}$$

$$\leq O(1)\frac{1}{n+1}\{1 + \int_1^{n+1}\frac{1}{x^2}dx\}$$

$$= O(1)\frac{1}{n+1}\{1 + 1 - \frac{1}{n+1}\} = O(1)\frac{1}{n+1}\{2 - \frac{1}{n+1}\},$$

i.e. $S_2 = O(\frac{1}{n})$. From this and equalities (7), (8) and (9), we obtain the proof of Theorem 2. □

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