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A NOTE ON GENERALIZED ALPHA-SKEW-NORMAL DISTRIBUTION

A.H. Handam
Department of Mathematics
Al Al-Bayt University
P.O. Box 130095, Al Mafraq, JORDAN

Abstract: The alpha-skew-normal distributions is suggested by Elal-Olivero in [4]. In this paper, we modified this distribution to a generalized alpha-skew-normal distribution. Some properties of $GASN$ are investigated.

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Key Words: alpha-skew-normal distribution, generalized alpha-skew-normal distribution, moment generating function

1. Introduction

Azzalini [1] defined the skew-normal distribution for a random variable Z with the parameter λ be: $g(z) = 2\phi(z)\Phi(\lambda z)$ $(-\infty < z < \infty)$, $\lambda \in R$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution function, respectively. The term skew normal (SN) refers to a parametric class of probability distributions that extend the normal distribution by an additional shape parameter that regulates the skewness, allowing for a continuous variation from normality to non-normality. Henze [5] gave a stochastic representation for this distribution, obtaining explicitly its odd moments. Also Azzalini [2] discussed stochastic representations in a more general context. On the other hand, there has been a number of works exploring bimodality arising from skew distributions, see [7], [3], [8].

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In [4], Elal-Olivero introduced a new family called alpha-skew-normal (ASN) and it is denoted by $\{ASN(\alpha) : \alpha \in R\}$ where α represents the asymmetric parameter with effect of uni-bimodality. This family of distributions is flexible enough to support both unimodal and bimodal shape. In this paper, we introduce a new class of skew normal distributions, which is a generaliztion of alpha-skew-normal distribution. This new class is called generalized alpha-skew-normal ($GASN$) and is denoted by $\{GASN(\alpha) : \alpha \in R\}$ where α represents the asymmetric parameter with effect of uni-bimodality, so that $GASN(0)$ corresponds to the normal distribution.

2. Generalized Alpha-Skew-Normal Distribution

In this section, we present the definition and some simple properties of $GASN(\alpha)$.

Definition 2.1. (see [4]) If a random variable X has density function,

$$f(x \mid \alpha) = \tfrac{(1-\alpha x)^2+1}{2+\alpha^2}\phi(x),\ x \in R,$$

where $\alpha \in R$, then we say that X is a alpha-skew-normal random variable with parameter α. We denote this as $X \sim ASN(\alpha)$.

Definition 2.2. We say that a random variable X has the generalized alpha-skew-normal distribution if its density is given by

$$f(x \mid \alpha) = \tfrac{(1-\alpha x)^{2n}+1}{A(\alpha)}\phi(x), \quad x \in R,$$

where $\alpha \in R$, $n \in N - \{0\}$ and $A(\alpha) = 2 + \sum\limits_{l=1}^{n} \binom{2n}{2l} \alpha^{2l} \prod\limits_{j=1}^{l} (2j-1)$. We denote this by $X \sim GASN\left(\alpha\right)$.

It is easy to show that the $GASN$ satisfies conditions of the pdf.

If $X \sim GASN\left(\alpha\right)$, the following properties are satisfied:

(1) $GASN\left(0\right) = N(0,1)$.

(2) If $\alpha \to \pm\infty$, then $X \xrightarrow{d} \left(\prod\limits_{j=1}^{n} (2j-1)\right)^{-1} x^{2n}\phi(x)$.

(3) $-X \sim GASN\left(-\alpha\right)$.

(4) if $n = 1$, then $GASN\left(\alpha\right) = ASN\left(\alpha\right)$.

Proposition 2.3. *Let Z be the standard normal distribution, then*

$$E\left(Z^{2k}\right)=\prod_{j=1}^{k}(2j-1) \text{ and } E\left(X^{2k-1}\right)=0,\ k=1,2,3,...$$

Theorem 2.4. *If $X\sim GASN(\alpha)$, then for $k=0,1,2,...,$ we have*

$$E\left(X^{2k}\right)=\frac{1}{A(\alpha)}\left(2\prod_{j=1}^{k}(2j-1)+\sum_{l=1}^{n}\binom{2n}{2l}\alpha^{2l}\prod_{j=1}^{k+l}(2j-1)\right)$$

and

$$E\left(X^{2k+1}\right)=\frac{1}{A(\alpha)}\left(-\sum_{l=1}^{n}\binom{2n}{2l-1}\alpha^{2l-1}\prod_{j=1}^{k+l-1}(2j-1)\right).$$

Proof.

$$E\left(X^{2k}\right)=\frac{1}{A(\alpha)}\int_{-\infty}^{\infty}x^{2k}\left((1-\alpha x)^{2n}+1\right)\phi(x)dx$$

$$=\frac{1}{A(\alpha)}\int_{-\infty}^{\infty}x^{2k}\left(2+\sum_{m=1}^{2n}\binom{2n}{m}(-\alpha x)^{m}\right)\phi(x)dx$$

$$=\frac{1}{A(\alpha)}\int_{-\infty}^{\infty}x^{2k}\left(2+\sum_{l=1}^{n}\binom{2n}{2l}(\alpha x)^{2l}+\sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha x)^{2l-1}\right)\phi(x)dx$$

$$=\frac{2E\left(Z^{2k}\right)}{A(\alpha)}+\frac{\sum_{l=1}^{n}\binom{2n}{2l}\alpha^{2l}E\left(Z^{2k+2l}\right)}{A(\alpha)}+\frac{\sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha)^{2l-1}E\left(Z^{2k+2l-1}\right)}{A(\alpha)}$$

$$=\frac{1}{A(\alpha)}\left(2\prod_{j=1}^{k}(2j-1)+\sum_{l=1}^{n}\binom{2n}{2l}\alpha^{2l}\prod_{j=1}^{k+l}(2j-1)\right)\quad\text{(by Proposition 2.3)},$$

on the other hand

$$E\left(X^{2k-1}\right)=\frac{1}{A(\alpha)}\int_{-\infty}^{\infty}x^{2k-1}\left((1-\alpha x)^{2n}+1\right)\phi(x)dx$$

$$=\frac{1}{A(\alpha)}\int_{-\infty}^{\infty}x^{2k-1}\left(2+\sum_{l=1}^{n}\binom{2n}{2l}(\alpha x)^{2l}+\sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha x)^{2l-1}\right)\phi(x)dx$$

$$= \frac{2E\left(Z^{2k-1}\right)}{A(\alpha)} + \frac{\sum_{l=1}^{n}\binom{2n}{2l}\alpha^{2l}E\left(Z^{2k+2l-1}\right)}{A(\alpha)} + \frac{\sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha)^{2l-1}E\left(Z^{2k+2l-2}\right)}{A(\alpha)}$$

$$= \frac{1}{A(\alpha)}\left(-\sum_{l=1}^{n}\binom{2n}{2l-1}\alpha^{2l-1}\prod_{j=1}^{k+l-1}(2j-1)\right) \quad \text{(by Proposition 2.3).}$$

□

Proposition 2.5. *[6]. If W is a normal distribution with mean μ and variance σ^2 , then*

$$E\left[W^k\right] = \begin{cases} \sigma^k \sum_{i=1}^{(k+1)/2} \frac{k!\mu^{2i-1}}{(2i-1)![(k+1)/2-i]!2^{(k+1)/2-i}\sigma^{2i-1}}, & k = 1,3,5,... \\ \sigma^k \sum_{i=0}^{k/2} \frac{k!\mu^{2i}}{(2i)!(k/2-i)!2^{k/2-i}\sigma^{2i}}, & k = 2,4,6,... \end{cases}$$

Theorem 2.6. *If $M_X(t)$ is the moment generating function of $X \sim GASN(\alpha)$, then*

$$M_X(t) = \frac{e^{t^2/2}}{A(\alpha)}\left[2 + \sum_{l=1}^{n}\binom{2n}{2l}\alpha^{2l}P(2l,t) + \sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha)^{2l-1}Q(2l-1,t)\right]$$

where

$$P(k,t) = \sum_{i=0}^{k/2}\frac{k!t^{2i}}{(2i)!(k/2-i)!2^{k/2-i}}$$

and

$$Q(k,t) = \sum_{i=1}^{(k+1)/2}\frac{k!t^{2i-1}}{(2i-1)!\left[(k+1)/2-i\right]!2^{(k+1)/2-i}}.$$

Proof.

$$M_X(t) = E\left(e^{tx}\right) = \frac{1}{A(\alpha)}\int_{-\infty}^{\infty} e^{tx}\left((1-\alpha x)^{2n}+1\right)\phi(x)dx$$

$$= \frac{1}{A(\alpha)}\int_{-\infty}^{\infty} e^{tx}\left(2 + \sum_{l=1}^{n}\binom{2n}{2l}(\alpha x)^{2l} + \sum_{l=1}^{n}\binom{2n}{2l-1}(-\alpha x)^{2l-1}\right)\phi(x)dx$$

$$= \frac{e^{t^2/2}}{A(\alpha)} \int_{-\infty}^{\infty} \left(2 + \sum_{l=1}^{n} \binom{2n}{2l} (\alpha x)^{2l} + \sum_{l=1}^{n} \binom{2n}{2l-1} (-\alpha x)^{2l-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

$$= e^{t^2/2} \left[\frac{2}{A(\alpha)} + \frac{\sum_{l=1}^{n} \binom{2n}{2l} \alpha^{2l} E\left(Y^{2l}\right)}{A(\alpha)} + \frac{\sum_{l=1}^{n} \binom{2n}{2l-1} (-\alpha)^{2l-1} E\left(Y^{2l-1}\right)}{A(\alpha)} \right]$$

$$\text{(where } Y \sim N(t,1))$$

$$= \frac{e^{t^2/2}}{A(\alpha)} \left[2 + \sum_{l=1}^{n} \binom{2n}{2l} \alpha^{2l} P(2l,t) + \sum_{l=1}^{n} \binom{2n}{2l-1} (-\alpha)^{2l-1} Q(2l-1,t) \right]$$

$$\text{(by Proposition 2.5)},$$

where

$$P(k,t) = \sum_{i=0}^{k/2} \frac{k! t^{2i}}{(2i)!(k/2-i)! 2^{k/2-i}}$$

and

$$Q(k,t) = \sum_{i=1}^{(k+1)/2} \frac{k! t^{2i-1}}{(2i-1)! \left[(k+1)/2-i\right]! 2^{(k+1)/2-i}}.$$

□

References

[1] A. Azzalini, Class of distributions which includes the normal ones, *Scandinavian Journal of Statistics*, **12** (1985), 171-178.

[2] A. Azzalini, Further results on a class of distributions which includes the normal ones, *Statistica*, **12** (1986), 199-208.

[3] A. Azzalini, A. Capitonio, Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution, *Journal of the Royal Statistical Society*, Series B, **65** (2003), 367-389.

[4] D. Elal-Olivero, Alpha-skew-normal distribution, *Proyecciones*, **29** (2010), 224-240.

[5] N. Henze, A probabilistic representation of the skew-normal distribution, *Scandinavian Journal of Statistics*, **13** (1986), 271-275.

[6] K. Krishnamoorthy, *Handbook of Statistical Distributions with Applications*, Chapman & Hall/CRC Press (2006).

[7] Y.Ma, M.G. Genton, Flexible class of skew-symmetric distribution, *Scandinavian Journal of Statistics*, **31** (2004), 459-468.

[8] H.S. Salinas, R.B. Arellano-Valle, H.W. Gómez, The extended skew-exponencial power distribution and its derivations, *Communications in Statistics-Theory and Methods*, **36** (2007), 1673-1689.