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SOME RESULTS ON t-BEST APPROXIMATION IN FUZZY ANTI-2-NORMED LINEAR SPACES

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Abstract: The aim of this paper to give the set of all t-best approximations on fuzzy anti-2-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [16].

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1. Introduction

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by Katsaras in [12]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [11]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [13].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in

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fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [12], Felbin [6], and Bag and Samanta [1]. Many authors studied on fuzzy normed linear space [5]. The concept of 2-norm on a linear space has been introduced and developed by Gähler in [7,8] and Gunawan and Mashadi [9]. Recently, Vaezpour and Karimi [16], studied on the set of all t-best approximations on fuzzy normed spaces and proved several theorems pertaining to this set.

In [10] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties.

In [14] Surender Reddy, studied on the set of all t-best approximations on fuzzy anti-normed spaces and proved several theorems pertaining to this set.

In [15] Surender Reddy introduced the notion of convergent sequence and Cauchy sequence in fuzzy anti-2-normed linear space.

In this paper, we give the set of all t-best approximations on fuzzy anti-2-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [16].

2. Preliminaries

Definition 1. Let X be a real linear space of dimension greater than one and let $\|\bullet,\bullet\|$ be a real valued function on $X\times X$ satisfying the following conditions

$2N_1$: $\|x,y\|=0$ if and only if x and y are linearly dependent

$2N_2$: $\|x,y\|=\|y,x\|$

$2N_3$: $\|\alpha x,y\|=|\alpha|\ \|x,y\|$, for every $\alpha\in R$

$2N_4$: $\|x,y+z\|\le\|x,y\|+\|x,z\|$

then the function $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X,\|\bullet,\bullet\|)$ is called a 2-normed linear space.

Example 2. *Let* $X=R^3$ *be a real linear space. Define* $\|\bullet,\bullet\|: X\times X\longrightarrow R$ *by* $\|x,y\|=\max\{|x_1y_2-x_2y_1|,|x_2y_3-x_3y_2|,|x_3y_1-x_1y_3|\}$, *where* $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ *are in* R^3. *Then* $(X,\|\bullet,\bullet\|)$ *is a 2-normed linear space.*

Definition 3. Let X be a linear space over a real field F. A fuzzy subset N of $X\times X\times R$ is called a fuzzy 2-norm on X if the following conditions are satisfied for all $x,y,z\in X$.

$(2-N_1)$: For all $t\in R$ with $t\le 0$, $N(x,y,t)=0$,

$(2-N_2)$: For all $t \in R$ with $t > 0$, $N(x, y, t) = 1$ if and only if x, y are linearly dependent

$(2-N_3)$: $N(x, y, t)$ is invariant under any permutation of x, y

$(2-N_4)$: For all $t \in R$ with $t > 0$, $N(x, cy, t) = N(x, y, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(2-N_5)$: For all $s, t \in R$, $N(x, y + z, s + t) \geq \min\{N(x, y, s), N(x, z, t)\}$

$(2-N_6)$: $N(x, y, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x, y, t) = 1$.

Then the pair (X, N) is called a fuzzy 2-normed linear space (briefly F-2-NLS).

Example 4. *Let $(X, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define*

$$\begin{aligned} N(x, y, t) &= \frac{t}{t + \|x, y\|}, \quad \text{if } t > 0, \quad t \in R, \quad x, y \in X \\ &= 0, \quad \text{if } t \leq 0, \quad t \in R, \quad x, y \in X. \end{aligned}$$

Then (X, N) is a fuzzy 2-normed linear space.

Definition 5. Let X be a linear space over a real field F. A fuzzy subset N of $X \times X \times R$ is called a fuzzy anti-2-norm on X if the following conditions are satisfied for all $x, y, z \in X$

$(a-2-N_1)$: For all $t \in R$ with $t \leq 0$, $N(x, y, t) = 1$,

$(a-2-N_2)$: For all $t \in R$ with $t > 0$, $N(x, y, t) = 0$ if and only if x, y are linearly dependent

$(a-2-N_3)$: $N(x, y, t)$ is invariant under any permutation of x, y

$(a-2-N_4)$: For all $t \in R$ with $t > 0$, $N(x, cy, t) = N(x, y, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(a-2-N_5)$: For all $s, t \in R$, $N(x, y+z, s+t) \leq \max\{N(x, y, s), N(x, z, t)\}$

$(a-2-N_6)$: $N(x, y, t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty} N(x, y, t) = 0$.

Then the pair (X, N) is called a fuzzy anti-2-normed linear space (briefly Fa-2-NLS).

Remark 6. *From $(a-2-N_3)$, it follows that in Fa-2-NLS,*

$(a-2-N_4)$: For all $t \in R$ with $t > 0$, $N(cx, y, t) = N(x, y, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(a-2-N_5)$: For all $s, t \in R$, $N(x+z, y, s+t) \leq \max\{N(x, y, s), N(z, y, t)\}$.

Example 7. *Let $(X, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define*

$$N(x, y, t) = \frac{\|x, y\|}{t + \|x, y\|}, \quad \text{if } t > 0, \quad t \in R, \quad x, y \in X$$

$$= \quad 1, \quad if \;\; t \leq 0, \quad t \in R, \quad x, y \in X.$$

Then (X, N) *is a Fuzzy anti-2-normed linear space.*

Definition 8. A sequence $\{x_k\}$ in a fuzzy anti-2-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1, x_k - x, t) < r$, for all $k \geq n_0$.

Theorem 9. *In a fuzzy anti-2-normed linear space* (X, N)*, a sequence* $\{x_k\}$ *converges to* $x \in X$ *if and only* $\lim_{k \to \infty} N(x_1, x_k - x, t) = 0$, $\forall\, t > 0$.

3. Main Results

Definition 10. Let (X, N) be a fuzzy anti-2-normed space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x_1, x - y, t) < r\}$$
$$B[x, r, t] = \{y \in X : N(x_1, x - y, t) \leq r\}$$

Definition 11. Let (X, N) be a fuzzy anti-2-normed space. A subset A of X is said to be open if there exists $r \in (0, 1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 12. Let (X, N) be a fuzzy anti-2-normed space. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$.

i.e., $\lim_{k \to \infty} N(x_1, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Definition 13. Let (X, N) be a fuzzy anti-2-normed space. A subset B of X is said to be closure of $A \subset B$ if for any $x \in B$, there exists a sequence $\{x_k\}$ in A such that $\lim_{k \to \infty} N(x_1, x_k - x, t) = 0$, $\forall\, t > 0$. We denote the set B by $\overline{A}$.

Definition 14. Let (X, N) be a fuzzy anti-2-normed space. A subset A of X is said to be compact if for any sequence $\{x_k\}$ in A has a sequence converging to an element of A.

Lemma 15. *If* (X, N) *be a fuzzy anti-2-normed space then*
(i) the function $(x, y) \longrightarrow x + y$ *is continuous*
(ii) the function $(\alpha, x) \longrightarrow \alpha x$ *is continuous*

Proof. (i) If $x_k \longrightarrow x$ and $y_k \longrightarrow y$, then as $k \longrightarrow \infty$,
$N(x_1, (x_k + y_k) - (x + y), t) \leq \max\{N(x_1, x_k - x, \frac{t}{2}), N(x_1, y_k - y, \frac{t}{2})\} \longrightarrow 0$

(ii) If $x_k \longrightarrow x$, $\alpha_k \longrightarrow \alpha$ and $\alpha_k \neq 0$ then

$$\begin{aligned} N(x_1, \alpha_k x_k - \alpha x, t) &= N(x_1, \alpha_k(x_k - x) + x(\alpha_k - \alpha), t) \\ &\leq \max\{N(x_1, \alpha_k(x_k - x), \tfrac{t}{2}), N(x_1, x(\alpha_k - \alpha), \tfrac{t}{2})\} \\ &= \max\{N(x_1, x_k - x, \tfrac{t}{2\alpha_k}), N(x_1, x, \tfrac{t}{2(\alpha_k - \alpha)})\} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$ □

Definition 16. Let (X, N) be a fuzzy anti-2-normed space and A is nonempty subset of X. Let $d(A, x, t) = \inf\{N(x_1, x - y, t) : y \in A\}$, where $x \in X$, $t > 0$. An element $y_0 \in A$ is said to be a t-best approximation of x from A if $N(x_1, y_0 - x, t) = d(A, x, t)$.

Definition 17. Let (X, N) be a fuzzy anti-2-normed space and A is nonempty subset of X. For $x \in X$, $t > 0$, we shall denote the set of all elements of t-best approximation of x from A by $P_A^t(x)$; i.e.,

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t-best approximation in A then A is called a t-proximinal (respectively t-chebyshev) set.

Definition 18. Let (X, N) be a fuzzy anti-2-normed space and A is nonempty subset of X. For $t > 0$, A is said to be t-boundedly compact if for each $x \in X$ and $0 < r < 1$, $B[x, r, t] \cap A$ is a compact subset of X.

Theorem 19. *Let (X, N) be a fuzzy anti-2-normed space and A is nonempty subset of X then*

(i) $d(A + y, x + y, t) = d(A, x, t)$, for all $x, y \in X$ and $t > 0$,

(ii) $P_A^t(x + y) = P_A^t(x) + y$, for all $x, y \in X$ and $t > 0$,

(iii) $d(\alpha A, \alpha x, t) = d(A, x, \frac{t}{|\alpha|})$, for all $x \in X$, $t > 0$ and $\alpha \in R\backslash\{0\}$,

(iv) $P_{\alpha A}^{|\alpha|t}(\alpha x) = \alpha P_A^t(x)$, for all $x \in X$, $t > 0$ and $\alpha \in R\backslash\{0\}$,

(v) A is t-proximinal (respectively t-chebyshev) if and only if $A + y$ is t-proximinal (respectively t-chebyshev) for any given $y \in X$,

(vi) A is t-proximinal (respectively t-chebyshev) if and only if αA is $|\alpha|t$-proximinal (respectively $|\alpha|t$-chebyshev) for any given $\alpha \in R\backslash\{0\}$.

Proof. (i) For $x, y \in X$ and $t > 0$,

$$\begin{aligned} d(A + y, x + y, t) &= \inf\{N(x_1, (z + y) - (x + y), t) : z \in A\} \\ &= \inf\{N(x_1, z - x, t) : z \in A\} = d(A, x, t). \end{aligned}$$

(ii) On using (i), it follows that, $y_0 \in P_{A+y}^t(x + y)$ if and only if $y_0 \in A + y$ and $d(A + y, x + y, t) = N(x_1, x + y - y_0, t)$ if and only if $y_0 - y \in A$ and $d(A, x, t) = N(x_1, x - (y_0 - y), t)$ if and only if $y_0 - y \in P_A^t(x)$, i.e., $y_0 \in P_A^t(x) + y$

(iii) We have $d(\alpha A, \alpha x, t) = \inf\{N(x_1, \alpha x - \alpha z, t) : z \in A\}$
$= \inf\{N(x_1, \alpha(x - z), t) : z \in A\}$
$= \inf\{N(x_1, x - z, \frac{t}{|\alpha|}) : z \in A\} = d(A, x, \frac{t}{|\alpha|})$.

(iv) On using (iii), it follows that $y_0 \in P^{|\alpha|t}_{\alpha A}(\alpha x)$ if and only if $y_0 \in \alpha A$ and $d(\alpha A, \alpha x, |\alpha|t) = N(x_1, \alpha x - y_0, |\alpha|t)$ if and only if $\frac{y_0}{\alpha} \in A$ and $N(x_1, x - \frac{y_0}{\alpha}, t) = d(A, x, t)$. However, this is equivalent to $\frac{y_0}{\alpha} \in P^t_A(x)$; i.e., $y_0 \in \alpha P^t_A(x)$.

(v) The proof of (v) is an immediate consequence of (ii).

(vi) The proof of (vi) follows from (iv). □

Corollary 20. *Let M be a nonempty subspace of X then*
(i) $d(M, x + y, t) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $y \in M$,
(ii) $P^t_M(x + y) = P^t_M(x) + y$, for all $t > 0$, $x \in X$ and $y \in M$,
(iii) $d(M, \alpha x, |\alpha t|) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $\alpha \in R\backslash\{0\}$,
(iv) $P^{|\alpha|t}_M(\alpha x) = \alpha P^t_M(x)$, for all $t > 0$, $x \in X$ and $\alpha \in R\backslash\{0\}$.

Proof. The proof of (i) and (ii) follows from theorem 19(i) and 19(ii) and the fact that if M is a subspace and $y \in M$ then $M + y = M$.

The proof of (iii) and (iv) follows from theorem 19(iii) and 19(iv) and the fact that if M is a subspace and $\alpha \neq 0$ then $\alpha M = M$ □

Definition 21. For $x \in X$, $0 < r < 1$, $t > 0$,

$$S[x, r, t] = \{y \in X : N(x_1, x - y, t) = r\} \quad \text{and} \quad e^t_A(x) = d(A, x, t).$$

Theorem 22. *Let (X, N) be a fuzzy anti-2-normed space, $A \subset X$, $x \in X\backslash\overline{A}$ and $t > 0$ then we have*

$$P^t_A(x) = A \cap B[x, e^t_A(x), t] = A \cap S[x, e^t_A(x), t]. \tag{1}$$

Proof. The inclusions;

$$P^t_A(x) \subseteq A \cap S[x, e^t_A(x), t] \subseteq A \cap B[x, e^t_A(x), t]. \tag{2}$$

are obvious by the definitions of $P^t_A(x)$ and $e^t_A(x)$.

Conversely, let $y \in A \cap B[x, e^t_A(x), t]$, then we have $y \in A$ and $N(x_1, y - x, t) \leq e^t_A(x) = d(A, x, t) \leq N(x_1, y - x, t)$. Therefore $y \in A$ and $N(x_1, y - x, t) = d(A, x, t)$, which implies that $y \in P^t_A(x)$. So, $A \cap B[x, e^t_A(x), t] \subset P^t_A(x)$. Hence by (2) we have (1) which completes the proof. □

Remark 23. *Let (X,N) be a fuzzy anti-2-normed linear space and A is nonempty subset of X, $x \in X\backslash\overline{A}$ and $t>0$ then we have*

$$A \cap B(x, e_A^t(x), t) = \emptyset, \tag{3}$$

because, if $y_0 \in A \cap B(x, e_A^t(x), t)$ then $d(A,x,t) \leq N(x_1, x-y_0, t) < d(A,x,t)$ which is impossible.

Corollary 24. *Let (X,N) be a fuzzy anti-2-normed space and A is nonempty subset of X, $x \in X\backslash\overline{A}$ with $P_A^t(x) \neq \emptyset$ and $0<r<1$ such that,*

$$\emptyset \neq A \cap B[x,r,t] \subseteq S[x,r,t] \tag{4}$$

then we have $r = e_A^t(x)$, and we can write $A \cap B[x,r,t] = P_A^t(x)$.

Proof. If $r < e_A^t(x)$ then by the definition of $e_A^t(x)$ we have $A \cap B[x,r,t] = \emptyset$, which contradicts (4). If $r > e_A^t(x)$, since $P_A^t(x) \neq \emptyset$, then by (1) we have $\emptyset \neq P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x,r,t)$, which contradicts (4), and this completes the proof. □

Definition 25. Let (X,N) be a fuzzy anti-2-normed space, $0<r<1$ and $t>0$. We shall say that a set $A \subset X$ supports the cell $B[x,r,t]$, or that A is a support set of the cell $B[x,r,t]$, if we have $d(A, B[x,r,t], t) = 1$ and $A \cap B(x,r,t) = \emptyset$.

Theorem 26. *Let (X,N) be a fuzzy anti-2-normed space, A is a nonempty subset of X, $x \in X\backslash\overline{A}$, $a_0 \in A$ and $t>0$. We have $a_0 \in P_A^t(x)$ if and only if the set A supports the cell $B = B[x, N(x_1, a_0 - x, t), t]$.*

Proof. Assume that $a_0 \in P_A^t(x)$. Hence $N(x_1, a_0 - x, t) = d(A,x,t)$. Then by (3), we have $A \cap B(x, N(x_1, a_0 - x, t), t) = \emptyset$, on the other hand, since $a_0 \in A \cap B[x, N(x_1, a_0 - x, t), t]$, we have $d(A,B,t) = 1$. Consequently, the set A supports the cell B. Conversely, suppose $a_0 \notin P_A^t(x)$, hence $N(x_1, a_0 - x, t) > d(A,x,t)$, and let $0<\varepsilon<1$ such that $N(x_1, a_0 - x, t) > d(A,x,t)+\varepsilon$. Then there exists an $a \in A$ such that $N(x_1, a_0 - x, t) > d(A,x,t)+\varepsilon > N(x_1, a-x, t)$, hence $a \in B(x, N(x_1, a_0 - x, t), t)$. Consequently, A does not support the cell B. □

Remark 27. *We recall that a set A in a topological space τ is said to be countably compact, if every countable open cover of A has a finite subcover, or, which is equivalent, if for every decreasing sequence $A_1 \supset A_2 \supset \dots$ of non-void closed subset of A we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$*

Theorem 28. *Let (X,N) be a fuzzy anti-2-normed space, τ be an arbitrary topology on X and $t>0$. If A is nonempty subset of X such that for $A\cap B[x,r,t]$ is $\tau-$countably compact, then A is t-proximinal.*

Proof. For all $n\in N$, $0<1-d(A,x,t)+\frac{d(A,x,t)}{n+1}<1$. put

$$A_n^t = A\cap B\left[x, 1-d(A,x,t)+\frac{d(A,x,t)}{n+1}, t\right], \quad (n=1,2,\dots).$$

Since for every $n\in N$, $d(A,x,t)\left(1-\frac{1}{n+1}\right)>d(A,x,t),$ obviously $A_1^t\supset A_2^t\supset\dots$ and each $A_n^t\neq\emptyset$. Hence there exists $a_n^t\in A$ such that

$$d(A,x,t)\left(1-\frac{1}{n+1}\right)>N(x_1,a_n^t-x,t).$$

It follows that $a_n^t\in A_n^t$. Now, since each A_n^t is $\tau-$countably compact and $\tau-$closed, we conclude that there exists an $a_0\in\bigcap_{n=1}^{\infty}A_n^t$. Then we have

$$d(A,x,t)\le N(x_1,a_0-x,t)\le d(A,x,t)\left(1-\frac{1}{n+1}\right), \quad (n=1,2,\dots),$$

hence $a_0\in P_A^t(x)$ which completes the proof. □

Definition 29. Let (X,N) be a fuzzy anti-2-normed space and A is nonempty subset of X. An element $y_0\in A$ is said to be an F-best approximation of $x\in X$ from A if it is a t-best approximation of x from A, for every $t>0$, i.e.,

$$y_0\in\bigcap_{t\in(0,\infty)}P_A^t(x).$$

The set of all elements of F-best approximations of $x\in X$ from A is denoted by $FP_A(x)$, i.e.,

$$FP_A(x)=\bigcap_{t\in(0,\infty)}P_A^t(x).$$

If each $x\in X$ has at least (respectively exactly) one F-best approximation in A then A is called a F-proximinal (respectively F-chebyshev) set.

Example 30. *Let $X=R^3$. Define $N:X\times X\times[0,\infty)\longrightarrow[0,1]$ by*

$$N(x_1,x_2,t) = \frac{\|x_1,x_2\|}{t}, \quad \text{if } t>0,\quad t\in R,\quad x_1,x_2\in X$$

$$= 1, \qquad \text{if } t \leq 0, \quad t \in R, \quad t \in R, \quad x_1, x_2 \in X,$$

where $\|x_1, x_2\| = \min_{1 \leq i \leq 2} \sum_{j=1}^{3} |x_{ij}|$. *Then* (X, N) *is a fuzzy anti-2-normed space. Let*

$$A = \left\{(a, b, c) \in R^3 : a^2 + b^2 \leq 1, \quad 0 \leq c \leq a^2 + b^2\right\}$$

and $x_1 = (1, 0, 0)$, $x = (0, 0, 4)$ *are in* X. *Let* $a_0 = (0, -1, 1)$ *and* $a_1 = (0, 1, 1)$ *are in* A, *Then for every* $t > 0$,

$$N(x_1, a_0 - x, t) = N(x_1, (0, -1, 1) - (0, 0, 4), t) = \frac{1}{t}$$

$$N(x_1, a_1 - x, t) = N(x_1, (0, 1, 1) - (0, 0, 4), t) = \frac{1}{t}.$$

On the other hand

$$\begin{aligned} d(A, x, t) &= d(A, (0, 0, 4), t) = \inf\{N(x_1, u - (0, 0, 4), t) : u \in A\} \\ &= \inf\{N(x_1, (a, b, c) - (0, 0, 4), t) : a^2 + b^2 \leq 1, \quad 0 \leq c \leq a^2 + b^2\} \\ &= \inf\left\{\frac{\min(|x_{11}| + |x_{12}| + |x_{13}|, |x_{21}| + |x_{22}| + |x_{23} - 4|}{t}\right\} \\ &= \frac{1}{t}. \end{aligned}$$

So, for every $t > 0$, $a_0 = (0, -1, 1)$ *and* $a_1 = (0, 1, 1)$ *are* t*-best approximations of* $(0, 0, 4)$ *from* A. *Hence* $a_0 = (0, -1, 1)$ *and* $a_1 = (0, 1, 1)$ *are* F*-best approximations of* $x = (0, 0, 4)$ *from* A. *Therefore* A *is not an* F*-Chebyshev set.*

Example 31. *Let* $X = R^3$. *Define* $N : X \times X \times R \longrightarrow [0, 1]$ *by*

$$\begin{aligned} N(x_1, x_2, t) &= \frac{\|x_1, x_2\|}{t + \|x_1, x_2\|}, \quad \text{if } t > 0, \quad t \in R, \quad x_1, x_2 \in X \\ &= 1, \qquad \text{if } t \leq 0, \quad t \in R, \quad x_1, x_2 \in X, \end{aligned}$$

where $\|x_1, x_2\| = \min_{1 \leq i \leq 2} \sum_{j=1}^{3} |x_{ij}|$. *Then* (X, N) *is a fuzzy anti-2-normed space. Let*

$$A = \left\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 \geq 1\right\}.$$

Then, for every $a = (x, y, z) \in R^3$ *where* $x^2 + y^2 + z^2 < 1$, *there exists a unique* $a_0 = (x_0, y_0, z_0) \in A$ *(especially in* ∂A*) which is an* F*-best approximation of* a *from* A. *So* A *is an* F*-proximinal set.*

Remark 32. *For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of A, and by $\mathcal{M}_A$ the set of all elements of the F-best approximation of the elements $x \in X$ from A. i.e.,*

$$\mathcal{M}_A = \bigcup_{x \in X} FP_A(x).$$

Theorem 33. *Let (X, N) be a fuzzy anti-2-normed space, A is nonempty subset of X and A be a F-best proximinal set in X then $\partial A \subset \overline{\mathcal{M}_A}$.*

Proof. If $\partial A = \emptyset$, the proof is obvious. If $\partial A \neq \emptyset$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $\varepsilon' < \varepsilon$ and the cell $B(a_0, \varepsilon', \frac{t}{2})$ contains at least one element $x \in X \backslash A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, A is F-proximinal). Then we have,

$$\begin{aligned} N(x_1, a_0 - \pi_A(x), t) &\leq \max\left\{N(x_1, a_0 - x, \tfrac{t}{2}), N(x_1, x - \pi_A(x), \tfrac{t}{2})\right\} \\ &= \max\left\{N(x_1, a_0 - x, \tfrac{t}{2}), N(x_1, A - x, \tfrac{t}{2})\right\} \\ &\leq \max\left\{N(x_1, a_0 - x, \tfrac{t}{2}), N(x_1, a_0 - x, \tfrac{t}{2})\right\} \\ &\leq \max\{\varepsilon', \varepsilon'\} = \varepsilon' \\ &< \varepsilon \end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \emptyset$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathcal{M}_A}$ which completes the proof. □

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