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RESULTS IN LINEAR 2-NORMED SPACES ANALOGOUS TO BAIRE'S THEOREM AND CLOSED GRAPH THEOREM

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Abstract: In this paper, the Baire's theorem for 2-Banach spaces is proved. The concept of continuous functions on linear 2- normed spaces is introduced and some relations between continuity and boundedness of linear functions are established. The result for a linear 2-normed space analogous to the closed graph theorem is proved.

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1. Introduction

The concept of linear 2-normed spaces and 2- metric spaces was introduced and investigated by S.Gähler in 1960's [7, 8]. The subject has been studied by great mathematicians like A. White,Y J Cho,R W Freese, S C Gupta, A H Siddique and others [2, 3, 9, 5] and they contributed a lot for the extension of this branch of mathematics. Recently many mathematicians came out with results in 2-normed spaces, analogous with that in classical normed spaces and Banach spaces. By a (K)-space, we mean a linear linear 2-normed space such that the

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2-metric induced by the 2-norm satisfies the (K) property [4]. The authors proved Baires theorem for 2-Banach (K)-spaces [6]. In this paper, we used the concept of Cauchy sequence and 2-Banach space as defined by A. White in [2] and proved the Baire's theorem for 2-Banach spaces. We introduced the concept of continuous functions on linear 2-normed spaces and established some results related to the continuity. In this setting we proved the closed graph theorem for a linear function defined on a 2- Banach space.

2. Preliminary Notes

The concept of 2-norm on a real linear space X of dimension greater than 1, is introduced as a 2 dimensional analogue of a norm, and is defined as a real valued function $\|.,.\|$, defined on $X \times X$ satisfying the following conditions:

For all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

$N1.$ $\|x, y\| = 0$ if and only if x and y are linearly dependent,

$N2.$ $\|x, y\| = \|y, x\|$,

$N3.$ $\|\alpha x, y\| = |\alpha| \|x, y\|$, and

$N4.$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|.,.\|)$ is called a linear 2-normed space.

It is observed that $\|.,.\|$ is nonnegative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$.

A simple and standard example of a 2-norm is the 2-norm $\|.,.\|$ on $\mathbb{R}^2$, defined by $\|a, b\| = |a_1 b_2 - a_2 b_1|$ where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. Geometrically this is the area of the parallelogram determined by the vectors a and b as the adjacent sides.

If $(X, \|.,.\|)$ is a linear 2-normed space, then for $e \in X$ the mapping $p_e(x) = \|x, e\|, x \in X$, defines a seminorm on X and in fact the collection $\{p_e,\ e \in X\}$ of seminorms makes X into a locally convex topological vector space. A typical basis element for this topology is a finite intersection, $\cap_{i=1}^{n} B_{e_i}(x, r_i)$ of open balls $B_{e_i}(x, r_i)$, where $B_{e_i}(x, r_i) = \{y \in X : \|y - x, e_i\| < r_i\},\ e_i \in X$ and $r_i > 0,\ i = 1, 2, \ldots, n$.

A sequence (x_n) in a linear 2-normed space $(X, \|.,.\|)$ is called a cauchy sequence if there exists two elements $y, z \in X$ such that y and z are linearly independent, $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$ and $\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0$.

A sequence (x_n) in a linear 2-normed space $(X, \|.,.\|)$ is convergent if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in X$.

A linear 2-normed space $(X, \|.,.\|)$ is called a 2-Banach space if every cauchy sequence in X is convergent.

By a 2-metric space (X, σ) we understand a set X in which for each triple $a, b, c \in X$ there corresponds a real number $\sigma(a, b, c)$ having the following conditions:

$M1$. For two distinct elements a and b in X there is an element c in X such that $\sigma(a, b, c) \neq 0$,

$M2$. $\sigma(a, b, c) = 0$ when two of the three elements are equal,

$M3$. $\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a)$, and

$M4$. $\sigma(a, b, c) \leq \sigma(a, b, d) + \sigma(a, d, c) + \sigma(d, b, c)$.

It is easly seen that $\sigma(a, b, c)$ is non negative.

For every linear 2-normed space $(X, \|.,.\|)$, the function defined on $X \times X \times X$ by $\sigma(x, y, z) = \|x - z, y - z\|, x, y, z \in X$ is a 2-metric and is called the 2-metric induced by the 2-norm $\|.,.\|$ [8].

Thus every linear 2-normed space $(X, \|.,.\|)$ will be considered to be a 2-metric space.

There is a well defined topology for a 2-metric space. For $\varepsilon > 0$ define the ε-neighborhood of two points a and b in X as the set $U_\varepsilon(a, b) = \{c \in X : \sigma(a, b, c) < \varepsilon\}$. Let $\mathscr{B}$ be the set of all intersections $\bigcap_i U_{\varepsilon_i}(a_i, b_i)$ of finitely many ε_i-neighborhoods of arbitrary points a_i and b_i in X. $\mathscr{B}$ forms a basis for the 2-metric topology of X and the system of all ε-neighborhoods of two points in X forms a subbasis for this topology. The family of all sets defined by $W_S(a) = \bigcap_i U_{\varepsilon_i}(a, b_i)$ with arbitrary finite pairs $S = \{(b_1, \varepsilon_1), \cdots, (b_n, \varepsilon_n)\}$ forms a complete system of neighborhoods of a. Let us denote this topology by τ_σ. We already noted that every linear 2-normed space $(X, \|.,.\|)$ is a 2-metric space. Suppose σ is the 2-metric induced by the 2-norm $\|.,.\|$, then from the very definition of the topology τ_σ, it coincides with the locally convex topology induced by the collection $\{p_e,\ e \in X\}$ of seminorms.

Theorem 2.1. (see [3]) *If $\mathscr{B}$ is a local base for a topological vector space X then every member of $\mathscr{B}$ contains the closure of some member of $\mathscr{B}$.*

3. Main Results

First of all we note that every linear 2-normed space X is regular. This result can be deduced from 2.1 or can directly be proved from the following proposition.

Proposition 3.1. *If* $(X, \|.,.\|)$ *is a 2-normed space,* $B_e(a,r) = \{x \in X : \|x-a,e\| < r\}$ *and* $B_e[a,r] = \{x \in X : \|x-a,e\| \leq r\}$*, then* $\overline{B_e(a,r)} \subseteq B_e[a,r]$

Proof. Let $y \notin B_e[a,r]$ and $\|y-a,e\| = r_1$. Then $r_1 > r$. Let $x \in B_e(a,r)$. Then we have

$$\begin{aligned}\|y-x,e\| &\geq ||y-a,e\| - |x-a,e\|| \\ &> r_1 - r > 0\end{aligned}$$

Therefore $x \notin B_z(y,\varepsilon)$ where $\varepsilon = r_1 - r$.That is there exists a neighborhood of y which does not intersects $B_e(a,r)$ implying that $y \notin \overline{B_e(a,r)}$. Therefore $\overline{B_e(a,r)} \subseteq B_e[a,r]$. □

The following theorem is a trivial one as it holds for any topological vector spaces.

Theorem 3.2. *Let* X *be a linear 2-normed space. Then the intersection of a finite number of dense open subsets of* X *is dense in* X*.*

Next we prove one of the main theorems of this paper.

Theorem 3.3. *(Baire's Theorem for 2-Banach spaces) If* X *is a 2-Banach space, then the intersection of a countable number of dense open subsets of* X *is dense in* X*.*

Proof. Let $D_1, D_2, \ldots$ be dense open subsets of X. Let $x_0 \in X$ and U_0 be a neighborhood of x_0. As X is regular there exists a neighborhood V_0 of x_0 such that $\overline{V}_0 \subset U_0$. As D_1 is dense in X, let $x_1 \in D_1 \cap V_0$. As $D_1 \cap U_0$ is open in X, by regularity of X, there exists a neighborhood V_1 of x_1 such that $\overline{V}_1 \subset D_1 \cap V_0$. Choose $U_1 = V_1 \cap B_{e_1}(x_1, \delta_1) \cap B_{e_2}(x_1, \delta_1)$ where e_1 and e_2 are two linearly independent elements in X and $\delta_1 > 0$. As D_2 is dense in X, let $x_2 \in D_2 \cap U_1$. As $D_2 \cap U_1$ is open in X, again by regularity of X, there exists a neighborhood V_2 of x_2 such that $\overline{V}_2 \subset D_2 \cap U_1$. Choose $U_2 = V_2 \cap B_{e_1}(x_2, \delta_2) \cap B_{e_2}(x_2, \delta_2)$, where $0 < \delta_2 < \delta_1$.Then $\overline{U}_2 \subset D_2 \cap U_1$.

Proceeding inductively suppose there exists a neighborhood V_n of x_n such that $\overline{V}_n \subset D_n \cap U_{n-1}$. Choose $U_n = V_n \cap B_{e_1}(x_n, \delta_n) \cap B_{e_2}(x_n, \delta_n)$ where $0 < \delta_n < \delta_{n-1}$. Then we have $\overline{U}_n \subset D_n \cap U_{n-1}$. As D_{n+1} is dense in X, let $x_{n+1} \in D_{n+1} \cap U_n$. As $D_{n+1} \cap U_n$ is open in X, again by regularity of X, there exists a neighborhood V_{n+1} of x_{n+1} such that $\overline{V}_{n+1} \subset D_{n+1} \cap U_n$. Choose $U_{n+1} = V_{n+1} \cap B_{e_1}(x_{n+1}, \delta_{n+1}) \cap B_{e_2}(x_{n+1}, \delta_{n+1})$ where $0 < \delta_{n+1} < \delta_n$. Then $\overline{U}_{n+1} \subset D_{n+1} \cap U_n$.

Thus we get a sequence (x_n) in X and a sequence (U_n) of open sets in X such that $x_n \in \overline{U}_n \subset U_m$ for all $n > m,\ m = 1, 2, \ldots$. Now for a positive integer

m, if $x_i, x_j \in U_m$ then $\|x_i - x_j, e_k\| < 2\delta_m$ for all $i, j > m$ and for $k = 1, 2$. By choosing (δ_n) so that $\lim_{n\to\infty} \delta_n = 0$, we see that the sequence (x_n) is a cauchy sequence. Since X is a 2-Banach space, (x_n) converges to some point $x \in X$ and $x \in \overline{U}_n, \forall n \in \mathbb{N}$. But $\overline{U}_n \subset U_m \subset (\cap_{i=1}^{m} D_i) \cap U_0, \forall\ n > m$ which implies $x \in (\cap_{i=1}^{\infty} D_i) \cap U_0$. As $x_0 \in X$ and U_0 are arbitrary, $\cap_{i=1}^{\infty} D_i$ is dense in X.

□

Definition 3.4. A linear function F from a 2-normed space $(X, \|.,.\|_X)$ into a 2-normed space $(Y, \|.,.\|_Y)$ is said to be bounded, if there exists $K > 0$ such that

$$\|F(x), F(y)\|_Y \leq K\|x, y\|_X,\ \forall x, y \in X$$

Definition 3.5. A linear function F from a 2-normed space $(X, \|.,.\|_X)$ into a 2-normed space $(Y, \|.,.\|_Y)$ is continuous at $x_0 \in X$ if for given $z \in X$ and $\varepsilon > 0$ there exist $\delta > 0$ and $z_1, \cdots, z_m \in X$ such that $\|F(y) - F(x_0), F(z)\|_Y < \varepsilon$ for every $y \in X$ for which $\|y - x_0, z_i\|_X < \delta$ for $i = 1, 2, \cdots, m$.

F is continuous if it is continuous at every point in X

Proposition 3.6. *Every bounded linear function is continuous.*

Proof. Let a linear function F from a 2-normed space $(X, \|.,.\|)$ into a 2-normed space $(Y, \|.,.\|)$ be a bounded function, i.e there exists $K > 0$ such that $\|F(x), F(z)\| \leq K\|x, z\|,\ \forall x, z \in X$. Then for $z \in X$ and $\varepsilon > 0$ there exist $\delta = \frac{\varepsilon}{K} > 0$ such that $\|F(x) - F(y), F(z)\| < \varepsilon$ whenever $\|x - y, z\| < \delta$. i.e, F is continuous. □

Let $F : X \to Y$ be a function from a linear 2-normed space X into a linear 2-normed space Y. We say that F is bounded on basis element $B = \cap_{i=1}^{n} B_{e_i}(x_0, \delta)$ if there exists $M > 0$ such that $\|F(x - x_0), F(e_i)\| < M$ whenever $x \in B$ (or $\|x - x_0, e_i\| < \delta\ \forall\ i = 1, \cdots, n$). We say F is bounded on a neighborhood U of x_0 if F is bounded on every basis element B containing x_0 and contained in U.

Theorem 3.7. *Let X and Y be linear 2-normed spaces and $F : X \to Y$ be a linear map.*

(i) F is bounded in some neighborhood of 0,

(ii) F is continuous at 0,

(iii) F is continuous on X.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Proof. $(i) \Rightarrow (ii)$: Suppose F is bounded on a neighborhood U of 0 and $B = \cap_{i=1}^{n} B_{e_i}(0, \delta)$ be a basis element contained in U. Then there exists M such that $\|F(y), F(e_i)\| < M$ for all $y \in B$. Let $x \in X$ and $\varepsilon > 0$ be given. Consider a basis element $B' = B \cap B_x(0, \delta)$. Then we have $\|F(y), F(x)\| < M$ for every $y \in B'$. Choose $z = (\frac{\varepsilon}{M})y$. Then $y \in B'$ if and only if $z \in \cap_{i=1}^{n} B_{e_i}(0, \delta') \cap B_x(0, \delta')$ where $\delta' = (\frac{\varepsilon}{M})\delta$. Thus we have $\|F(z), F(x)\| < \varepsilon$ for every z such that $\|z, e_i\| < \delta'$ and $\|z, x\| < \delta'$. Therefore by definition F is continuous at 0.

$(ii) \Rightarrow (iii)$: Assume F is continuous at 0. Let $z \in X$ and $\varepsilon > 0$. There exists $\delta > 0$ and elements $z_1, \cdots, z_m \in X$ such that $\|F(y), F(z)\| < \varepsilon$ whenever $\|y, z_i\| < \delta$ for $i = 1, \cdots, m$. Then for every x for which $\|y - x, z_i\| < \delta$ for $i = 1, \cdots, m$, we have $\|F(y) - F(x), F(z)\| < \varepsilon$. i.e F is continuous at x. □

Definition 3.8. Assume X and Y are 2-normed spaces. A map $F : X \to Y$ is said to be closed if $x_n \to x$ in X and $F(x_n) \to y$ in Y imply that $y = F(x)$.

Remark 3.9. If F is continuous then it is closed.

Proposition 3.10. *If $F : X \to Y$ is a bijective closed map then F^{-1} is also closed.*

Proof. If $y_n \to y$ in Y and $F^{-1}(y_n) \to x$ in X. Letting $x_n = F^{-1}(y_n)$ we have $x_n \to x$ in X and $F(x_n) \to y$ in Y. Since F is closed $y = F(x)$, *i.e* $x = F^{-1}(y)$, as desired. □

Lemma 3.11. (see [1]) *Let X be a linear space over $\mathbb{K}$. Consider the subsets U and V and $k \in \mathbb{K}$ such that $U \subset V + kU$. Then for each $x \in U$ there is a sequence v_n in V such that*

$$x - (v_1 + kv_2 + \cdots + k^{n-1}v_n) \in k^n U, n = 1, 2, ...$$

Theorem 3.12. *(Closed graph theorem for linear 2-normed spaces) Let X and Y be two 2-Banach spaces. If $F : X \to Y$ is a closed linear map, then F is continuous.*

Proof. Let $\{z_1, z_2, \cdots, z_r\}$ be a fixed subset of X such that z_1 and z_2 are two linearly independent elements in X for which $F(z_1)$ and $F(z_2)$ are linearly independent in Y. If such a pair z_1 and z_2 does not exist then $F(x)$ and $F(y)$ would be linearly dependent for every pair x and y in X, resulting that $\|F(x), F(y)\| = 0$ for every pair $x, y \in X$ and consequently F would be continuous.

For each $n \in \mathbb{N}$ let

$$V_n = \{x \in X : \|F(x), F(z_i)\| \leq n, \text{ for } i = 1, \cdots, r\}$$

We prove that some V_n contains a neighborhood of 0 in X. Clearly,

$$X = \cup_{n=1}^{\infty} V_n = \cup_{n=1}^{\infty} \overline{V}_n,$$

therefore

$$\cap_{n=1}^{\infty} \overline{V}_n^c = \phi$$

As X is a 2-Banach space, by Baire's theorem 3.3 at least one $\overline{V}_n^c$ will not be dense in X. Hence there is a positive integer p and some x_0 in X such that a neighborhood of x_0 is contained in $\overline{V}_p$. We assume that this neighborhood is $B_{x_0} = \cap_{i=1}^{q} B_{e_i}(x_0, \delta)$. Choosing $B'_{x_0} = \cap_{i=1}^{q} B_{e_i}(x_0, \delta) \cap_{i=1}^{m} B_{z_i}(x_0, \delta)$ we have $B'_{x_0} \subset \overline{V}_p$. So we can assume without loss of generality that $\{z_1, z_2, \cdots, z_r\} \subset \{e_1, \cdots, e_q\}$ and we can take $B_{x_0} = \cap_{i=1}^{q} B_{e_i}(x_0, \delta) \subset \overline{V}_p$.

We claim that $B_0 = \cap_{i=1}^{q} B_{e_i}(0, \delta) \subset V_{4p}$. First we prove that $\cap_{i=1}^{q} B_{e_i}(0, \delta) \subset \overline{V}_{2p}$. Suppose $x \in B_0$, then $\|x, e_i\| < \delta$, for $i = 1, \cdots, q$ and $x + x_0 \in B_{x_0}$. Also $x_0 \in \overline{V}_p$. So there are sequences (a_n) and (b_n) in V_p such that $(a_n) \to x + x_0$ and $(b_n) \to x_0$, resulting into $(a_n - b_n) \to x$. Since

$$\|F(a_n - b_n), F(z_i)\| \leq \|F(a_n), F(z_i)\| + \|F(b_n), F(z_i)\| \leq 2p, \quad \text{for } i = 1, \cdots, r$$

we have $(a_n - b_n)$ is a sequence in V_{2p}. As the sequence $(a_n - b_n)$ converges to x, there exist $N_0 \in \mathbb{N}$ such that $\|(a_n - b_n) - x, e_i\| < \frac{\delta}{2}$ for all $n \geq N_0$. Choose $x_1 = a_{N_0} - b_{N_0}$. Then we have for every $x \in B_0$ there is $x_1 \in V_{2p}$ such that $\|x - x_1, e_i\| < \frac{\delta}{2}$ for $i = 1, \cdots, q$.

Thus $B_0 \subset V_{2p} + \frac{1}{2} B_0$, for if $x \in B_0$, $x = x_1 + (x - x_1) \in V_{2p} + \frac{1}{2} B_0$.

The lemma 3.11 with $U = B_0$, $V = V_{2p}$, $k = \frac{1}{2}$ follows that there is a sequence v_n in V_{2p} such that

$$x - (v_1 + \frac{v_2}{2} + ... + \frac{v_n}{2^{n-1}}) \in \frac{1}{2^n} B_0, \text{ for } n = 1, 2, \ldots$$

Let $w_n = v_1 + \frac{v_2}{2} + \cdots + \frac{v_n}{2^{n-1}}$ for $n = 1, 2, \ldots$, then $\|x - w_n, z_i\| < \frac{\delta}{2^n}$.

So w_n is a cauchy sequence in X, for $\|w_m - w_n, z_i\| \leq \|x - w_n, z_i\| + \|x - w_m, z_i\| < \frac{\delta}{2^n} + \frac{\delta}{2^m} \to 0$ as $m, n \to \infty$, valid for two linearly independent z_i's. Since X is complete w_n converges to some $y \in X$.

As $\lim_{n\to\infty} \|x - w_n, z_i\| = \lim_{n\to\infty} \|y - w_n, z_i\| = 0$, we have $|x - y, z_i\| \leq |x - w_n, z_i\| + |x - w_n, z_i\| \to 0$ when $n \to \infty$.Therefore $|x - y, z_i\| = 0$ which shows that $x - y$ is a scalar multiple of at least two linearly independent z_i's and consequently $y = x$.

Also for all $n > m$ we have,

$$\|F(w_n) - F(w_m), F(z_i)\| \leq \|F\left(\sum_{j=m+1}^{n} \frac{v_j}{2^{j-1}}\right), F(z_i)\|$$

$$\leq \sum_{j=m+1}^{n} \left(\frac{\|F(v_j), F(z_i)\|}{2^{j-1}} \right)$$
$$\leq \frac{4p}{2^m} \to 0 \text{ as } m \to \infty \text{ for } i = 1, \cdots, r.$$

Since $F(z_i)$, $i = 1, 2$ are linearly independent, $F(w_n)$ is a cauchy sequence in Y. Since Y is 2- Banach $F(w_n)$ is convergent in Y and as F is closed $\lim_{n\to\infty} F(w_n) = F(x)$.

Now let $m = 0$ and $w_0 = 0$ we get $\|F(w_n), F(z_i)\| \leq 4p$ for all $n \geq 1$ and $i = 1, \cdots, r$.

$$\|F(x), F(z_i)\| \leq 4p, \ \forall i = 1, \cdots, r.$$

Since $x \in B_0$ is arbitrary $B_0 \subset V_{4p}$. By theorem 3.7, F is continuous. □

From the proposition 3.10 and the theorem 3.12 we have the following corollary:

Corollary 3.13. *Let X and Y be two 2-Banach spaces. If $F : X \to Y$ is a closed bijective linear map, then F is an isomorphism.*

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