International Journal of Pure and Applied Mathematics
Volume 74 No. 4 2012, 519-525
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

TACTICAL CONFIGURATION, PBIB DESIGNS AND ASSOCIATION SCHEME ARISING FROM SRNT GRAPHS

Anwar Alwardi[1 §], Ahmad N. Al-Kenani[2], N.D. Soner[3]

[1,3]Department of Studies in Mathematics
University of Mysore
Mysore, 570006, INDIA

[2]Department of Mathematics
King Abdulaziz University
P.O. Box 80219, Jeddah, 21589, SAUDI ARABIA

Abstract: In this paper we construct spacial types of partial balanced incomplete block designs from Petersen, and Clebsch graph with parameters $(16, 5, 0, 2)$, and then we generalize this result for all strongly regular graphs without triangles.

AMS Subject Classification: 05E30
Key Words: partial balanced incomplete block designs, clebsch graph, strongly regular graphs, net

1. Introduction

The strongly regular graphs and its relation with partial balanced incomplete block designs(PBIBD's) and Association Scheme were studied in [1] and [3] they have shown that the strongly regular graphs are emerged from PBIBD with two association schemes. Very recently papers [1] and [7] which show the relation between the dominating sets of strongly regular graphs and PBIBD's. In this paper, we construct spacial type of partial balanced incomplete block designs, in Petersen graph and folded 5- cube$(16, 5, 0, 2)$ graph and we generalize this result to all SRNT graphs.

Received: November 14, 2011

§Correspondence author

We refer to [4], [5] and [6] for the necessary background about strongly regular graphs, and PBIBD's

2. Definitions and Notations

Definition 1. A strongly regular graph with no triangles (SRNT graph) G with the parameters $(n,k,0,\mu)$ is k- regular graph with n vertices such that for any two adjacent vertices have no common neighbours, and any two non-adjacent vertices have μ common neighbours.

Lemma 2. *If G is an strongly regular graph with parameters $(n,k,0,\mu)$ then $k(k-1)=\mu(n-k-1)$*

Proof. It is clear from [3] that for any strongly regular graph(n,k,λ,μ) we have $k(k-\lambda-1)=\mu(n-k-1)$ and in strongly regular graph without triangles we have $\lambda=0$, then directly we get $k(k-1)=\mu(n-k-1)$. □

Definition 3. Given v objects a relation satisfying the following conditions is said to be an association scheme with m classes:

(i) any two objects are either first associates, or second associates,..., or m^{th} associates, the relation of association being symmetric.

(ii) each object α has n_i i^{th} associates, the number n_i being independent of α.

(iii) if two objects α and β are i^{th} associates, then the number of objects which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β. Also p^i_{jk}=p^i_{kj}.

Definition 4. The PBIB design is arrangement of v objects into b sets (called blocks) of size k where $k<v$ such that

(i) every object is contained in exactly r blocks.

(ii) each block contains k distinct objects.

(iii) Any two objects which are i^{th} associates occur together in exactly λ_i blocks.

Definition 5. A triple (V,B,I) is called incidence structure, where V and B are two non empty disjoint sets and $I\subseteq V\times B$. The elements of V are called points, the elements of B are called blocks, and the elements of I are called flags.

Definition 6. A $t-(v,k,\lambda)$ design, is an incidence structure with v points, k points on a block, and any subset of t points is contained in exactly λ blocks, where $v > k$, $\lambda > 0$. The number of blocks is b and the number of blocks on a point is r. The design$D$ is resolvable if its blocks can be partitioned into r parallel classes, such that each parallel class partitions the point set of D. Blocks in the same parallel class are parallel. Clearly each parallel class has $m = v/k$ blocks. D is affine resolvable, or simply affine, if it can be resolved so that any two nonparallel blocks meet in μ points, where $\mu = k/m = k2/v$ is constant. Affine 1-designs are also called nets. The dual design of a design D denoted $D^{\square}$. If D and $D^{\square}$ are both affine, we call D a (μ, m) symmetric net.

Definition 7. A 1- design $1-(v,k,r)$ is called tactical configuration (or simply configuration) with parameters v, r, k and $b := vr/k$.

3. Results

Proposition 8. *Let u and v be any two adjacent vertices in Petersen graph in Figure. 1 then a design with point set $P = \{N(u) \bigcup N(v)\}$-{u,v}and block set B contains those vertices which has distance 2 from u and v is PBIB design, and this design is symmetric*$(1,2)$ *net.*

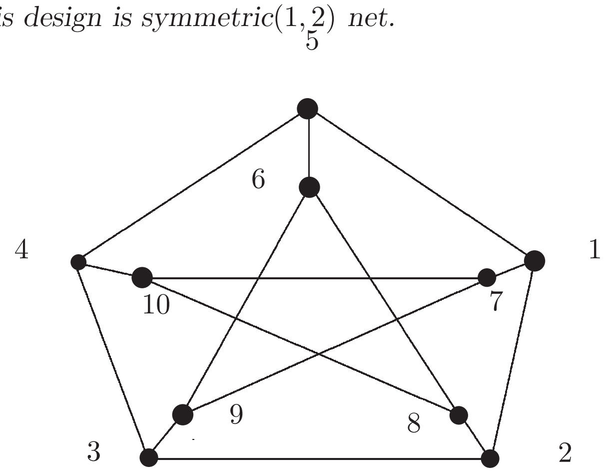

Figure 1

Proof. By fixing any two adjacent vertices in Petersen graph for example 1 and 2 see Figure.1. Then the point set is $P = \{3, 5, 7, 8\}$ and the block set is

$\{4, 6, 9, 10\}$, where
$4 := \{5, 3\}$,
$6 := \{5, 8\}$,
$9 := \{3, 7\}$,
$10 := \{7, 8\}$.

we can define 2-association scheme on the points as follows

The two points in P are first associates, if they do not belong to any block set, second associates, if they belong to exactly two blocks, and with this definition we give the table of association scheme below

Elements	First Associates	Second Associates
3	8	5,7
5	7	3,8
7	5	3,8
8	3	5,7

With this association scheme. One can verify the design $D(P, B)$ is a PBIB design $(v\prime, b\prime, r\prime, k\prime, \lambda_1, \lambda_2) = (4, 4, 2, 2, 0, 1)$ and $P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can also prove the design $D(P, B)$ is $1 - (4, 2, 2)$ is affine design which its dual is also affine design with the same parameters; that is D is a symmetric $(1, 2)$ net.

□

Proposition 9. *Let u and v be any two adjacent vertices in Clebsch graph in Figure. 2 then a design with point set $P = \{N(u) \bigcup N(v)\}$-$\{u$,$v\}$and block set B contains those vertices which has distance 2 from u and v is PBIB design.*

Proof. By fixing in two adjacent points in clebsch graph for example 1 and 2 see Figure. 2. Then the point set is $P = \{3, 5, 6, 8, 9, 10, 12, 13\}$ and the block set is $\{4, 7, 11, 14, 15, 16\}$, where
$4 := \{3, 5, 6, 8\}$,
$7 := \{3, 5, 12, 13\}$,
$11 := \{5, 6, 10, 13\}$,
$14 := \{3, 8, 9, 12\}$,
$15 := \{9, 10, 12, 13\}$,
$16 := \{6, 8, 10, 14\}$.

Now it is clear that any point in the set P appear in 3 block sets and we can see any two points u, v in P one of the following statements are true

(I) u, v does not belong to block set.

(II) u, v belong to exactly one block.
(III) u, v belong to exactly two blocks.

On the basis of the properties above, we can define 3-association scheme on the points as follows

The two points in P are first associates, if they do not belong to any block set, second associates, if they belong to exactly one block set and third associates, if they belong to exactly two block sets. and with this definition we give the table of association scheme below

Elements	First associates	Second associates	Third associates
3	10	6,9,13	5,8,12
5	9	8,10,12	3,6,13
6	12	3,9,13	5,8,10
8	13	5,12,10	3,9,6
9	5	3,6,13	8,10,12
10	3	5,8,12	6,9,13
12	6	5,8,10	3,9,13
13	8	3,6,9	5,10,12

With this association scheme, one can easily verify that, the design $D = (P, B)$ which define above is a PBIB design with the parameters $(v\prime, b\prime, r\prime, k\prime, \lambda_1, \lambda_2, \lambda_3) = (8, 6, 3, 4, 0, 1, 2)$ and $P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}$ □

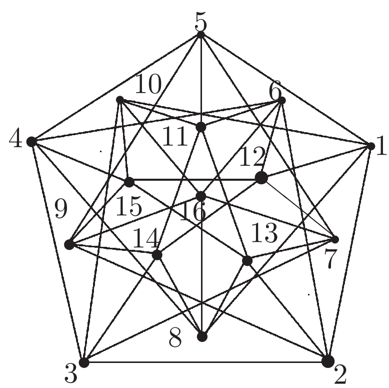

Figure. 2

It is easy to prove that the PBIB design arising from Clebsch graph above is a tactical configuration with the parameters $1-(8, 4, 3)$, and so we can using the same method for other SRNT graphs from Hoffman-singleton graph(50,7,0,1) we get the tactical configuration $1-(12, 2, 6)$, from Gewirtz graph $(56, 10, 0, 2)$, we get the tactical configuration $1-(18,4,8)$, and from M_{22} graph $(77,16,0,4)$, we get the tactical configuration $1-(30,8,12)$.

So we can make generalization for all strongly regular graphs without triangles by the next theorem.

Lemma 10. *If G is strongly regular graph without triangles with the parameters $(n,k,0,\mu)$. Then $(2k-2)(k-\mu) = 2\mu(n-2k)$.*

Proof. We have $(2k-2)(k-\mu) = 2(k^2+(1-k)\mu-k)$, and by Lemma 1.2 we have $n = \frac{k(k-1)}{\mu} + k + 1$, by substitute in $2\mu(n-2k)$, we get $2\mu(n-2k) = 2\mu(\frac{k(k-1)}{\mu}+k+1-2k) = 2(k^2+(1-k)\mu-k)$. Hence $(2k-2)(k-\mu) = 2\mu(n-2k)$.

□

Theorem 11. *Let G be SRNT graph with the parameters $(n,k,0,\mu)$ and let v,u be any two adjacent vertices and let D be the design which its point set is $P = \{N(v)\bigcup N(u)\} - \{u,v\}$ and block set B contains those vertices which has distance 2 from v and u. Then D is a tactical configuration $1-(2k-2, 2\mu, k-\mu)$.*

Proof. First we will prove that each point in P is appear in $k-\mu$ blocks, let x be any point in P, then either in $N(u)$ or in $N(v)$ and x does not belong to both $N(v)$ and $N(u)$. suppose x is in $N(v)$, then we know that X is of degree k in G that is the point x adjacent with μ vertices in $N(u)$ because x and u are non adjacent, hence x adjacent with $k-\mu$ vertices in the set B, then every point in P is appear in $k-\mu$ blocks. Clearly to prove that eash block contains 2μ points, let C be any block set in B, then C as a vertex in G has μ common neigbourhoods with the vertex v and it has μ common neigbourhoods with the vertex u so C as a block set contains 2μ from the point set P, and also we can prove that the nuber of blocks $b = n-2k$, also by lemma 3.3 we have $(2k-2)(k-\mu) = 2\mu(n-2k)$. Hence $1-(2k-2, 2\mu, k-\mu)$ is a tactical configuration.

□

References

[1] Anwar Alwardi, N. D. Soner, Partial balanced incomplete block designs arising from some minimal dominating sets of SRNT graphs, *International Journal of Mmathematical Archive*, **2**, No. 2 (2011), 233-235.

[2] R.C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.*, **13** (1963), 389-419.

[3] A.E. Brouwer. Strongly regular graphs, In: *Handbook of Combinatorial Designs* (Ed-s: C. Colbourn, J. Dinitz), CRC Press (1996).

[4] A.E. Brouwer, J.H. Van Lint, Strongly regular graphs and partial geometries, In: *Enumeration and Design*, Academic Press, Waterloo, Ont. (1982), 85-122.

[5] P.J. Cameron, J. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press (1991).

[6] C.D. Godsil, G.F. Royle, *Algebraic Graph Theory*, Springer (2001).

[7] H.B. Walikar, H.S. Ramane, B.D. Acharya, H.S. Shekhareppa, S. Arumugum, Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles, *AKCE J. Graphs Combin.*, **4**, No. 2 (2007), 223-232.