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ON THE EXISTENCE OF SOLUTIONS OF A NONLOCAL QUASILINEAR ELLIPTIC EQUATION

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Abstract: In this paper, we study the existence of solutions of a nonlocal quasilinear elliptic problem. Our analysis is based on Galerkin method.

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1. Introduction

In this paper, we study the following nonlocal elliptic problem

$$
\begin{gathered}
-\left(\int_\Omega f(u)\right)^\beta \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x,u,\nabla u)\right) = g(x,u) - C_0 u, \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{1}
$$

where $\Omega \in R^N$ is a bounded smooth domain.

If the quasilinear term $\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x,u,\nabla u)\right)$ is replaced by Δu, and $g(x,u) - C_0 u$ is replaced by $(f(u))^\alpha$, then the equation

$$
-\left(\int_\Omega f(u)\right)^\beta \Delta u = (f(u))^\alpha \tag{2}
$$

arises in numerous physical models such as: systems of particles in thermody-

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namical equilibrium via gravitational (Coulomb) potential, 2-D fully turbulent behavior of real flow, thermal runaway in Ohmic Heating, shear bands in metal deformed under high strain rates, see [1] for references of these applications.

Nonlocal elliptic problems similar to (2) have been investigated by several authors, see [1]-[4]. In these literatures, they study the existence of solutions by variational method or Galerkin method.

However, to the best of our knowledge, we didn't find any literature studying nonlocal quasilinear elliptic problem like (1). In this work, we prove the existence of weak solutions to problem (1) based on Galerkin method. The techniques we use to deal with the quasilinear term are inspired by the work of F. Guglielmino, F. Nicolosi and P. Dr*á*bek, see [5, p. 41-65].

We need the following results whose proof may be found in [6, p. 53]

Lemma 1. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ a continuous function with $\langle F(x), x\rangle \geq 0$, for x satisfying $|x| = R > 0$, where $\langle x, y\rangle$ is the usual inner product of $\mathbb{R}^m$. Then, there exists $z_0 \in \overline{B}_R(0)$ such that $F(z_0) = 0$.*

We point out that, by a weak solution of (1), we mean a function $u \in H_0^1(\Omega)$ such that

$$\left(\int_\Omega f(u)\right)^\beta \int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u)\frac{\partial\varphi}{\partial x_i}dx = \int_\Omega (g(x, u) - C_0 u)\,\varphi dx \qquad (3)$$

for all $\varphi \in H_0^1(\Omega)$.

This paper is organized as follows: Section 2 is devoted to derive apriori estimates for solutions of problem (1). In Section 3, we apply Galerkin method to prove the existence of weak solutions of problem (1).

2. Apriori Estimates

Throughout this paper, we make the following assumptions

(H1) The function $a_i(x, \xi) : \ \Omega \times (-\infty, \infty)^{N+1} \to [0, \infty)$, where $\xi = (\xi_0, \xi_1, \cdots, \xi_N)$, satisfies

$$\sum_{i=1}^N a_i(x, \xi)\xi_i \geq c \sum_{i=1}^N |\xi_i|^2$$

where c is a positive constant.

(H2) There exists positive constants C_1, C_2 and C_3 such that

$$|a_i(x,\xi)| \leq C_1 + C_2|\xi_0| + C_3 \sum_{i=1}^{N} |\xi_i|$$

(H3) There exists positive constant α such that

$$\left[a(x,\xi_0,\hat{\xi_1}) - a(x,\xi_0,\hat{\xi_2})\right] \cdot \left[\hat{\xi_1} - \hat{\xi_2}\right] \geq \alpha \left|\hat{\xi_1} - \hat{\xi_2}\right|^2$$

for any $\hat{\xi_1},\ \hat{\xi_2}$ in $\mathbb{R}^N$, where the function $a = (a_1, \cdots, a_N)$.

(H4) The function $f:\ (-\infty, +\infty) \to (0, +\infty)$ satisfies $f_\infty \geq f \geq f_0 > 0$, where f_0 and f_∞ are positive constants.

(H5) The function $g:\ \Omega \times (-\infty, +\infty) \to (0, +\infty)$ satisfies $g_\infty > g > 0$, where g_∞ is a positive constant.

We use the techniques from [5, p.44] to prove the following lemma.

Lemma 2. *Assume that (H1)-(H5) hold and $C_0 > 0,\ \beta > 0$, let u be a weak solution of (1), then*

$$\|u\|_\infty \leq L$$

where $L = \frac{g_\infty}{C_0} \max\{|\Omega|, 1\}$.

Proof. In (3), let $\varphi = u|u|^r,\ r > 0$. We have

$$\left(\int_\Omega f(u)\right)^\beta \int_\Omega \sum_{i=1}^{N} a_i(x,u,\nabla u)(r+1)|u|^r \frac{\partial u}{\partial x_i} dx + \int_\Omega C_0 u^2 |u|^r dx \\ - \int_\Omega g(x,u) u|u|^r dx = 0$$

By (H1) and (H5) we get

$$\left(\int_\Omega f(u)\right)^\beta \int_\Omega c(r+1)|u|^r |\nabla u|^2 dx + C_0 \int_\Omega |u|^{2+r} dx \leq g_\infty \int_\Omega |u|^{r+1} dx$$

Hence,

$$C_0 \int_\Omega |u|^{2+r} dx \leq g_\infty \int_\Omega |u|^{r+1} dx - \left(\int_\Omega f(u)\right)^\beta \int_\Omega c(r+1)|u|^r |\nabla u|^2 dx$$

Since $c,\ r > 0$ and $f > 0$, we have

$$\int_\Omega |u|^{2+r} dx \le \frac{g_\infty}{C_0} \int_\Omega |u|^{r+1} dx$$

Apply Hölder's Inequality, we obtain

$$\begin{aligned}\int_\Omega |u|^{2+r} dx &\le \frac{g_\infty}{C_0} \left[\int_\Omega \left(|u|^{r+1}\right)^{\frac{r+2}{r+1}} dx \right]^{\frac{r+1}{r+2}} |\Omega|^{\frac{1}{r+2}} \\ &= \frac{g_\infty}{C_0} \left(\int_\Omega |u|^{r+2} dx \right)^{\frac{r+1}{r+2}} |\Omega|^{\frac{1}{r+2}}\end{aligned}$$

Then

$$\left(\int_\Omega |u|^{r+2} dx \right)^{\frac{1}{r+2}} \le \frac{g_\infty}{C_0} |\Omega|^{\frac{1}{r+2}} \le \frac{g_\infty}{C_0} \max\{|\Omega|, 1\} = L. \tag{4}$$

If $\|u\|_\infty \le L$ is not true, then there exists $\eta > 0$, and a set $A \subset \Omega,\ |A| > 0$ ($|A|$ means measure of A), such that $u(x) \ge L + \eta$ for $x \in A$.

Then

$$\begin{aligned}\left(\int_\Omega |u|^{r+2} dx \right)^{\frac{1}{r+2}} &\ge \left(\int_A |u|^{r+2} dx \right)^{\frac{1}{r+2}} \\ &\ge (L+\eta)|A|^{\frac{1}{r+2}}\end{aligned}$$

Let $r \to \infty$, then we derive a contradiction with (4). Hence $\|u\|_\infty \le L$. The proof is completed. □

Lemma 3. *Assume that (H1)-(H5) hold and $C_0 > 0,\ \beta > 0$, let u be a weak solution of (1), then*

$$\int_\Omega |\nabla u|^2 dx \le M$$

where $M = \frac{g_\infty L |\Omega|}{c(f_0|\Omega|)^\beta}$, and L is given in Lemma 2.

Proof. In (3), let $\varphi = u$, then

$$\left(\int_\Omega f(u) \right)^\beta \int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial u}{\partial x_i} dx + \int_\Omega C_0 u^2 dx - \int_\Omega g(x,u) u dx = 0$$

By (H1), (H4) and (H5), we have

$$(f_0|\Omega|)^\beta c \int_\Omega |\nabla u|^2 dx + \int_\Omega C_0 u^2 dx \le \int_\Omega g_\infty |u| dx$$

$$
\begin{aligned}
(f_0|\Omega|)^{\beta} c \int_{\Omega} |\nabla u|^2 dx &\le g_{\infty} \int_{\Omega} |u| dx \\
&\le g_{\infty} L|\Omega|
\end{aligned}
$$

Therefore,

$$
\int_{\Omega} |\nabla u|^2 dx \le \frac{g_{\infty} L|\Omega|}{c(f_0|\Omega|)^{\beta}} = M. \qquad \square
$$

3. Main Results

In this section, we prove the main theorem via Galerkin method.

Theorem 4. *Assume (H1)-(H5) hold, and* $C_0 > 0,\ \beta > 0$, *then problem (1) has at least one weak solution.*

Proof. Let $\{\varphi_1, \cdots, \varphi_m, \cdots\}$ be an orthonormal basis of the Hilbert space $H_0^1(\Omega)$ endowed with the norm

$$
\|u\|^2 = \int_{\Omega} |\nabla u|^2
$$

For each $m \in \mathbb{N}$, let $V_m = \operatorname{span}\{\varphi_1, \cdots, \varphi_m\}$ be the finite dimensional Hilbert space spanned by $\varphi_1, \cdots, \varphi_m$. Each $u \in V_m$ is written as $u = \sum_{i=1}^m \xi_i \varphi_i$. We use on V_m the norm $\|u\|_m := \sum_{i=1}^m |\xi_i|$.

$(V_m,\ \|\cdot\|_m)$ and $(\mathbb{R}^m,\ |\cdot|)$ are isomorphic (we use $|\cdot|$ to indicate Euclidian norm in $\mathbb{R}^m$) through the following map.

$$
u = \sum_{i=1}^{m} \xi_i \varphi_i \to T(u) = \xi = (\xi_1, \xi_2, \cdots, \xi_m)
$$

Consider the function $F:\ \mathbb{R}^m \to \mathbb{R}^m$, defined by

$$
\begin{aligned}
F(\xi) &= (F_1(\xi), \cdots, F_m(\xi)), \\
F_j(\xi) &= \left(\int_{\Omega} f(u) \right)^{\beta} \int_{\Omega} \sum_{i=1}^{N} a_i(x, u, \nabla u) \frac{\partial \varphi_j}{\partial x_i} dx \\
&\quad + \int_{\Omega} C_0 u \varphi_j dx - \int_{\Omega} g(x, u) \varphi_j dx
\end{aligned}
$$

where $j = 1, \cdots, m$ and $u = \sum_{i=1}^m \xi_i \varphi_i$.

Then

$$\langle F(\xi),\xi\rangle = \left(\int_\Omega f(u)\right)^\beta \int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial u}{\partial x_i}dx + \int_\Omega C_0 u^2 dx - \int_\Omega g(x,u)u\,dx$$

By (H1), (H4) and (H5), we have

$$\langle F(\xi),\xi\rangle \geq c\,(f_0|\Omega|)^\beta \int_\Omega |\nabla u|^2 dx - g_\infty \int_\Omega |u|dx$$

Apply Poincar$\acute{e}$ and H$\ddot{o}$lder inequality we get

$$\langle F(\xi),\xi\rangle \geq c\,(f_0|\Omega|)^\beta \|u\|^2 - g_\infty C\|u\| > 0$$

for $\|u\| = r,\ r$ large enough, independently of m.

By Lemma 1, there exists $u_m \in V_m,\ \|u_m\| \leq r$, such that

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial \varphi_j}{\partial x_i} + \int_\Omega C_0 u_m \varphi_j - \int_\Omega g(x,u_m)\varphi_j = 0$$

where $j = 1,2,\cdots,m$

This implies that

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial \varphi}{\partial x_i} + \int_\Omega C_0 u_m \varphi - \int_\Omega g(x,u_m)\varphi = 0 \quad (5)$$

for all $\varphi \in H_0^1(\Omega)$.

Next, we prove that the sequence $\{u_m\} \subset H_0^1(\Omega)$ has a convergent subsequence which converges to a solution of (1).

Since $\|u_m\| \leq r$, then there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, we have

$$\begin{aligned} &u_m \rightharpoonup u \text{ weakly in } H_0^1(\Omega),\\ &u_m \to u \text{ strongly in } L^2(\Omega),\\ &u_m(x) \to u(x) \text{ almost everywhere in } \Omega \end{aligned}$$

In (5), let $\varphi = u_m - u$, then

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial (u_m-u)}{\partial x_i} + \int_\Omega C_0 u_m(u_m-u) \qquad (6)$$

$$- \int_\Omega g(x, u_m)(u_m - u) = 0$$

We can write (6) as

$$\left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N (a_i(x, u_m, \nabla u_m) - a_i(x, u_m, \nabla u)) \frac{\partial(u_m - u)}{\partial x_i}$$
$$+ \int_\Omega \sum_{i=1}^N a_i(x, u_m, \nabla u) \frac{\partial(u_m - u)}{\partial x_i}$$
$$= \int_\Omega g(x, u_m)(u_m - u) - \int_\Omega C_0 u_m(u_m - u)$$

By (H2) we have

$$|a_i(x, u_m, \nabla u)| \leq C_1 + C_2|u_m| + C_3 \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|$$

Then by Hölder Inequality and some algebraic operations, we can easily find out that $\int_\Omega |a_i(x, u_m, \nabla u)|^2$ is bounded by the expression involving $|\Omega|$, $\int_\Omega |u_m|^2$ and $\int_\Omega |\nabla u|^2$. From the proof of Lemma 2 in Section 2, we easily see that $\|u_m\|_\infty \leq L$. So we know that $a_i(x, u_m, \nabla u)$ belongs to $L^2(\Omega)$, then the weak convergence of u_m to u in $H_0^1(\Omega)$ implies that

$$\lim_{m\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, u_m, \nabla u) \frac{\partial(u_m - u)}{\partial x_i} dx = 0$$

Apply Dominated Convergence Theorem, it follows that

$$\int_\Omega |C_0 u_m(u_m - u)| \to 0$$
$$\int_\Omega |g(x, u_m)(u_m - u)| \to 0$$

Then

$$\lim_{m\to\infty} \left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N (a_i(x, u_m, \nabla u_m) - a_i(x, u_m, \nabla u)) \frac{\partial(u_m - u)}{\partial x_i} = 0 \tag{7}$$

By (H3) and (H4) we have

$$0 \leq (f_0|\Omega|)^\beta \alpha \int_\Omega |\nabla u_m - \nabla u|^2$$

$$\leq \left(\int_\Omega f(u_m)\right)^\beta \int_\Omega \sum_{i=1}^N (a_i(x,u_m,\nabla u_m) - a_i(x,u_m,\nabla u))\frac{\partial(u_m-u)}{\partial x_i}$$

then (7) implies that $\lim_{m\to\infty}\int_\Omega |\nabla u_m - \nabla u|^2 = 0$. Hence we have $u_m \to u$ strongly in $H_0^1(\Omega)$. and $\nabla u_m \to \nabla u$ almost everywhere in Ω. Then due to the growth condition (H2) of a_i and the boundness of $\|u_m\|_\infty$ and $\|u_m\|$, we can apply Dominated Convergence Theorem to $\int_\Omega \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial\varphi}{\partial x_i}dx$ in (5) to obtain

$$\int_\Omega \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial\varphi}{\partial x_i}dx \to \int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx$$

Similarly, we have

$$\int_\Omega C_0 u_m \varphi dx \to \int_\Omega C_0 u\varphi dx$$

$$\int_\Omega g(x,u_m)\varphi dx \to \int_\Omega g(x,u)\varphi dx$$

Therefore we can pass the limit in (5), and obtain the weak solution of problem (1). □

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