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USING LAGUERRES QUADRATURE IN WEIGHTED RESIDUAL METHOD FOR PROBLEMS WITH SEMI INFINITE DOMAIN

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Abstract: Natural roots of Laguerre Polynomials were used for both partition and collocation points in the weighted residual method for solving boundary value problems involving semi infinite domain. Numerical integration methods such as Trapezoidal rule, Simpson $\frac{1}{3}$ rule and shifted Laguerre formular were used where applicable, results obtained were compared with the exact solution when they are available and with referenced solution where they are not. Different problems were considered and it is observed that the choice of natural roots produce better results in most cases.

AMS Subject Classification: 26A33
Key Words: natural roots, Laguerre polynomials, trapezoidal rule, Simpson rule, partition method, weighted residual

1. Introduction

Many physical problems linear or non linear differential equations in science and engineering such as heat flow and fluid flow in an infinite region are usually subjected to boundary conditions at infinity.

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In Odejide and Aregbesola [6], method of weighted residual was used to solve problems in semi infinite domains. Partition method was used to minimize the residuals. The partition of the infinite domain were carried out in an arbitrary manner, where Simpson $\frac{1}{3}$[5] rule was used to integrate the finite sub-intervals while the remaining part was integrated using shifted Laguerre formular[4].

In this paper, after forcing the Trial functions on the boundary conditions as in the case of [9], an additional condition of forcing the residual to zero at $x = 0$ is included, with both partition and collocation methods used to minimize the residual using the roots of Laguerre polynomials as partition and collocation points.

2. Method of Weighted Residuals

The idea of method weighted residuals is to seek an approximate solution in form of a polynomial to the differential equation of the form

$$L[u(x)] = f \text{ in the domain } \Omega, \tag{1}$$

$$B_\mu[u] = \Omega \text{ on } \partial\Omega, \tag{2}$$

where $L[u]$ denotes a general differential operator (linear or non linear) involving spatial derivatives of dependent variable u as in [7] and [8], f is a known function of position, $B_\mu[u]$ represents the appropriate number of boundary conditions and Ω is the domain with the boundary $\partial\Omega$.

A trial function of the form

$$\phi = \phi_0 + \sum_{i=1}^{m} c_i\phi_i \tag{3}$$

is assumed, where c_i are constants to be determine which satisfies the given boundary condition (2). The trial function is chosen in such a way that it satisfy all the given boundary condition including those at infinity. The function ϕ_0 satisfies the given boundary conditions, while all the $phi_i s$ satisfy the homogeneous boundary conditions.

Substitution of equation (3) into equation (1) gives the residual function $R(x)$. The idea is to minimize the residual function as close as possible to zero along the whole domain. The following methods were used.

n	x_k	w_k
6	0.22284660	0.45896467
	1.18893210	0.41700083
	2.99273633	0.11337338
	5.77514357	0.1039920
	9.83746742	0.00026102
	15.98287398	0.00000090

Table 1

2.1. Partition Method

In this case the whole domain is sub divided into sub intervals, the roots of Laguerre polynomial were used as the sub division points. The residual is then integrated in each interval and then equated to zero.

The integration of the type $\int_a^b R(x)$ were carried out using Simpson $\frac{1}{3}$ or Trapezoidal rule, while those of the type $\int_a^{\infty} R(x)$ were carried out using shifted Laguerre formular.

$$\int_b^{\infty} R(x) = e^{-b} \sum_{k=1}^{n} w_k e^{x_k} e^b R(x_k + b), \tag{4}$$

where

$$w_k = \frac{(n!)^2}{x_k (L_n'(x_k))^2} \tag{5}$$

and the arguments x_k are the zeros of the nth Laguerre polynomial

$$L_n(x) = e^x \frac{d^n}{dx^n}(e^{-x} x^n). \tag{6}$$

Table 1 shows the roots of Laguerre polynomial x_k and the corresponding weight function w_k for $n = 6$

The resulting non linear equation were solved to obtain parameter c_i.

2.2. Collocation Method

In this case, after obtaining the residual by substituting the assumed trial function into the given differential equation, The residual is wvaluated at the roots of Laguerre polynomial x_k and equate the results to zero.

3. Numerical Examples

Example 1. Consider the heat and mass problem obtained by E. Magyari and B. Keller [2]

$$f''' + ff'' - 2f'^2 = 0 \tag{7}$$

subject to

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0. \tag{8}$$

Assuming the trial function

$$f = \sum_{i=0}^{m} c_i e^{\frac{-i\eta}{4}} \tag{9}$$

Using the procedure discussed in section two for both partition and collocation methods, The results for $f''(0)$ are presented in Table 2 with the roots of Laguerre polynomials used as both partition and collocation points as well as when the points were chosen arbitrarily as 1,2,3.... The graph of the corresponding minimised residual function $R(\eta)$ are shown in Figure 1 for $n = 8$.

$$\begin{aligned} f =& 0.905643755497581 + 0.0000403802230176764\, e^{-0.250000000000000\,\eta} \\ &- 0.00119958523255079\, e^{-0.500000000000000\,\eta} \\ &- 0.0272866998399317\, e^{-0.750000000000000\,\eta} \\ &- 1.69994025086324\, e^{-1.0\,\eta} + 3.87521375587331\, e^{-1.25000000000000\,\eta} \\ &- 8.66120476536236\, e^{-1.50000000000000\,\eta} \\ &+ 12.6098467093495\, e^{-1.75000000000000\,\eta} \\ &- 11.5185895057369\, e^{-2.0\,\eta} + 5.81983327444472\, e^{-2.25000000000000\,\eta} \\ &- 1.30235706835310\, e^{-2.50000000000000\,\eta}. \end{aligned}$$

From Table 2 above the results obtained through the use of the Laguerre roots as partition points are better than those obtained using arbitrary points as partition points.

	n	Computed Value	Referenced Value	abs(error)
Collocation Lag.roots	6	- 1.28144470711322	- 1.281808	$3.6329288678\times 10^{-4}$
Partition Arbtr. Pts.	6	- 1.2898962744	- 1.281808	$8.088274372\times 10^{-3}$
Partition Lag.roots	6	- 1.281781913525	- 1.281808	2.60864748×10^{-5}
Partition Lag.roots	8	- 1.28180382147588	- 1.281808	4.17852412×10^{-6}

Table 2

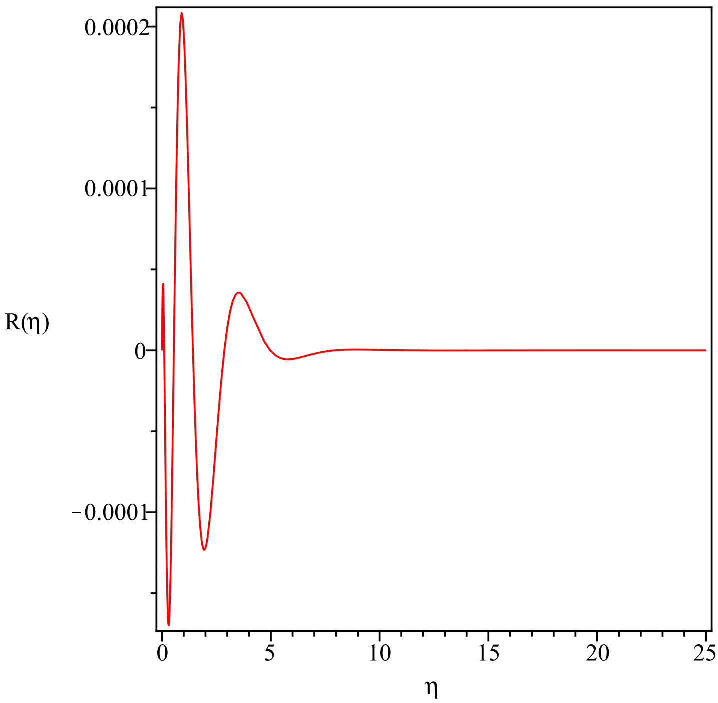

figure 1

Example 2. The fluid problem obtained by A. Mostapha etal[1]

$$f''' + ff'' - (D + Re)f' - (1 + \alpha)f'^2 = 0 \tag{10}$$

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0 \tag{11}$$

whose exact is

$$\frac{1}{\sqrt{1+D+Re}}(1-e^{-\eta\sqrt{1+D+Re}}) \tag{12}$$

when $\alpha = 0, Re = 0, D = 1$. then we have

$$f''' + ff'' - f' - f'^2 = 0 \tag{13}$$

with the same boundary conditions.

Assuming the trial function

$$f = \sum_{i=0}^{m} c_i e^{\frac{-i\eta}{4}} \tag{14}$$

Using the procedure discussed in section two for both partition and collocation methods, with the roots of Laguerre polynomials used as both partition and collocation points and when the points are patitioned arbitrarily. The results obtained in each case for $f''(0)$ are presented in Table 3 and the graph of the corresponding minimised residual function $R(\eta)$ are shown in Figure 2.

$$\begin{aligned} f =& 0.707106776963606 - 0.0000000993002113012473\,\mathrm{e}^{-0.25\,\eta} \\ & + 0.0000606607654697207\,\mathrm{e}^{-0.50\,\eta} \\ & - 0.00138052521759966\,\mathrm{e}^{-0.75\,\eta} + 0.0165585439916541\,\mathrm{e}^{-1.0\,\eta} \\ & - 0.165401527915718\,\mathrm{e}^{-1.25\,\eta} \\ & - 0.723558568975734\,\mathrm{e}^{-1.50\,\eta} + 0.252560111753016\,\mathrm{e}^{-1.75\,\eta} \\ & - 0.116877021098166\,\mathrm{e}^{-2.0\,\eta} \\ & + 0.0360839162512813\,\mathrm{e}^{-2.25\,\eta} - 0.00515137351569660\,\mathrm{e}^{-2.50\,\eta}. \end{aligned}$$

From Table 3 above the results obtained through the use of the Laguerre roots as partition points are better than those obtained using arbitrary points as partition points.

	n	Computed Value	Referenced Value	abs(error)
Collocation Lag.roots	6	- 1.41421423585	- 1.414213562373	6.7347651×10^{-7}
Partition Arbtr. Pts.	6	- 1.414213436075	- 1.414213562373	1.2629744×10^{-7}
Partition Lag.roots	6	- 1.4142135391431	- 1.414213562373	2.32300×10^{-8}
Partition Lag.roots	8	- 1.414213566478	- 1.414213562373	4.10518×10^{-9}

Table 3

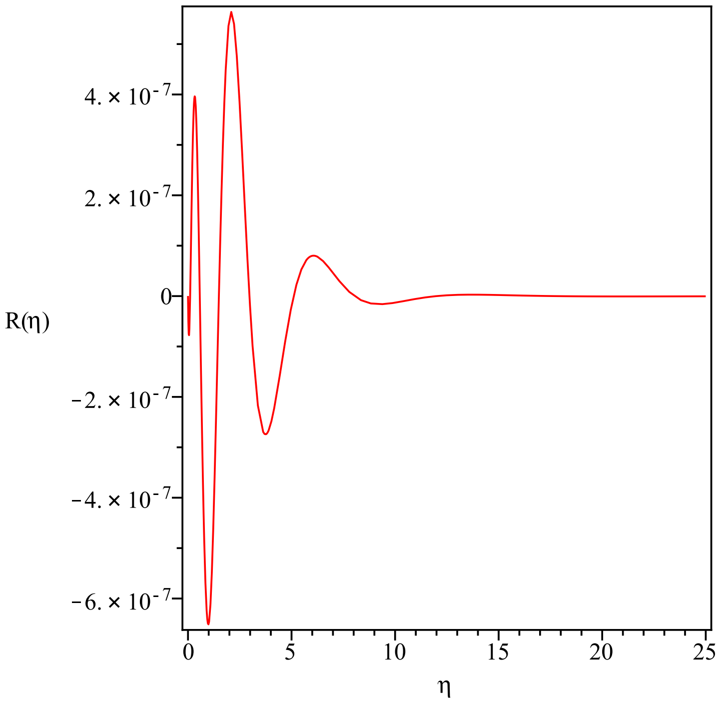

figure 2

Example 3. Consider also the Blasius Equation govern by

$$f''' + \frac{1}{2} f f'' = 0 \tag{15}$$

	n	Computed Value	Referenced Value	abs(error)
Collocation Lag.roots	6	0.333118687	0.33205734	1.061347×10^{-3}
Partition Arbtr. Pts.	6	0.356744046	0.33205734	2.4686706×10^{-2}
Partition Lag.roots	6	0.33213118778	0.33205734	7.3847285×10^{-5}
Partition Lag.roots	8	0.332093421	0.33205734	3.34214×10^{-5}

Table 4

$$f(0) = 0, f'(0) = 0, f'(\infty) = 1 \tag{16}$$

.

Assuming the trial function

$$f = b\eta + \sum_{i=0}^{m} c_i e^{\frac{-i\eta}{4}} \tag{16}$$

Using the procedure discussed in section two for both partition and collocation methods, with the roots of Laguerre polynomials used as both partition and collocation points and when the points are patitioned arbitrarily. The results obtained in each case for $f''(0)$ are presented in table 4 and the graph of the corresponding minimised residual function $R(\eta)$ is shown in Figure 3.

$$\begin{aligned} f =& \eta - 1.72161948095799 + 0.0528503111178819\,\mathrm{e}^{-0.25\,\eta} \\ &- 1.73241230065504\,\mathrm{e}^{-0.50\,\eta} \\ &+ 24.4545187124244\,\mathrm{e}^{-0.75\,\eta} - 164.287563937946\,\mathrm{e}^{-1.0\,\eta} \\ &+ 570.144223716973\,\mathrm{e}^{-1.25\,\eta} \\ &- 1073.28359346677\,\mathrm{e}^{-1.50\,\eta} + 1183.28537312973\,\mathrm{e}^{-1.75\,\eta} \\ &- 772.093370286151\,\mathrm{e}^{-2.0\,\eta} \\ &+ 277.887469520799\,\mathrm{e}^{-2.25\,\eta} - 42.7058759185615\,\mathrm{e}^{-2.50\,\eta}. \end{aligned}$$

From Table 4 above the results obtained through the use of the Laguerre roots as partition points are better than those obtained using arbitrary points as partition points.

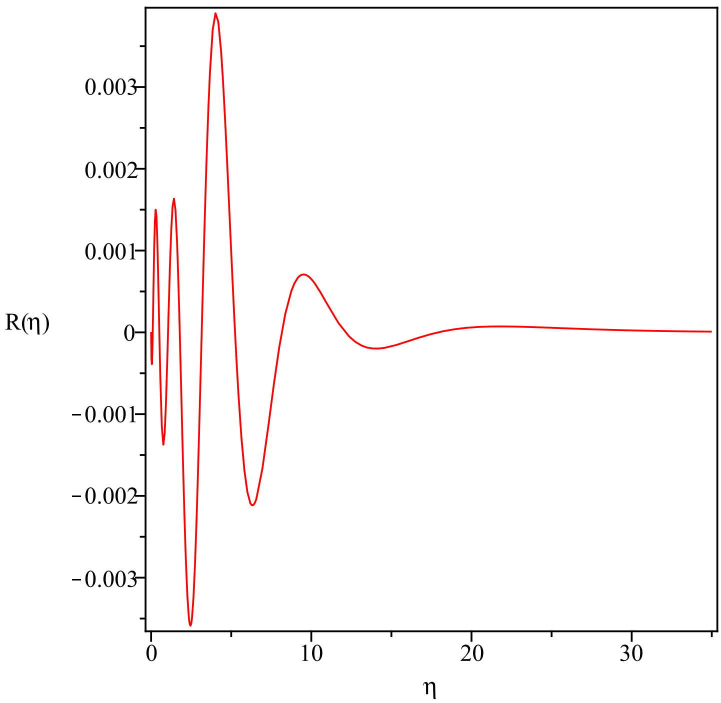

figure 3

Example 4. Consider also the Fakner Skan Equation govern by

$$f''' + ff'' + (1 - f'^2) = 0 \tag{17}$$

$$f(0) = 0, f'(0) = 0, f'(\infty) = 1 \tag{18}$$

.

Assuming the trial function

$$f = b\eta + \sum_{i=0}^{m} c_i e^{\frac{-i\eta}{4}} \tag{19}$$

Using the procedure discussed in section two for both partition and collocation methods, with the roots of Laguerre polynomials used as both partition and collocation points and when the points are patitioned arbitrarily. The results obtained in each case for $f''(0)$ are presented in table 5 and the graph of the corresponding minimised residual function $R(\eta)$ is shown in Figure 4.

$$f = \eta - 0.647923692196899 + 0.00609382120515324\,\mathrm{e}^{-0.25\,\eta}$$

	n	Computed Value	Referenced Value	abs(error)
Collocation Lag.roots	6	1.2348672972	1.232587936	2.2793612×10^{-3}
Partition Arbtr. Pts.	6	1.23148775	1.232587936	1.10017615×10^{-3}
Partition Lag.roots	6	1.232654211	1.232587936	6.6275231×10^{-5}
Partition Lag.roots	8	1.232556966	1.232587936	3.096985×10^{-5}

Table 5

$$
\begin{aligned}
&- 0.234670446284887\, e^{-0.50\,\eta} \\
&+ 3.62186988407087\, e^{-0.75\,\eta} - 27.0549250147286\, e^{-1.0\,\eta} \\
&+ 110.076546338999\, e^{-1.25\,\eta} \\
&- 258.630135567887\, e^{-1.50\,\eta} + 355.212741870772\, e^{-1.75\,\eta} \\
&- 277.489851128898\, e^{-2.0\,\eta} \\
&+ 114.757504223308\, e^{-2.25\,\eta} - 19.6172502883602\, e^{-2.50\,\eta}.
\end{aligned}
$$

From Table 5 above the results obtained through the use of the Laguerre roots as partition points are better than those obtained using arbitrary points as partion points.

4. Conclusion

A weighted residual method using the Laguerre's roots as partition points is presented for the boundary value problems of a class of nonlinear third-order differential equation on semi-infinite intervals. The main idea is to minimize the residual function to a value very close to zero and this has been closely achieved as shown in the figures for all the cases considered. The solutions of the equations are obtained to a near exact analytical solution. Unlike the other methods such as Runge-Kutta method, where solutions are obtained at the preassigned points or the series solutions which are valid within the radius of convergence, the solution obtained through the weighted residual method is valid at every point in the domain.

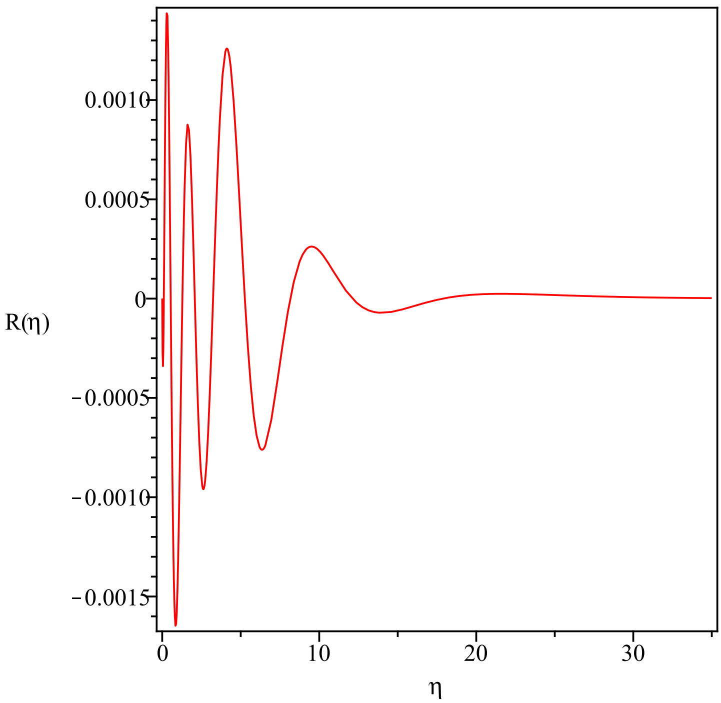

figure 4

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